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# ON ESTIMATION OF HIGH QUANTILES FOR CERTAIN CLASSES OF DISTRIBUTIONS

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**Abstract:** We investigate the rate of convergence of the direct-simulation estimator $\hat{x}_p(n)$ of a large quantile $x_p$ of the Pareto and Gamma distributions. The upper bound of the probability $P\{|\hat{x}_p(n) - x_p| \geqslant \varepsilon\}$ is determined.

**Keywords:** High Quantile Estimation, Negative Dependence, the Pareto Distribution, Gamma Distribution.

**MSC:** 62F12, 62G32.

## 1. INTRODUCTION

Estimation of large quantiles of an unknown distribution function is a statistical problem of great practical importance. Let us mention estimation of the Value-at-Risk parameter for a given financial portfolio as an important problem that directly involves high quantile estimators. Different estimators of high quantiles based on the upper order statistics of a sample were proposed and many important properties were proved. See, for example, Feldman and Tucker [7], Dekkers and de Haan [5], Embrechts et al. [6], Matthus and Beirlant [13] and references therein. In this paper we consider the rate of convergence of the direct-simulation estimator of large quantiles and the aim of this paper is to calculate the rate of convergence of the Pareto and Gamma distributions. Applications of that distributions in theory as in empirical analyzes are well known. For example, it is well established that the burst and idle times for on/off traffic are modeled by the Pareto and Gamma distributions, respectively. Also, the inter arrival times between on/off-traffic is the convolution of the Pareto and Gamma random variables. For details see Nadarajah and Kotz [14]. The Pareto distribution is widely applied in different fields such as finance, insurance, physics, hydrology, geology, climatology, astronomy. Recently much attention has been paid to the statistical distribution of certain socio-economic quantities such as annual personal income of individuals (Pareto's law is one of the two functions most often used to describe the size distribution of income), magnitudes of earthquakes, the size of human settlements, number of hits at web sites, the assets of firms as well as standardized price returns on individual stocks or stock indexes. The intellectual antecedents of these studies can be found in the works of Pareto, Gibrat and others, and for other references see Champernowne [4] , Quandt [15] , Singh and Maddala [18], Levy and Solomon [11], Levy [10], Reed [16] , Aoyama et al. [1], Fujiwara et al. [8]. The second distribution which has been considered in this paper is the Gamma distribution. It is a special case of the generalized inverse Gaussian distribution (that is nonnegative process for modeling changing volatility). This distribution is self-decomposable and may serve as building blocks in the various dynamic models, which has been discussed in paper Barndorff-Nielsen et al. [3]. A review of the definitions and properties of the generalized inverse Gaussian distribution is given in Schiryaev [17]. The importance of the Gamma distribution is also a fact that the Variance Gamma processes are special classes of subordinated processes extensively studied in finance. They have been first introduced in literature by Madan and Seneta [12] as model for stocks return.

## 2. PRELIMINARIES AND NOTATION

Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with the common distribution function $F$. Define the empirical distribution function

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} I(X_k \le x), \quad -\infty < x < \infty,$$

where $I(X_k \le x)$ denotes the indicator of the event $\{X_k \le x\}$. The quantile $x_p$ of the distribution function $F$ is defined by $x_p = \inf\{x : F(x) \ge p\}$, for all $p \in (0, 1)$. In this paper we shall consider the following estimator of $x_p$:

$$\widehat{x_p}(n) = \inf\{t : F_n(t) \ge p\}. \tag{1}$$

That is direct-simulation estimator. The following notion of negative dependence will be used in what follows.

**Definition 2.1.** *(see [19]) Random variables $X_1, X_2, \dots, X_n$, are called negatively dependent if the following two inequalities hold for all $x_1, x_2, ..., x_n$:*

$$P\{X_1 \le x_1, ..., X_n \le x_n\} \le P\{X_1 \le x_1\} \cdot ... \cdot P\{X_n \le x_n\},$$

$$P\{X_1 \ge x_1, ..., X_n \ge x_n\} \le P\{X_1 \ge x_1\} \cdot ... \cdot P\{X_n \ge x_n\}.$$

*A sequence of random variables $(X_n)$ is negatively dependent if for all $n \geqslant 2$, $1 \le j_1 < \cdots < j_n$ and $x_1, \dots, x_n \in \mathbb{R}$ the following inequalities hold true:*

$$P\{X_{j_1} \le x_1, ..., X_{j_n} \le x_n\} \le P\{X_{j_1} \le x_1\} \cdot ... \cdot P\{X_{j_n} \le x_n\},$$

$$P\{X_{j_1} \ge x_1, ..., X_{j_n} \ge x_n\} \le P\{X_{j_1} \ge x_1\} \cdot ... \cdot P\{X_{j_n} \ge x_n\}.$$

