Yugoslav Journal of Operations Research
26 (2016), Number 2, 221-242
DOI: 10.2298/YJOR140615011L

RETAILER'S OPTIMAL ORDERING POLICIES FOR EOQ MODEL WITH IMPERFECTIVE ITEMS UNDER A TEMPORARY DISCOUNT

Wen Feng LIN
Department of Aviation Service and Management,China University of Science and Technology, Taipei, Taiwan
linwen@cc.cust.edu.tw
Horng Jinh CHANG
Graduate Institute of Management Sciences,Tamkang University, Tamsui, Taiwan
chj@mail.tku.edu.tw

Received: June 2014 / Accepted: March 2015

Abstract: In this article, we study inventory models to determine the optimal special order and maximum saving cost of imperfective items when the supplier offers a temporary discount. The received items are not all perfect and the defectives can be screened out by the end of 100% screening process. Three models are considered according to the special order that occurs at regular replenishment time, non-regular replenishment time, and screening time of economic order quantity cycle. Each model has two sub-cases to be discussed. In temporary discount problems, in general, there are integer operators in objective functions. We suggest theorems to find the closed-form solutions to these kinds of problems. Furthermore, numerical examples and sensitivity analysis are given to illustrate the results of the proposed properties and theorems.

Keywords: Economic Order Quantity, Temporary Discount, Imperfective Items, Inventory.

MSC: 90B05.

1. INTRODUCTION

The economic order quantity (EOQ) model is popular in supply chain management. The traditional EOQ inventory model supposed that the inventory parameters (for example: cost per unit, demand rate, setup cost or holding cost) are constant during sale period. Schwarz [32] discussed the finite horizon EOQ model, in which the costs of the model were static and the optimal ordering number could be found during the finite horizon. In real life, there are many reasons for suppliers to offer a temporary price discount to retailers. The retailers may engage in purchasing additional stock at reduced price and sell at regular price later. Lev and Weiss [22] considered the case where the cost parameters may change, and the horizon may be finite as well as infinite. However, the lower and upper bounds they used did not guarantee that the boundary conditions could be met. Tersine and Barman [35] incorporated quantity and freight discounts into the lot size decision in a deterministic EOQ system. Ardalan [3] investigated optimal ordering policies for a temporary change in both price and demand, where demand rate was not constant. Tersine [34] proposed a temporary price discount model, the optimal EOQ policy was obtained by maximizing the difference between regular EOQ cost and special ordering quantity cost during sale period. Martin [26] revealed that Tersine's [34] representation of average inventory in the total cost was flawed, and suggested the true representation of average inventory. But Martin [26] sacrificed the closed-form solution in solving objective function, and used search methods to find special order quantity and maximum gain. Wee and Yu [38] assumed that the items deteriorated exponentially with time and temporary price discount purchase occurred at the regular and non-regular replenishment time. Sarker and Kindi [31] proposed five different cases of the discount sale scenarios in order to maximize the annual gain of the special ordering quantity. Kovalev and Ng [21] showed a discrete version of the classic EOQ problem, they assumed that the time and product were continuously divisible and demand occurred at a constant rate. Cárdenas-Barrón [6] pointed out that there were some technical and mathematical expression errors in Sarker and Kindi [31] and presented the closed form solutions for the optimal total gain cost. Li [23]presented a solution method which modified Kovalev and Ng's [21]search method to find the optimal number of orders. Cárdenas-Barrónet al. [8] proposed an economic lot size model where the supplier was offered a temporary discount, and they specified a minimum quantity of additional units to purchase. García-Lagunaet al. [16] illustrated a method to obtain the solution of the classic EOQ and economic production quantity models when the lot size must be an integer quantity. They obtained a rule to discriminate between the situation in which the optimal solution is unique and the situation when there are two optimal solutions. Chang et al. [13] used closed-form solutions to solve Martin's [26] EOQ model with a temporary sale price and Wee and Yu's [38] deteriorating inventory model with a temporary price discount. Chang and Lin [12] deal with the optimal ordering policy for deteriorating inventory when some or all of the cost parameters may change over a finite horizon. Taleizadeh et al. [33] developed an inventory control model to determine the optimal order and shortage quantities of a perishable item when the supplier offers a special sale. Other authors also considered similar issues, see Abad [1], Khoujaand Park [20], Wee et al. [37], Cárdenas-Barrón [5], Andriolo et al.[2], etc.

