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STRICT BENSON PROPER-ε-EFFICIENCY IN VECTOR OPTIMIZATION WITH SET-VALUED MAPS

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Abstract: In this paper the notion of Strict Benson proper-ε-efficient solution for a vector optimization problem with set-valued maps is introduced. The scalarization theorems and ε-Lagrangian multiplier theorems are established under the assumption of ic-cone-convexlikeness of set-valued maps.

Keywords: Ic-cone-convexlikeness, Set-valued Maps, strict Benson proper-ε-efficiency, scalarization, ε-Lagrangian Multipliers.

MSC: 90C26, 90C29, 90C30, 90C46.

1. INTRODUCTION

In the study of vector optimization, the theory of efficiency plays an important role. Kuhn and Tucker [8] and later Geoffrion [6] observed that certain efficient points exhibit some abnormal properties and to eliminate such anomalous solutions in large set of efficient solutions, they introduced the concept of proper efficiency. Borwein [2] and Benson [1] proposed proper efficiency for vector

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maximization problem over cones. Chen and Rong [4] and Li [9] gave characterization of Benson proper efficiency for vector optimization problems. Cheng and Fu [5] introduced the concept of strong efficiency in locally convex spaces. Various authors have studied approximate efficient solutions for vector optimization problems. Some of them are Liu [11], Chen and Huang [3] and Rong and Wu [12]. Li, Xu and Zhu [10] introduced ε-strictly efficient solutions for set-valued optimization problem.

The purpose of this paper is to introduce the notion of Strict Benson proper-ε-efficient solution for vector optimization problems with set-valued maps as a generalization of Benson proper efficient solution [1]. We study the relationship of Strict Benson proper-ε-efficient solution with ε-strict efficient solution given by Li et al. [10]. An alternative theorem is presented in section 3 for ic-cone-convexlikeness set-valued maps, which were introduced by Sach [13], and scalarization theorems and ε-Lagrangian Multiplier theorems are established in sections 4 and 5.

2. DEFINITIONS AND NOTATIONS

Let X be locally convex topological vector space and Y, Z be real locally convex Hausdorff topological vector spaces; let $D \subset Y$, $E \subset Z$ be pointed closed convex cones. For a set $A \subset Y$, we write $\text{cone}\, A = \{\lambda a : \lambda \geq 0, a \in A\}$.
The closure and interior of the set A are denoted by $\text{cl}\, A$ and $\text{int}\, A$. A convex subset B of $\text{cone}\, A$ is a base of A if $0 \notin \text{cl}\, B$ and $A = \text{cone}\, B$. Let Y^* be the dual space of Y, the positive dual cone D^* of $D \subset Y$ is defined as $D^* = \{f \in Y^* : f(y) \geq 0$ for all $y \in D\}$. The set $D^\#$ of strictly positive functions is defined as
$D^\# = \{f \in Y^* : f(y) > 0 \text{ for all } y \in D \setminus \{0\}\}$.
For a set-valued map $F : X \to 2^Y$ the domain of F, denoted by $\text{dom}\, F$, is defined as $\text{dom}\, F = \{x \in X : F(x) \neq \phi\}$, and the image of F, denoted as $\text{im}F$, is defined as $\text{im}F = F(X) = \bigcup_{x \in X} F(x)$.

Benson [1] introduced the following definition of proper efficiency.

Definition 2.1. If S is non empty set in Y and D is a convex cone in Y, then $y \in S$ is called Benson proper efficient point of S over D written as $y \in \text{BPMin}[S, D]$ if
$\text{clcone}(S + D - y) \cap (-D) = \{0\}$ (1)

Now we introduce the notion of Strict Benson proper-ε-efficient point of a set S over a cone D.

Definition 2.2. Let S be a non empty set in Y, D be a convex cone in Y and $\varepsilon \in D$, then $\bar{y} \in S$ is called a Strict Benson proper-ε-efficient point of S over D written as $\bar{y} \in \text{BP-}\varepsilon\text{-Min}[S, D]$ if $\text{clcone}(S + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$. (2)

It is easy to verify that $\mathrm{BPMin}[S, D] \subset \mathrm{BP}\text{-}\varepsilon\text{-}\mathrm{Min}[S, D]$.

The following example illustrates the proper containment, that is, there exists $\bar{y} \notin \mathrm{BP}\text{-}\varepsilon\text{-}\mathrm{Min}[S, D]$ but $\bar{y} \in \mathrm{BPMin}[S, D]$.