**Lemma 2.2.** (see [19]) *If $X_i$, $i = 1, ..., n$ are negatively dependent with $E|X_i| < +\infty$, $i = 1, ..., n$, then*

$$E(X_i X_j) \leq E(X_i)E(X_j), i \neq j, i, j = 1, ..., n.$$

*Furthermore, if $X_i, i = 1, ..., n$ are non-negative and $E(X_1 \cdot ... \cdot X_n) < +\infty$, then*

$$E(X_1 \cdot ... \cdot X_n) \leq E(X_1) \cdot ... \cdot E(X_n).$$

The proof of this lemma can be found in Xing Jin and Michael C. Fu [19].

The following two results show that $\widehat{x_p}(n)$ converges to $x_p$ exponentially fast in probability as n goes to infinity. These results were proved in Xing Jin and Michael C. Fu [19]. We shall use them for our calculation in the next section.

**Lemma 2.3.** (see [19]) *Let $\{Y_n, n \geq 1\}$ be negatively dependent and identically distributed random variables, with moment generating function $M(\lambda) = E[\exp(\lambda Y_1)]$. Let $S_n = \sum_{i=1}^{n} Y_i$. If $M(\lambda)$ exists in a neighborhood $(-\epsilon, \epsilon)$ of $\lambda = 0$ for some $\epsilon > 0$, then*

$$P\{S_n/n \geq x\} \leq e^{-n\Delta_+(x,n)}, \quad \text{for all } x > E(Y_1), \tag{2}$$

$$P\{S_n/n \leq x\} \leq e^{-n\Delta_-(x,n)}, \quad \text{for all } x < E(Y_1), \tag{3}$$

*where*

$$\begin{aligned} \Delta_+(x,n) &= \sup_{0 \leq \lambda \leq \epsilon} \Big(\lambda x - \frac{\ln E\{\exp[\lambda S_n]\}}{n}\Big) \\ &\geq \sup_{0 \leq \lambda \leq \epsilon} (\lambda x - \ln E\{\exp[\lambda Y_1]\}) > 0, \end{aligned}$$

*and*

$$\begin{aligned} \Delta_-(x,n) &= \sup_{-\epsilon \leq \lambda \leq 0} \Big(\lambda x - \frac{\ln E\{\exp[\lambda S_n]\}}{n}\Big) \\ &\geq \sup_{-\epsilon \leq \lambda \leq 0} (\lambda x - \ln E\{\exp[\lambda Y_1]\}) > 0. \end{aligned}$$

*Conversely, if $E[|Y_1|] < +\infty$ and for any $x > E(Y_1)$, there exist $\alpha(x) > 0$ such that*

$$P\{S_n/n \geq x\} \leq e^{-n\alpha(x)},$$

*and for any $x < E(Y_1)$, there exist $\beta(x) > 0$ such that*

$$P\{S_n/n \leq x\} \leq e^{-n\beta(x)},$$

*then the moment generating function $M(\lambda)$ exists in neighborhood $(-\epsilon, \epsilon)$ of $\lambda = 0$ for some $\epsilon > 0$.*

**Theorem 2.4.** (see [19]) *If the distribution function F is strictly increasing and* $\{Y_n, n \geq 1\}$ *are negatively dependent, then*

$$P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\} \leq e^{-n\Delta_+(\epsilon,n)} + e^{-n\Delta_-(\epsilon,n)}, \quad \textit{for all } \epsilon > 0, \tag{4}$$

*where*

$$\Delta_+(\epsilon, n) = \sup_{-\infty<\lambda\leq 0} \Big(\lambda p - \frac{\ln E\{\exp[\lambda \sum_{i=1}^n I(Y_i \leq x_p + \epsilon)]\}}{n}\Big),$$

$$\Delta_-(\epsilon, n) = \sup_{0\leq\lambda<+\infty} \Big(\lambda p - \frac{\ln E\{\exp[\lambda \sum_{i=1}^n I(Y_i \leq x_p - \epsilon)]\}}{n}\Big).$$

*And, moreover, the rate is enhanced by negatively dependence in the sense that*

$$\Delta_+(\epsilon, n) \geq \sup_{-\infty<\lambda\leq 0} (\lambda p - \ln E\{\exp[\lambda I(Y \leq x_p + \epsilon)]\}) > 0,$$

$$\Delta_-(\epsilon, n) \geq \sup_{0\leq\lambda<+\infty} (\lambda p - \ln E\{\exp[\lambda I(Y \leq x_p - \epsilon)]\}) > 0,$$