In traditional EOQ model, the assumption that all items are perfect in each ordered lot is not pertinent. Because of defective production or other factors, there may be a percentage of imperfect quantity in received items. Salameh and Jaber [30] investigated an EOQ model which contains a certain percentage of defective items in each lot. The percentage is a continuous random variable with known probability density function. Their model assumes that shortage of stock is not allowed. Cárdenas-Barrón [7] modified the expression of optimal order size in Salameh and Jaber [30]. Goyal and Cárdenas-Barrón [17] presented a simple approach to determine Salameh and Jaber's [30] model. Papachristos and Konstantaras [29] pointed out that the proportion of the imperfects is a random variable, and that the sufficient condition to avoid shortage may not really prevent occurrence in Salameh and Jaber [30]. Wee et al. [39] and Eroglu and Ozdemir [15] extended imperfect model by allowing shortages backordered. Maddah and Jaber [25] proposed a new model and used renewal-reward theorem to derive the exact expression for the expected profit per unit time in Salameh and Jaber [30]. Hsu and Yu [18] investigated an EOQ model with imperfective items under a one-time-only sale, where the defective rate is known. However, Hsu and Yu's [18] representation of holding cost is true whenever the ratio of special order quantity to economic order quantity is an integer value. Ouyang et al.[27]developed an EOQ model where the supplier offers the retailer trade credit in payment, products received are not all perfect, and the defective rate is known. Wahab and Jaber[36] extended Maddah and Jaber [25] by introducing different holding cost for the good and defective items. Chang [10] present a new model for items with imperfect quality, where lot-splitting shipments and different holding costs for good and defective items are considered. Other authors also considered similar issues, see Chang [9], Chung and Huang [14], Chang and Ho [11], Lin [24], Khan et al. [19], Bhowmick and Samanta [4], Ouyang et al.[28], etc.

In this article, we extend Hsu and Yu [18], considering that the end of special order process is not coincident with the regular economic order process. We also propose theorems to find closed form solutions when integer operators are involved in objective function. The remainder of this paper is organized as follows. In Section 2, we described the notation and assumptions used throughout this paper. In Section 3, and Section 4, we establish mathematical models and propose theorems to find maximum saving cost and optimal order quantity. In Section 5, we give numerical examples to illustrate the proposed theorems and the results. In Section 6, we summarize and conclude the paper.

2. NOTATION AND ASSUMPTIONS

Notation:

λ	the demand rate
c	the purchasing cost per unit
b	the holding cost rate per unit/per unit time
a	the ordering cost per order
p	the defective percentage for each order
w	the screening cost per unit
s	the screening rate, $s > \lambda$.
k	the discount price of purchasing cost per unit
Q_p	the order size for purchasing cost c per unit

Q_{sj}	the special order quantity, $j = 1, 2, \cdots, 6$.
T_p	EOQ model's optimal period under regular price
T_{sj}	special order model's optimal period under reduced price , $j = 1, 2, \cdots, 6$.
$TC_s^{(j)}$	the total cost corresponding to special order policy, $j = 1, 2, \cdots, 6$.
$TC_n^{(j)}$	the total cost without special order, $j = 1, 2, \cdots, 6$.
$D^{(j)}(Q_{sj})$	the saving cost for Case (1) to Case (6) , $j = 1, 2, \cdots, 6$.
q_{j0}	the remnant stock level at time T , $j = 1, 2, \cdots, 6$.
$\lceil \ \rceil$	integer operator, integer value equal to or greater than its argument
$\lfloor \ \rfloor$	integer operator, integer value equal to or less than its argument
*	the superscript representing optimal value

Assumptions:

1. The demand rate is constant and known.
2. The rate of replenishment is infinite.
3. Based on past statistics, the defective rate is small and known.
4. For shortage is not allowed, the sufficient condition is $\lambda / s < 1 - p$.
5. In Model 1, the purchasing cost for the first regular order quantity is $c - k$.
6. The defective items are withdrawn from inventory when all order quantities are inspected.
7. The time horizon is infinite.