Example 2.3. Let $Y = \mathbb{R}^2$, $D = \{(x, y) : x \leq y, y \geq 0\}$, $\varepsilon = \left(\frac{3}{2}, \frac{3}{2}\right)$,
$S = \left\{\left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{5}{4}, 1\right), \left(\frac{5}{2}, \frac{5}{2}\right), (1, 1)\right\}$, $\bar{y} = (1, 1)$, then $S+\varepsilon-\bar{y} = \left\{\left(1, \frac{5}{4}\right), \left(\frac{7}{4}, \frac{3}{2}\right), (3, 3), \left(\frac{3}{2}, \frac{3}{2}\right)\right\}$ and $\mathrm{clcone}(S + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$.
Thus, $\bar{y} \in \mathrm{BP}\text{-}\varepsilon\text{-}\mathrm{Min}[S, D]$.
Also, $S - \bar{y} = \left\{\left(\frac{-1}{2}, \frac{-1}{4}\right), \left(\frac{1}{4}, 0\right), \left(\frac{3}{2}, \frac{3}{2}\right), (0, 0)\right\}$ which shows that $\left(\frac{-1}{2}, \frac{-1}{4}\right) \in \mathrm{clcone}(S + D - \bar{y}) \cap (-D \setminus \{0\})$ and $\mathrm{clcone}(S + D - \bar{y}) \cap (-D \setminus \{0\}) \neq \phi$.
Thus, $\bar{y} \notin \mathrm{BP}\text{-}\varepsilon\text{-}\mathrm{Min}[S, D]$.

Li, Xu and Zhu [10] introduced ε-strictly minimal efficient point which is defined as follows.

Definition 2.4. Let S be a non empty subset of Y, D be a convex cone in Y, B be a base of D, $\varepsilon \in D$, then $\bar{y} \in S$ is called an ε-strictly minimal efficient point of S with respect to B, written as $\bar{y} \in \varepsilon\text{-}\mathrm{Fmin}[S, B]$ if there is a neighborhood U of 0 such that

$$\mathrm{clcone}(S + \varepsilon - \bar{y}) \cap (U - B) = \phi \tag{3}$$

It is shown in the following theorem that every ε-strict minimal efficient point of S with respect to B is Strict Benson proper-ε-efficient point of S over D.

Theorem 2.5. $\varepsilon\text{-}\mathrm{Fmin}[S, B] \subset \mathrm{BP}\text{-}\varepsilon\text{-}\mathrm{Min}[S, D]$.

Proof. Let $\bar{y} \in \varepsilon\text{-}\mathrm{Fmin}[S, B]$, which implies that there is neighborhood U of 0 such that $\mathrm{clcone}(S + \varepsilon - \bar{y}) \cap (U - B) = \phi$.
Now, to show $\bar{y} \in \mathrm{BP}\text{-}\varepsilon\text{-}\mathrm{Min}[S, D]$, we have to prove $\mathrm{clcone}(S+\varepsilon-\bar{y}) \cap (-D \setminus \{0\}) = \phi$.
On the contrary, suppose that there exists $y^* \in \mathrm{clcone}(S + \varepsilon - \bar{y}) \cap (-D \setminus \{0\})$. It follows that $y^* \in \mathrm{clcone}(S + \varepsilon - \bar{y})$ and $y^* \in (-D \setminus \{0\})$, which gives that $y^* = -d$, for $d \in D \setminus \{0\}$. Since B is a base of D, therefore $D = \mathrm{cone}\, B$, which gives that $d = \lambda b$, for $\lambda \geq 0$, $b \in B$. It follows that $y^* = -d = -\lambda b$.
Clearly, $\frac{y^*}{\lambda} = -b \in (U - B)$ and also, $\frac{y^*}{\lambda} \in \mathrm{clcone}(S + \varepsilon - \bar{y})$. Thus, $\frac{y^*}{\lambda} \in \mathrm{clcone}(S + \varepsilon - \bar{y}) \cap (U - B)$, which gives a contradiction.

Remark 2.6. The following example illustrates that the set of Strict
Benson proper-ε-efficient points is not contained in the set of ε-strictly minimal efficient points.