*where the right-hand "sup" quantiles are the rates for i.i.d. samples.*

**Remark 2.5.** *Let us notice* $p_n = P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\}$*. If we use the fact that probabilities* $p_n$ *are finite,*

$$\sum_{n=1}^{\infty} p_n \leq \frac{1}{1 - e^{-\Delta_+}} + \frac{1}{1 - e^{-\Delta_-}} < +\infty,$$

*and Borel-Cantelli lemma, then the probability that infinitely many of them occur is* 0*, that is*

$$P\{\lim_{n\to+\infty} \widehat{x_p}(n) = x_p\} = 1.$$

## 3. THE CASES OF THE PARETO AND GAMMA DISTRIBUTIONS

In this section we shall determine the rate of convergence for the Pareto and Gamma distributions.

**Theorem 3.1.** *Let* $\{Y_n, n \geq 1\}$ *be negatively dependent random variables with the common Pareto distribution*

$$\overline{F}(x) = Kx^{-\alpha},\ x \geq \sqrt[\alpha]{K},\ K, \alpha > 0,$$
$$F(x) = 1 - Kx^{-\alpha}.$$

*The rate of convergence for standard quantile estimator* $\widehat{x_p}(n)$ *in this case is given by*

$$\begin{aligned} P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\} \quad \leq \quad & \Big(\frac{pK(x_p+\epsilon)^{-\alpha}}{(1 - K(x_p+\epsilon)^{-\alpha})(1-p)}\Big)^{-pn} \cdot \Big(\frac{K(x_p+\epsilon)^{-\alpha}}{1-p}\Big)^{n} \\ + \quad & \Big(\frac{pK(x_p-\epsilon)^{-\alpha}}{(1 - K(x_p-\epsilon)^{-\alpha})(1-p)}\Big)^{-pn} \cdot \Big(\frac{K(x_p-\epsilon)^{-\alpha}}{1-p}\Big)^{n}. \end{aligned}$$

*Proof.* Since the Pareto distribution is strictly increasing we may us the Theorem(2.4) and obtain

$$P\{|\widehat{x_p}(n) - x_p| \geq \epsilon\} \leq e^{-n\Delta_+(\epsilon,n)} + e^{-n\Delta_-(\epsilon,n)}, \quad \text{for all } \epsilon > 0,$$

where

$$\begin{aligned} \Delta_+(\epsilon, n) &\geq \sup_{-\infty<\lambda\leq 0} (\lambda p - \ln E\{\exp[\lambda I(Y \leq x_p + \epsilon)]\}) = \Delta_+, \\ \Delta_-(\epsilon, n) &\geq \sup_{0\leq\lambda<+\infty} (\lambda p - \ln E\{\exp[\lambda I(Y \leq x_p - \epsilon)]\}) = \Delta_-, \end{aligned}$$

and $p = P[Y \leq x_p]$. Let us determine $\Delta_+$ and $\Delta_-$. We may denote

$$p^+ = P[Y \leq x_p + \epsilon] = F(x_p + \epsilon) = 1 - K(x_p + \epsilon)^{-\alpha}.$$

The distribution of the indicator $I(Y \leq x_p + \epsilon)$ is given by

$$I(Y \leq x_p + \epsilon) : \begin{pmatrix} 0 & 1 \\ 1 - p^+ & p^+ \end{pmatrix},$$

where $p^+ = 1 - K(x_p + \epsilon)^{-\alpha}$ and $1 - p^+ = K(x_p + \epsilon)^{-\alpha}$. Now we may calculate

$$\Delta_+ = \sup_{-\infty<\lambda\leq 0} (\lambda p - \ln(e^\lambda p^+ + 1 - p^+)).$$

The maximum of the function $\lambda p - \ln(e^\lambda p^+ + 1 - p^+)$ is attained for $\lambda = \ln \frac{p(1-p^+)}{(1-p)p^+}$, and since $p < p^+$ it is always negative. Consequently we obtain that

$$\Delta_+ = p \ln \frac{p(1 - p^+)}{(1 - p)p^+} - \ln \frac{1 - p^+}{1 - p},$$

$$\Delta_+ = p \ln \frac{pK(x_p + \epsilon)^{-\alpha}}{(1 - K(x_p + \epsilon)^{-\alpha})(1 - p)} - \ln \frac{K(x_p + \epsilon)^{-\alpha}}{(1 - p)}.$$

Similarly, let us denote $p^- = P[Y \leq x_p - \epsilon] = F(x_p - \epsilon) = 1 - K(x_p - \epsilon)^{-\alpha}$. We may calculate

$$\Delta_- = \sup_{0\leq\lambda<+\infty} (\lambda p - \ln(e^\lambda p^- + 1 - p^-)).$$