3. MODEL FORMULATION

When suppliers offer a temporary discount to retailers, retailers typically respond with ordering additional items to take advantage of the price reduction. Saving cost is the difference between total cost when special order is taken and total cost when special cost is not taken. According to the time that supplier offers a temporary reduction to retailers, there are three models to be discussed. Model 1 considers the case when special order occurs at regular replenishment time. Model 2, special order occurs at non-regular replenishment time and before the end of screening time. Model 3, special order occurs at non-regular replenishment time and after the end of screening time.

3.1. Model 1

According to the special period length ends before or after screening time of last regular EOQ period length, we have following two sub-cases to be discussed. (i) Case (1)

: $t_n \le T_{s1} < t_n + Q_p / s$, as shown in Fig. 1. (ii) Case (2) : $t_n + Q_p / s < T_{s2} < t_{n+1}$, as shown in Fig. 2. The procurement cost for special order policy is $a + (c-k)Q_{sj}$, the screening cost is wQ_{sj}, and the holding cost is $(c-k)b[(1-p)^2 / 2\lambda + p / s]Q_{sj}^2$, where $j = 1, 2$. The total cost corresponding to special order policy during $0 \le t \le T_{sj}$, $j = 1, 2$, is

$$TC_s^{(j)}(Q_{sj}) = a + (c-k)Q_{sj} + wQ_{sj} + (c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right]Q_{sj}^2 \tag{1}$$

In Model 1, the first order is taken using regular EOQ at reduced price $c-k$, others are taken at regular price c. For Case (1), the total cost without special order during the identical period length T_{s1} is

$$\begin{aligned} TC_n^{(1)}(Q_{s1}) &= a\left\lceil \frac{T_{s1}}{T_p} \right\rceil + (c-k)Q_p + c(Q_{s1} - Q_p) + wQ_{s1} + (c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right]Q_p^2 \\ &\quad + cb\left\{\left\lfloor \frac{T_{s1}}{T_p} - 1 \right\rfloor\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right]Q_p^2 + \frac{1}{2}(Q_p + q_1)(T_{s1} - t_n)\right\} \end{aligned} \tag{2}$$

For Case (2), the total cost without special order during the identical period length T_{s2} is

$$\begin{aligned} TC_n^{(2)}(Q_{s2}) &= a\left\lceil \frac{T_{s2}}{T_p} \right\rceil + (c-k)Q_p + c(Q_{s2} - Q_p) + wQ_{s2} \\ &\quad + (c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right]Q_p^2 + cb\left\{\left\lceil \frac{T_{s2}}{T_p} - 1 \right\rceil\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right]Q_p^2 - \frac{q_2^2}{2\lambda}\right\} \end{aligned} \tag{3}$$

The saving cost of Case (1) and Case (2) is

$$D^{(j)}(Q_{sj}) = TC_n^{(j)}(Q_{sj}) - TC_s^{(j)}(Q_{sj}) \qquad j = 1, 2 \tag{4}$$

Since $T_{sj} = Q_{sj}(1-p)/\lambda$, $T_p = Q_p(1-p)/\lambda$, $t_n = \left\lfloor \frac{T_{sj}}{T_p} \right\rfloor T_p$, $q_j = \left\lceil \frac{T_{sj}}{T_p} \right\rceil Q_p - Q_{sj}$ and $a = cb[(1-p)^2 / 2\lambda + p / s]Q_p^2$, where $j = 1, 2$, we get