Example 2.7. Let $Y = \mathbb{R}^2$, $S = \left\{\left(\frac{-1}{2}, \frac{1}{2}\right), \left(-2, \frac{3}{4}\right), \left(\frac{-1}{10}, \frac{3}{2}\right)\right\}$, $D = \{(x, y) : x \leq 0, y \leq 0\}$, $B = \{(x, y) : x + y + 1 = 0, x \leq 0, y \leq 0\}$ be a base of $\mathrm{cone}\, D$, $\varepsilon = \left(\frac{-1}{2}, 0\right)$, $\bar{y} = \left(\frac{-1}{2}, \frac{1}{2}\right)$

and $U = \{(x,y) : x^2 + y^2 < \frac{1}{4}\}$, then $S + \varepsilon - \bar{y} = \left\{\left(\frac{-1}{2}, 0\right), \left(-2, \frac{1}{4}\right), \left(\frac{-1}{10}, 1\right)\right\}$ and clcone$(S + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$, which gives that $\bar{y} \in$ BP-ε-Min$[S, D]$.
Also, $\left(\frac{-1}{10}, 1\right) \in$ clcone$(S + \varepsilon - \bar{y}) \cap (U - B)$, which shows that $\bar{y} \notin \varepsilon$-Fmin$[S, B]$. Thus BP-$\varepsilon$-Min$[S, D] \not\subset \varepsilon$-Fmin$[S, B]$.

3. THEOREM OF ALTERNATIVE

A theorem of the alternative will be established in this section for ic-D-convexlike set-valued maps, which were introduced by Sach [13] and are defined as follows.

Definition 3.1. A set-valued map $F : X \to 2^Y$ is called ic-D-convexlike on X if intcone$(F(X) + D)$ is a convex cone and $F(X) + D \subset$ clintcone$(F(X) + D)$.

Theorem 3.2. Let $\operatorname{int} D \neq \phi$ and let the set-valued map $F : X \to 2^Y$ be ic-D-convexlike on X then, one and only one of the following statements is true:
(I) there exists $x \in X$ such that $F(x) \cap (-\operatorname{int} D) \neq \phi$
(II) there exists $\mu \in (D^* \setminus \{0\})$ such that $\mu(y) \geq 0$, for all $y \in F(X)$.

Proof. Assume that both (I) and (II) hold. Then there exist $x \in X$, $y \in F(x)$ such that $y \in -\operatorname{int} D$, which gives that $\mu(y) < 0$ for every $\mu \in D^* \setminus \{0\}$. This contradicts (II). Thus (I) and (II) cannot hold simultaneously.
Now, we show that if (I) is not true, then (II) holds.
Suppose that $F(X) \cap (-\operatorname{int} D) = \phi$. (4)
Now we claim that intcone$(F(X)+D) \cap (-\operatorname{int} D) = \phi$. Indeed, let $y^* \in$ intcone$(F(X)+D) \cap (-\operatorname{int} D)$, then there exist $x \in X, d \in D, \lambda > 0$ such that $y^* = \lambda(F(x)+d) \in -\operatorname{int} D$, which gives that $\frac{y^*}{\lambda} - d \in F(x)$. Since $y^* \in -\operatorname{int} D$, $\lambda > 0$ therefore, $\frac{y^*}{\lambda} \in -\operatorname{int} D$, which implies that $\frac{y^*}{\lambda} - d \in -\operatorname{int} D$. Thus, $\frac{y^*}{\lambda} - d \in F(X) \cap (-\operatorname{int} D)$, which contradicts (4).
By the assumption F is ic-D-convexlike on X, we have that intcone$(F(X) + D)$ is a convex cone. Thus, by separation theorem for convex sets in topological vector spaces as given by Jahn [7], there exists $\mu \in Y^* \setminus \{0\}$ such that
$\mu(y + td) \geq 0$ for all $y \in F(X)$, $d \in D$ and $t > 0$. (5)
We assert that $\mu(d) \geq 0$ for all $d \in D$ otherwise, suppose that there exists $\bar{d} \in D$ with $\mu(\bar{d}) < 0$. Then we will have $\mu(y + t\bar{d}) = \mu(y) + t\mu(\bar{d}) < 0$, for given y and sufficiently large t, which contradicts (5). Thus, $\mu \in D^* \setminus \{0\}$. Letting $t \to 0$ in (5), we obtain $\mu(y) \geq 0$ for all $y \in F(X)$. This implies that (II) is true.