The maximum of the function $\lambda p - \ln(e^\lambda p^- + 1 - p^-)$ is attained for $\lambda = \ln \frac{p(1-p^-)}{(1-p)p^-}$, and since $p > p^-$ it is always positive. Now we may calculate

$$\Delta_- = p \ln \frac{p(1 - p^-)}{(1 - p)p^-} - \ln \frac{1 - p^-}{1 - p},$$

$$\Delta_- = p \ln \frac{pK(x_p - \epsilon)^{-\alpha}}{(1 - K(x_p - \epsilon)^{-\alpha})(1 - p)} - \ln \frac{K(x_p - \epsilon)^{-\alpha}}{(1 - p)}.$$

Finally we obtain

$$\begin{aligned} P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\} &\leq e^{-n\Delta+} + e^{-n\Delta-} \\ &= \Big(\frac{pK(x_p+\epsilon)^{-\alpha}}{(1-K(x_p+\epsilon)^{-\alpha})(1-p)}\Big)^{-pn} \cdot \Big(\frac{K(x_p+\epsilon)^{-\alpha}}{1-p}\Big)^{n} \\ &+ \Big(\frac{pK(x_p-\epsilon)^{-\alpha}}{(1-K(x_p-\epsilon)^{-\alpha})(1-p)}\Big)^{-pn} \cdot \Big(\frac{K(x_p-\epsilon)^{-\alpha}}{1-p}\Big)^{n}, \end{aligned}$$

and the proof is completed. □

Also, we can analyze more general case, for example general Pareto distribution, $F(x) = L(x)x^{-\alpha}$, where $\alpha > 0$ and $L(x)$ is slowly varying function. In that case we can obtain the next result:

$$\begin{aligned} P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\} &\leq e^{-n\Delta+} + e^{-n\Delta-} \\ &= \Big(\frac{p(1-L(x_p+\epsilon)(x_p+\epsilon)^{-\alpha})}{(1-p)L(x_p+\epsilon)(x_p+\epsilon)^{-\alpha}}\Big)^{-pn} \cdot \Big(\frac{1-L(x_p+\epsilon)(x_p+\epsilon)^{-\alpha}}{1-p}\Big)^{n} \\ &+ \Big(\frac{p(1-L(x_p-\epsilon)(x_p-\epsilon)^{-\alpha})}{(1-p)L(x_p-\epsilon)(x_p-\epsilon)^{-\alpha}}\Big)^{-pn} \cdot \Big(\frac{1-L(x_p-\epsilon)(x_p-\epsilon)^{-\alpha}}{1-p}\Big)^{n}. \end{aligned}$$

The proof in this case is analog as the proof for Pareto distribution and we will omit it here.

**Theorem 3.2.** *Let* $\{Y_n, n \geq 1\}$ *be negatively dependent random variables with the common Gamma density*

$$f(x,\alpha,\beta) = x^{\alpha-1}\frac{\beta^{\alpha}e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0, \alpha > 0, \beta > 0.$$

*The rate of convergence for standard quantile estimator* $\widehat{x_p}(n)$ *in this case is given by:*

$$\begin{aligned} P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\} &\leq \Big(\frac{pA}{\Gamma(\alpha)(1-\frac{A}{\Gamma(\alpha)})(1-p)}\Big)^{-pn} \cdot \Big(\frac{A}{\Gamma(\alpha)(1-p)}\Big)^{n} \\ &+ \Big(\frac{pB}{\Gamma(\alpha)(1-\frac{B}{\Gamma(\alpha)})(1-p)}\Big)^{-pn} \cdot \Big(\frac{B}{\Gamma(\alpha)(1-p)}\Big)^{n}, \end{aligned}$$

*where*

$$A = e^{-\beta(x_p+\epsilon)}\Big([\beta(x_p+\epsilon)]^{\alpha-1} + (\alpha-1)[\beta(x_p+\epsilon)]^{\alpha-2} + o([\beta(x_p+\epsilon)]^{\alpha-2})\Big),$$
$$B = e^{-\beta(x_p-\epsilon)}\Big([\beta(x_p-\epsilon)]^{\alpha-1} + (\alpha-1)[\beta(x_p-\epsilon)]^{\alpha-2} + o([\beta(x_p-\epsilon)]^{\alpha-2})\Big).$$

*Proof.* Since the Gamma distribution is strictly increasing we may use the same notation as in Section 2 and we shall determine

$$\Delta_+ = \sup_{-\infty<\lambda\leq 0} (\lambda p - \ln E\{\exp[\lambda I(Y \leq x_p + \epsilon)]\}),$$
$$\Delta_- = \sup_{0\leq\lambda<+\infty} (\lambda p - \ln E\{\exp[\lambda I(Y \leq x_p - \epsilon)]\}).$$

Let us denote $p^+ = P[Y \leq x_p + \epsilon]$. If we use substitution $y = \beta x$ we will obtain

$$\begin{aligned} p^+ &= P[Y \leq x_p + \epsilon] = \int_0^{x_p+\epsilon} x^{\alpha-1}\frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)}dx \\ &= \frac{1}{\Gamma(\alpha)}\int_0^{\beta(x_p+\epsilon)} y^{\alpha-1}e^{-y}dy \\ &= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} y^{\alpha-1}e^{-y}dy - \int_{\beta(x_p+\epsilon)}^{+\infty} y^{\alpha-1}e^{-y}dy\Big) \\ &= \frac{1}{\Gamma(\alpha)}\Big(\Gamma(\alpha) - A\Big) = 1 - \frac{A}{\Gamma(\alpha)}, \end{aligned} \tag{5}$$

where

$$A = e^{-\beta(x_p+\epsilon)}\Big([\beta(x_p+\epsilon)]^{\alpha-1} + (\alpha-1)[\beta(x_p+\epsilon)]^{\alpha-2} + o([\beta(x_p+\epsilon)]^{\alpha-2})\Big).$$

The last equality follows from the result

$$\int_t^{+\infty} s^{r-1}e^{-s}ds = e^{-t}\Big(t^{r-1} + (r-1)t^{r-2} + o(t^{r-2})\Big),$$

which can be found in Dekkers and de Haan [5]. Now we may obtain

$$\Delta_+ = \sup_{-\infty<\lambda\leq 0}\Big(\lambda p - \ln\Big(e^\lambda p^+ + 1 - p^+\Big)\Big),$$

$$\Delta_+ = \sup_{-\infty<\lambda\leq 0}\Big(\lambda p - \ln\Big(e^\lambda\Big(1 - \frac{A}{\Gamma(\alpha)}\Big) + \frac{A}{\Gamma(\alpha)}\Big)\Big).$$

Maximum is attained for $\lambda = \ln\frac{p(1-p^+)}{(1-p)p^+}$, and since $p < p^+$ it is always negative. And we may calculate

$$\Delta_+ = p\ln\frac{pA}{\Gamma(\alpha)(1-\frac{A}{\Gamma(\alpha)})(1-p)} - \ln\frac{A}{\Gamma(\alpha)(1-p)}. \tag{6}$$

Similarly let us denote $p^- = P[Y \leq x_p - \epsilon]$. Now we may obtain

$$\Delta_- = \sup_{0\leq\lambda<+\infty}(\lambda p - \ln(e^\lambda p^- + 1 - p^-)).$$

In this case the maximum is attained for $\lambda = \ln\frac{p(1-p^-)}{(1-p)p^-}$, and since $p > p^-$ it is always positive. Similarly as we have obtained $p^+$ ( equation (5)) we may obtain

$$p^- = 1 - \frac{B}{\Gamma(\alpha)},$$

where

$$B = e^{-\beta(x_p-\epsilon)}\Big([\beta(x_p-\epsilon)]^{\alpha-1} + (\alpha-1)[\beta(x_p-\epsilon)]^{\alpha-2} + o([\beta(x_p-\epsilon)]^{\alpha-2})\Big).$$

As we have calculated $\Delta_+$ ( equation (6)) we may calculate

$$\Delta_- = p \ln \frac{pB}{\Gamma(\alpha)(1-\frac{B}{\Gamma(\alpha)})(1-p)} - \ln \frac{B}{\Gamma(\alpha)(1-p)}. \tag{7}$$

Finally we obtain result

$$\begin{aligned} P\{\left|\widehat{x_p}(n) - x_p\right| \geq \epsilon\} &\leq e^{-n\Delta_+} + e^{-n\Delta_-} \\ &= \Big(\frac{pA}{\Gamma(\alpha)(1-\frac{A}{\Gamma(\alpha)})(1-p)}\Big)^{-pn} \cdot \Big(\frac{A}{\Gamma(\alpha)(1-p)}\Big)^{n} \\ &+ \Big(\frac{pB}{\Gamma(\alpha)(1-\frac{B}{\Gamma(\alpha)})(1-p)}\Big)^{-pn} \cdot \Big(\frac{B}{\Gamma(\alpha)(1-p)}\Big)^{n}, \end{aligned} \tag{8}$$

and the proof is completed. □

## 4. NUMERICAL EXAMPLES AND DATA ANALYSIS

### *4.1. Numerical examples*

In this subsection we present some numerical examples to see the performance of rate of convergence from the Section 3. Table 4.1.1–4.1.4 contain results that are related to the Pareto distribution and Theorem 3.1. We take two values of $K$ and $\alpha$ and three values of $\epsilon$. The whole approach can be applied on any other parameters setting. For each parameters setting we compute rate of convergence by using the appropriate formula.