4. SCALARIZATION

We consider the following set-valued optimization problem:
(VP) $\text{D-}\min_{x \in X_0} F(x)$
s.t. $G(x) \cap (-E) \neq \phi$

where $X_0 \subset X$ is a nonempty set, $E \subset Z$ is a pointed closed convex cone, $-E = \{-x : x \in E\}$, $F : X_0 \to 2^Y$, $G : X_0 \to 2^Z$ are set-valued maps. The set of feasible solutions of (VP) is denoted by $V = \{x \in X_0 : G(x) \cap (-E) \neq \phi\}$.

We now introduce Strict Benson proper-ε-efficient solution of (VP).

Definition 4.1. A point $\bar{x} \in V$ is said to be Strict Benson proper-ε-efficient solution of (VP) if $F(\bar{x}) \cap \text{BP-}\varepsilon\text{-Min}[F(V), D] \neq \phi$.
A pair $(\bar{x}, \bar{y})$ is said to be Strict Benson proper-ε-minimizer of (VP) if $\bar{y} \in F(\bar{x}) \cap \text{BP-}\varepsilon\text{-Min}[F(V), D]$.

Corresponding to the set-valued optimization problem (VP), we associate the following scalar optimization problem:
(SP)μ $\quad \min_{x \in V} (\mu F)(x)$
where $\mu \in D^* \setminus \{0\}$

Definition 4.2. Let $\bar{x} \in V$, $\bar{y} \in F(\bar{x})$, then $\bar{x}$ is said to be an ε-minimal solution of (SP)μ, if $\mu(\bar{y}) \leq \mu(y) + \mu(\varepsilon)$ for all $y \in F(V)$ and $(\bar{x}, \bar{y})$ is said to be an ε-rninimizer pair of (SP)μ.

The fundamental results characterizing Strict Benson proper-ε-minimizer of (VP) in terms of ε- minimizer of (SP)μ are now discussed.

Theorem 4.3. Let $\mu \in D^{\#}$ be fixed. If $(\bar{x}, \bar{y})$ is an ε-minimizer pair of (SP)μ, then $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer pair of (VP).

Proof. Since $(\bar{x}, \bar{y})$ is an ε-minimizer of (SP)μ, therefore
$\mu(\bar{y}) \leq \mu(y) + \mu(\varepsilon)$, for all $y \in F(V)$. (6)
Now, we shall show that $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer pair of (VP).
It is enough to show that, $\text{clcone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$. Indeed, if there exists $y^* \in \text{clcone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus \{0\})$, then, there exist $\{y_n\} \subset F(V)$ and $\{\lambda_n\} \subset \mathbb{R}_+$ such that $y^* = \lim_{n \to \infty} \lambda_n (y_n + \varepsilon - \bar{y})$ and $y^* \in -D \setminus \{0\}$
By using (6), we have $\mu(y^*) = \lim_{n \to \infty} \lambda_n \mu(y_n + \varepsilon - \bar{y}) \geq 0$ (7)
Since $\mu \in D^{\#}$ and $y^* \in -D \setminus \{0\}$, therefore $\mu(y^*) < 0$, which contradicts (7).
Thus, we conclude $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer pair of (VP).

Below we give an example to illustrate the above theorem.

Example 4.4. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}^2$ and $D = \{(x, y) : x \geq y, y \geq 0\}$,
$E = \{(x, y) : y \geq x, x \leq 0\}$ and $\varepsilon = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Define $F : X \to 2^Y$, as $F(x) = \begin{cases} [0,1]\times[0,1] & \text{if } x \geq 0 \\ [0,x^2]\times[0,x^2] & \text{if } x < 0 \end{cases}$ and define $G : X \to 2^Z$, as
$G(x) = \begin{cases} \left([0,x], \frac{x}{2}\right) & \text{if } x \geq 0 \\ [-2,-1]\times[-2,-1] & \text{if } x < 0 \end{cases}$
The feasible set of the problem (VP) is $V = \{x : x \geq 0\}$.
Let $\mu = \left(\frac{3}{2}, \frac{3}{2}\right) \in D^{\#}$, $\bar{x} = 1$ and $\bar{y} = \left(\frac{1}{2}, 0\right) \in F(\bar{x})$.
Then, $F(V) = [0,1]\times[0,1]$, $\mu(\bar{y}) = \frac{3}{4}$, $\mu(\varepsilon) = \frac{3}{2}$ and $\mu(\bar{y}) \leq \mu(y)+\mu(\varepsilon)$, for all $y \in F(V)$, which implies that, $(\bar{x}, \bar{y})$ is an ε-minimizer pair of (SP)μ.
Since $\varepsilon - \bar{y} = \left(0, \frac{1}{2}\right)$, therefore $\text{clcone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$.
Thus, $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer pair of (VP).

Theorem 4.5. Let $\bar{F} : X \to 2^Y$ be defined as $\bar{F}(x) = F(x) + \varepsilon - \bar{y}$ for all $x \in X$ and $\bar{F}$ be ic-D-convexlike on V. If $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer pair of (VP), then there exists $\mu \in D^* \setminus \{0\}$ such that $(\bar{x}, \bar{y})$ is an ε-minimizer pair of (SP)μ.

Proof. Let $(\bar{x}, \bar{y})$ be a strict Benson proper-ε-minimizer pair of (VP). Then $\bar{x} \in V$ and $\bar{y} \in F(x) \cap \text{BP-}\varepsilon\text{-Min}[F(V), D]$, which gives that $\text{clcone}(F(V)+\varepsilon-\bar{y})\cap(-D\setminus\{0\}) = \phi$. It follows that $(F(V) + \varepsilon - \bar{y}) \cap (-\operatorname{int} D) = \phi$.
By assumption $\bar{F}$ is ic-D-convexlike on V, then by Theorem 3.1 there exists $\mu \in D^* \setminus \{0\}$ such that $\mu(z) \geq 0$ for all $z \in \bar{F}(V)$, which gives that $\mu(y + \varepsilon - \bar{y}) \geq 0$, for all $y \in F(V)$.
Thus, $\mu(y) + \mu(\varepsilon) \geq \mu(\bar{y})$, for all $y \in F(V)$. Hence, $(\bar{x}, \bar{y})$ is an ε-minimizer pair of (SP)μ.

5. ε-LAGRANGIAN MULTIPLIER THEOREMS

In this section we present two ε-Lagrangian Multiplier theorems which show that a Strict Benson proper-ε-minimizer of the constrained set-valued vector optimization problem (VP) is exactly a Strict Benson proper-ε-minimizer for an appropriate unconstrained set-valued vector optimization problem under certain conditions.
Let $L(Z, Y)$ be the space of continuous linear operators from Z to Y,
and let $L_+(Z, Y) = \{T \in L(Z, Y) : T(E) \subset D\}$.
Denote by (F, G) the set-valued map from X to $Y \times Z$ defined by $(F, G)(x) = F(x) \times G(x)$, for all $x \in X$.
If $\mu \in Y^*$, $T \in L(Z, Y)$, we define $\mu F : X \to 2^{\mathbb{R}}$ and $F + TG : X \to 2^Y$ as
$(\mu F)(x) = \mu(F(x))$ and $(F + TG)(x) = F(x) + T(G(x))$, respectively.
The set-valued Lagrange map of (VP), $L : X_0 \times L_+(Z, Y) \to 2^Y$ is defined as
$L(x, T) = F(x) + T(G(x))$, where $(x, T) \in X_0 \times L_+(Z, Y)$.

We consider the following unconstrained set-valued minimization problem associated with (VP) for a fixed $T \in L_+(Z, Y)$
$\overline{(VP)}_T \quad D\text{-}\min_{x\in X_0} L(x, T)$

Sach [13] gave the following result for ic-cone convexlike set-valued maps.

Lemma 5.1. Let intcone(im$F$ +D) $\neq \phi$, then F is an ic-D-convexlike if and only if k intcone(im$F$ +D) + $(1-k)$ cone(im$F$ +D) $\subset$ intcone(im$F$ +D), for all $k \in (0,1)$.

We establish the following result by using the above lemma.

Lemma 5.2. If (F,G) is ic-D-convexlike on X and intcone(im(F,G) + $D \times E$) $\neq \phi$ then for $\mu \in D^* \setminus \{0\}$, $(\mu F, G)$ is ic-$(\mathbb{R}_+ \times E)$-convexlike on X.