Table 4.1.1 Rate of convergence for Pareto distribution and $K = 2, \alpha = 3, p = 0.32, x_p = 1.432761$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $4^2$ | 0.595463 | 0.260649 | 0.856112 |
| | $5^2$ | 0.444846 | 0.122345 | 0.567191 |
| | $7^2$ | 0.204405 | 0.016281 | 0.220686 |
| | $10^2$ | 0.039160 | 0.000224 | 0.039384 |
| | $13^2$ | 0.004187 | 0.000001 | 0.004188 |
| $\epsilon = 0.01$ | $14^3$ | 0.293799 | 0.262761 | 0.556560 |
| | $16^3$ | 0.160676 | 0.136011 | 0.296687 |
| | $19^3$ | 0.946808 | 0.035409 | 0.982217 |
| | $22^3$ | 0.008625 | 0.005593 | 0.014218 |
| | $25^3$ | 0.000935 | 0.000495 | 0.001430 |
| $\epsilon = 0.001$ | $12^5$ | 0.315207 | 0.312308 | 0.627515 |
| | $14^5$ | 0.082465 | 0.080834 | 0.163299 |
| | $10^6$ | 0.009660 | 0.009308 | 0.018968 |
| | $11^6$ | 0.000269 | 0.000252 | 0.000521 |
| | $12^6$ | 0.000001 | 0.000001 | 0.000002 |

Table 4.1.2 Rate of convergence for Pareto distribution and $K = 2, \alpha = 3, p = 0.99, x_p = 5.8480355$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $9^5$ | 0.468399 | 0.443661 | 0.912060 |
| | $10^5$ | 0.276812 | 0.252510 | 0.529322 |
| | $12^5$ | 0.040924 | 0.032559 | 0.073483 |
| | $15^5$ | 0.000058 | 0.000029 | 0.000087 |
| | $10^6$ | 0.000003 | 0.000001 | 0.000004 |
| $\epsilon = 0.01$ | $14^6$ | 0.368872 | 0.366333 | 0.735205 |
| | $15^6$ | 0.221193 | 0.218895 | 0.440088 |
| | $16^6$ | 0.108372 | 0.196717 | 0.305089 |
| | $19^6$ | 0.001967 | 0.001884 | 0.003851 |
| | $21^6$ | 0.000012 | 0.000011 | 0.000023 |
| $\epsilon = 0.001$ | $13^8$ | 0.338286 | 0.338067 | 0.676353 |
| | $14^8$ | 0.140734 | 0.140569 | 0.281303 |
| | $15^8$ | 0.033323 | 0.331290 | 0.364613 |
| | $16^8$ | 0.003323 | 0.003312 | 0.006635 |
| | $17^8$ | 0.000094 | 0.000094 | 0.000188 |

Table 4.1.3 Rate of convergence for Pareto distribution and $K = 5, \alpha = 5, p = 0.32, x_p = 1.490363$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | 10 | 0.483973 | 0.011269 | 0.495242 |
| | 13 | 0.389285 | 0.002934 | 0.392219 |
| | 15 | 0.336691 | 0.001196 | 0.337887 |
| | 20 | 0.234230 | 0.000127 | 0.234357 |
| | 27 | 0.140935 | 0.000006 | 0.140941 |
| $\epsilon = 0.01$ | $10^3$ | 0.325466 | 0.178036 | 0.603502 |
| | $12^3$ | 0.143751 | 0.109498 | 0.253249 |
| | $15^3$ | 0.022631 | 0.013300 | 0.035931 |
| | $17^3$ | 0.004027 | 0.001857 | 0.005884 |
| | $10^4$ | 0.000013 | 0.000003 | 0.000016 |
| $\epsilon = 0.001$ | $10^5$ | 0.305191 | 0.299674 | 0.604865 |
| | $11^5$ | 0.147876 | 0.143595 | 0.291471 |
| | $13^5$ | 0.012197 | 0.011398 | 0.023595 |
| | $15^5$ | 0.000122 | 0.000126 | 0.000228 |
| | $10^6$ | 0.000007 | 0.000006 | 0.000013 |