Proof. Let (F,G) be ic-D-convexlike on X. Then by using Lemma 5.1
k(intcone(im$F$ +D), intcone(im$G$ +E)) + $(1-k)$(cone(im$F$ +D), cone(im$G$ +E)) $\subset$ (intcone(im$F$ +D), intcone(im$G$ +E)), for all $k \in (0,1)$ which gives that
k intcone(im$F$ +D) + $(1-k)$ cone(im$F$ +D) $\subset$ intcone(im$F$ +D), for all $k \in (0,1)$ and
k intcone(im$G$ +E) + $(1-k)$ cone(im$G$ +E) $\subset$ intcone(im$G$ +E), for all $k \in (0,1)$.
Now, it is enough to show k intcone(im $\mu F + \mathbb{R}_+$) + $(1-k)$ cone(im $\mu F + \mathbb{R}_+$) $\subset$ intcone(im $\mu F + \mathbb{R}_+$), for all $k \in (0,1)$.
Let $y^* \in k$ intcone(im $\mu F + \mathbb{R}_+$) + $(1-k)$ cone(im $\mu F + \mathbb{R}_+$), then there exists $\lambda_1 > 0$, $\lambda_2 \geq 0$, $x_1, x_2 \in X$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ and $r_1, r_2 \in \mathbb{R}_+$ such that $y^* = k\lambda_1(\mu(y_1) + r_1) + (1-k)\lambda_2(\mu(y_2) + r_2)$, which gives that $y^* = \mu(k\lambda_1 y_1 + (1-k)\lambda_2 y_2) + k\lambda_1 r_1 + (1-k)\lambda_2 r_2$
Now $k\lambda_1 y_1 + (1-k)\lambda_2 y_2 \in k\lambda_1(F(x_1) + D) + (1-k)\lambda_2(F(x_2) + D)$
$\subset k$ intcone(im$F$ +D) + $(1-k)$ cone(im$F$ +D) $\subset$ intcone(im$F$ +D), for all $k \in (0,1)$.
This gives that there exists $\lambda_3 > 0$, $x_3 \in X$, $y_3 \in F(x_3)$ and $d_3 \in D$ such that
$k\lambda_1 y_1 + (1-k)\lambda_2 y_2 = \lambda_3(y_3 + d_3)$.
Then, $y^* = \mu(\lambda_3(y_3 + d_3)) + k\lambda_1 r_1 + (1-k)\lambda_2 r_2 = \lambda_3\mu(y_3) + \lambda_3\mu(d_3) + k\lambda_1 r_1 + (1-k)\lambda_2 r_2$
$\in \lambda_3(\mu(F(x_3)) + \mu(d_3) + (k\lambda_1 r_1 + (1-k)\lambda_2 r_2)/\lambda_3)$
$\subset$ intcone(im $\mu F + \mathbb{R}_+$), as $\mu(d_3) \in \mathbb{R}_+$ and $(k\lambda_1 r_1 + (1-k)\lambda_2 r_2)/\lambda_3 \in \mathbb{R}_+$.
Thus, $(\mu F, G)$ is ic-$(\mathbb{R}_+ \times E)$-convexlike on X.

We now give ε-Lagrangian multiplier theorems:

Theorem 5.3. Let Y be locally convex space, D be closed convex pointed cone with a non empty interior. Let $\bar{F} : X \to 2^Y$ be defined as $\bar{F}(x) = F(x) + \varepsilon - \bar{y}$ for all $x \in X$, $\bar{F}$ be ic-D-convexlike on V and (F,G) be ic-$(D \times E)$-convexlike on X_0 and intcone((imF, G) + $D \times E$) $\neq \phi$. Further, let (VP) satisfy the generalized slater constraint qualification, that is, im$G \cap (-\operatorname{int} E) \neq \phi$. If $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer of (VP) and $0 \in T(G(\bar{x}))$, then there exist $T \in L_+(Z,Y)$ and $\mu \in D^* \setminus \{0\}$ such that $(\bar{x}, \bar{y})$ is an ε-minimizer pair of the following scalar set-valued optimization problem $\overline{(VP)}_\mu \quad \min_{x \in X_0} \mu(F(x) + T(G(x))$

If $\mu \in D^\#$ then $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer of $\overline{(VP)}_T$.

Proof. Since $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer of (VP), therefore by Theorem 4.2 there exists $\mu \in D^* \setminus \{0\}$ such that

$\mu(F(x) + \varepsilon - \bar{y}) \geq 0$, for all $x \in V$ (8)
Let us define $H : X_0 \to 2^{\mathbb{R}\times Z}$ as $H(x) = \mu(F(x) + \varepsilon - \bar{y}) \times G(x) = (\mu F, G)(x) + (\mu(\varepsilon) - \mu(\bar{y}), 0)$. Since (F, G) is ic-$(D\times E)$-convexlike on X_0, and intcone$[(\text{im}F, G)+D\times E] \neq \phi$, therefore by Lemma 5.2 H is ic-$(\mathbb{R}_+ \times E)$ convexlike on X_0.
Further, (8) implies that the system $x \in X_0$, $H(x) \cap (-\operatorname{int}(\mathbb{R}_+ \times E)) \neq \phi$ has no solution. Hence by Theorem 3.1, there exists $(\lambda, \psi) \in \mathbb{R}_+ \times E^* \setminus \{0, 0\}$, $y \in F(x)$, $z \in G(x)$ such that $\lambda\mu(y + \varepsilon - \bar{y}) + \psi(z) \geq 0$ for all $x \in X_0$ (9)
We claim that $\lambda \neq 0$.
On the contrary, suppose that $\lambda = 0$ then, we have $\psi \in E^* \setminus \{0\}$. By generalized slater constraint qualification, there exists $x_1 \in X_0$ such that
$G(x_1) \cap (-\operatorname{int} E) \neq \phi$. Thus, there exists $z_1 \in G(x_1)$ such that $z_1 \in (-\operatorname{int} E)$, which gives that $\psi(z_1) < 0$ but on substituting $\lambda = 0$ and taking $x = x_1$ and $z = z_1$ in (9), we have $\psi(z_1) \geq 0$, which is a contradiction. Hence $\lambda > 0$.
Since $\mu \in D^* \setminus \{0\}$. We can choose $d \in D \setminus \{0\}$ such that $\lambda\mu(d) = 1$.
We define the operator $T : Z \to Y$ as $T(z) = \psi(z)d$ (10)
then $T \in L_+(Z, Y)$ and $0 \in \psi(G(\bar{x}))d = T(G(\bar{x}))$.
Hence, $\bar{y} \in F(\bar{x}) + T(G(\bar{x}))$.
From (9) and (10), we obtain

$$\begin{aligned}\lambda\mu(y + \varepsilon + T(z)) &= \lambda\mu(y) + \lambda\mu(\varepsilon) + \psi(z)\lambda\mu(d) = \lambda\mu(y) + \lambda\mu(\varepsilon) + \psi(z)\\ &\geq \lambda\mu(\bar{y}), \text{ for all } x \in X_0 \text{ which gives that}\end{aligned}$$

$\mu(\bar{y}) \leq \mu(y + T(z)) + \mu(\varepsilon)$ for all $x \in X_0$, $y \in F(x)$ and $z \in G(x)$.
Hence, $(\bar{x}, \bar{y})$ is an ε-minimizer pair of set-valued optimization problem $\overline{(VP)}\mu$.
If $\mu \in D^\#$, then by using Theorem 4.1, we get that $(\bar{x}, \bar{y})$ is Strict Benson proper-ε-minimizer of $\overline{(VP)}_T$.

We now establish the converse of Theorem 5.1.

Theorem 5.4. Let $\bar{x} \in V$, $\bar{y} \in F(\bar{x})$. If there exists $T \in L_+(Z, y)$ such that $0 \in T(G(x))$, and $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer $\overline{(VP)}_T$, then $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer of (VP).

Proof. Since $0 \in T(G(\bar{x}))$, and $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer of $\overline{(VP)}_T$, therefore, $\bar{y} \in F(\bar{x}) + T(G(\bar{x}))$ and $\operatorname{clcone}(F(V) + T(G(V)) + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$. (11)
Now we shall show that $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer pair of (VP).
For that, it is enough to show that $\operatorname{cone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \phi$.
On the contrary, if $y^* \in \operatorname{cone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus \{0\})$ then there exists $x \in V$, $y \in F(x)$, $k > 0$ such that $y^* = k(y + \varepsilon - \bar{y})$ and $y^* \in (-D \setminus \{0\})$.
Since $0 \in T(G(\bar{x}))$, therefore, $y^* \in \operatorname{clcone}(F(V) + T(G(V)) + \varepsilon - \bar{y})$, which contradicts (11).
Hence, $(\bar{x}, \bar{y})$ is a Strict Benson proper-ε-minimizer of (VP).

6. CONCLUSION

The objective of this paper is to introduce the notion of Strict Benson proper-ε -efficient solution for vector optimization problem with set-valued maps to generalize the notion of Benson proper efficiency and establish an alternative theorem. We also obtain scalarization theorems and ε-Lagrangian multiplier theorems under the assumption of ic-cone-convexlikeness.

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