Table 4.1.4 Rate of convergence for Pareto distribution and $K = 5, \alpha = 5, p = 0.99, x_p = 2.8854$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $13^2$ | 0.536247 | 0.214966 | 0.751213 |
| | $15^2$ | 0.436202 | 0.129164 | 0.565366 |
| | $19^2$ | 0.264179 | 0.037487 | 0.301666 |
| | $10^3$ | 0.025038 | 0.000122 | 0.025160 |
| | $11^3$ | 0.007388 | 0.000005 | 0.007393 |
| $\epsilon = 0.01$ | $13^2$ | 0.382301 | 0.349354 | 0.731655 |
| | $16^2$ | 0.233040 | 0.203303 | 0.436343 |
| | $10^3$ | 0.003381 | 0.001983 | 0.005364 |
| | $11^3$ | 0.000514 | 0.000253 | 0.000767 |
| | $13^3$ | 0.000004 | 0.000001 | 0.000005 |
| $\epsilon = 0.001$ | $15^2$ | 0.263636 | 0.260492 | 0.524128 |
| | $18^2$ | 0.146638 | 0.144127 | 0.290765 |
| | $25^2$ | 0.024642 | 0.023835 | 0.048477 |
| | $11^3$ | 0.000376 | 0.000350 | 0.000726 |
| | $13^3$ | 0.000002 | 0.000002 | 0.000004 |

From Table 4.1.1–4.1.4 we can see that rate of convergence decreases as the sample size increases. That is also an expected result.

Tables 4.2.1–4.2.4 contain results that are related to the Gamma distribution and Theorem 3.2. We take two values of $\alpha$ and $\beta$ and three vales of $\epsilon$. The whole approach can be applied on any other parameters setting. For each parameters setting, we compute rate of convergence by using the appropriate formula.

Table 4.2.1 Rate of convergence for Gamma distribution and $\alpha = 1, \beta = 2, p = 0.95, x_p = 1.497866$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $10^3$ | 0.375500 | 0.321375 | 0.696875 |
| | $12^3$ | 0.271522 | 0.140643 | 0.412165 |
| | $13^3$ | 0.116257 | 0.082586 | 0.198843 |
| | $17^3$ | 0.008129 | 0.003784 | 0.011913 |
| | $10^4$ | 0.000056 | 0.000012 | 0.000068 |
| $\epsilon = 0.01$ | $10^5$ | 0.351729 | 0.346295 | 0.698024 |
| | $11^5$ | 0.185851 | 0.181248 | 0.367099 |
| | $12^5$ | 0.074271 | 0.071449 | 0.145720 |
| | $2 * 13^5$ | 0.005516 | 0.005105 | 0.010621 |
| | $10^6$ | 0.000029 | 0.000025 | 0.000054 |
| $\epsilon = 0.001$ | $10^7$ | 0.439389 | 0.348647 | 0.788036 |
| | $11^7$ | 0.128837 | 0.128304 | 0.257141 |
| | $12^7$ | 0.023099 | 0.022924 | 0.046023 |
| | $13^7$ | 0.001362 | 0.001344 | 0.002706 |
| | $10^8$ | 0.000027 | 0.000026 | 0.000053 |

Table 4.2.2 Rate of convergence for Gamma distribution and $\alpha = 1, \beta = 2, p = 0.99, x_p = 2.302588$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $6 * 10^3$ | 0.321812 | 0.272792 | 0.594604 |
| | $10^4$ | 0.151125 | 0.114741 | 0.265866 |
| | $2 * 10^4$ | 0.022839 | 0.013165 | 0.036004 |
| | $3 * 10^4$ | 0.003451 | 0.001511 | 0.004962 |
| | $6 * 10^5$ | 0.000001 | 0.000002 | 0.000003 |
| $\epsilon = 0.01$ | $14^5$ | 0.339887 | 0.334887 | 0.674774 |
| | $10^6$ | 0.134459 | 0.130804 | 0.265263 |
| | $11^6$ | 0.028592 | 0.027230 | 0.055822 |
| | $12^6$ | 0.002500 | 0.002303 | 0.004803 |
| | $13^6$ | 0.000062 | 0.000054 | 0.000116 |
| $\epsilon = 0.001$ | $12^7$ | 0.485174 | 0.484566 | 0.969740 |
| | $10^8$ | 0.132861 | 0.132397 | 0.265258 |
| | $11^8$ | 0.013211 | 0.013112 | 0.026323 |
| | $12^8$ | 0.000170 | 0.000168 | 0.000338 |
| | $3 * 11^8$ | 0.000002 | 0.000002 | 0.000004 |

Table 4.2.3 Rate of convergence for Gamma distribution and $\alpha = 1, \beta = 3, p = 0.95, x_p = 0.9985774$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $20^2$ | 0.426816 | 0.345625 | 0.772441 |
| | $25^2$ | 0.264394 | 0.190139 | 0.454533 |
| | $10^3$ | 0.119015 | 0.070228 | 0.189243 |
| | $15^3$ | 0.000759 | 0.000128 | 0.000887 |
| | $17^3$ | 0.000029 | 0.000002 | 0.000031 |
| $\epsilon = 0.01$ | $35^3$ | 0.366289 | 0.358154 | 0.724443 |
| | $7 * 10^4$ | 0.194033 | 0.187047 | 0.381080 |
| | $10^5$ | 0.096090 | 0.091186 | 0.187276 |
| | $12^5$ | 0.002942 | 0.002582 | 0.005524 |
| | $14^5$ | 0.000003 | 0.000002 | 0.000005 |
| $\epsilon = 0.001$ | $5 * 10^6$ | 0.306406 | 0.305569 | 0.611975 |
| | $8 * 10^6$ | 0.150687 | 0.150029 | 0.300716 |
| | $10^7$ | 0.093884 | 0.093373 | 0.187257 |
| | $12^7$ | 0.000208 | 0.000204 | 0.000412 |
| | $3 * 11^7$ | 0.000001 | 0.000001 | 0.000002 |

Table 4.2.4 Rate of convergence for Gamma distribution and $\alpha = 1, \beta = 3, p = 0.99, x_p = 1.53505672$

| | n | $\Delta_+$ | $\Delta_-$ | $e^{-n\Delta_+} + e^{-n\Delta_-}$ |
|---|---|---|---|---|
| $\epsilon = 0.1$ | $3 * 10^3$ | 0.290920 | 0.219950 | 0.510870 |
| | $7 * 10^3$ | 0.127729 | 0.029203 | 0.156932 |
| | $10^4$ | 0.016315 | 0.006423 | 0.022738 |
| | $11^4$ | 0.002416 | 0.000617 | 0.003033 |
| | $13^4$ | 0.000008 | 0.000001 | 0.000009 |
| $\epsilon = 0.01$ | $3 * 10^5$ | 0.259284 | 0.252168 | 0.511452 |
| | $5 * 10^5$ | 0.105429 | 0.100651 | 0.206080 |
| | $10^6$ | 0.011115 | 0.010131 | 0.021246 |
| | $11^6$ | 0.000345 | 0.000293 | 0.000638 |
| | $12^6$ | 0.000001 | 0.000001 | 0.000002 |
| $\epsilon = 0.001$ | $3 * 10^7$ | 0.256091 | 0.255367 | 0.511458 |
| | $5 * 10^7$ | 0.103274 | 0.102788 | 0.206062 |
| | $10^8$ | 0.010665 | 0.010565 | 0.021230 |
| | $15^7$ | 0.000427 | 0.000420 | 0.000847 |
| | $11^8$ | 0.000059 | 0.000058 | 0.000117 |

From Table 4.2.1–4.2.4 we can see that rate of convergence decrease as the sample size increase. That is also expected result.

*4.2. Data Analysis*

In this subsection we analyze real data set and demonstrate how the proposed results can be used in practice. The data set X represent the failure time of the air conditioning system of an airplane (in hours): 33, 47, 55, 56, 104, 176, 182, 220, 239, 246, 320 and it is reported by Bain and Engelhardt [2]. X can be model with Gamma(1,152.5) distribution. Jovanović and Rajić [9] studied validity of the Gamma distribution for that data and they computed Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function, and KS statistic is approximately 0.23 with p value grater than 0.05. It is clear that the Gamma model fits quite well this data set.
We obtain $\widehat{x_p}(n)$ by using formula (1) for $p = 0.90$ and $n = 11$, and we obtain $\widehat{x_{0.90}}(11) = 246$. It is possible to calculate quantile $x_p$ for Gamma(1,152.5) distribution and probability $p = 0.90$, it is $x_{0.90} = 351.1442$. Now, using result (8) we can calculate that there is 94.95% chances that quantile $x_{0.90}$ deviates from direct-simulation estimator $\widehat{x_{0.90}}(11)$, for more than 160.

## 5. CONCLUSION

In this paper is considered the estimation of the probability
$P\{|\hat{x}_p(n) - x_p| \geqslant \varepsilon\}$ of the direct-simulation estimator $\hat{x}_p(n)$ of a large quantile $x_p$. Some results for rate of convergence for Pareto and Gamma distributions are determined. That results show that rate of convergence decreases as the samplesize increases.

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