$$\begin{aligned} D^{(1)}(Q_{s1}) &= -(c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right]Q_{s1}^2 + kQ_{s1} - \left[kQ_p + (1+\frac{k}{c})a\right] + a\left\lceil \frac{Q_{s1}}{Q_p} \right\rceil \\ &\quad + a\left\lfloor \frac{Q_{s1}}{Q_p} \right\rfloor + \frac{1-p}{2\lambda}cb\left[\left\lceil \frac{Q_{s1}}{Q_p} \right\rceil Q_p + Q_p - Q_{s1}\right]\left(Q_{s1} - \left\lfloor \frac{Q_{s1}}{Q_p} \right\rfloor Q_p\right) \end{aligned} \tag{5}$$

$$D^{(2)}(Q_{s2}) = -(c-k)b\left[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}\right]Q_{s2}^2 + kQ_{s2} - \left[kQ_p + (1+\frac{k}{c})a\right] + 2a\left\lceil\frac{Q_{s2}}{Q_p}\right\rceil$$
$$-\frac{cb}{2\lambda}\left(\left\lceil\frac{Q_{s2}}{Q_p}\right\rceil Q_p - Q_{s2}\right)^2 \tag{6}$$

In Model 1, if the defective percentage for each order is zero, the screening rate quickly tends to infinite and the screening cost is zero, Model 1 is the same as Martin (1994) model. Martin (1994) considered the ordering cost is aQ_{s1}/Q_p, in this paper, the ordering cost is $a\left\lceil\frac{Q_{s1}}{Q_p}\right\rceil$.

3.2. Model 2

According to the special period length ends before or after screening time of last regular EOQ period length, we have following two sub-cases to be discussed. (i) Case (3) : $t_n \le T_{s3} < t_n + Q_p/s$, as shown in Fig. 3. (ii) Case (4) : $t_n + Q_p/s < T_{s4} < t_{n+1}$, as shown in Fig. 4. Retailer places an economic order quantity Q_p at $t=0$, the remnant stock level at $t=T$ is q_{j0}, $j=3,4$. Because supplier offers a temporary discount at $t=T$, retailer additionally places a special order quantity Q_{sj}, $j=3,4$. The procurement cost is $2a + cQ_p + (c-k)Q_{sj}$, the screening cost is $w(Q_p+Q_{sj})$, $j=3,4$, and the holding cost is

$$cb\left[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}\right]Q_p^2 + (c-k)b\left\{(Q_{sj}p+Q_pp)(\frac{Q_{sj}+Q_p}{s}-\frac{Q_p-q_{j0}}{\lambda}) + \frac{(Q_{sj}+q_{j0}-Q_{sj}p-Q_pp)^2}{2\lambda} - \left[Q_pp(\frac{Q_p}{s}-\frac{Q_p-q_{j0}}{\lambda}) + \frac{(q_{j0}-Q_pp)^2}{2\lambda}\right]\right\} \quad j=3,4$$

The total cost corresponding to special order policy during $0 \le t \le T_{sj}$, $j=3,4$, is

$$TC_s^{(j)}(Q_{sj}) = 2a + cQ_p + (c-k)Q_{sj} + w(Q_p+Q_{sj}) + cb\left[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}\right]Q_p^2$$
$$+(c-k)b\left\{\left[(Q_{sj}p+Q_pp)(\frac{Q_{sj}+Q_p}{s}-\frac{Q_p-q_{j0}}{\lambda}) + \frac{(Q_{sj}+q_{j0}-Q_{sj}p-Q_pp)^2}{2\lambda}\right]\right. \tag{7}$$
$$\left.-\left[Q_pp(\frac{Q_p}{s}-\frac{Q_p-q_{j0}}{\lambda}) + \frac{(q_{j0}-Q_pp)^2}{2\lambda}\right]\right\} \quad j=3,4$$

For Case (3) and Case (4), if there is no temporary price discount occurs, the total cost without special order during the identical period length T_{sj}, $j=3,4$, is

$$TC_n^{(3)}(Q_{s3}) = a\left\lceil \frac{T_{s3}}{T_p} \right\rceil + c(Q_{s3}+Q_p) + w(Q_{s3}+Q_p) + cb\left\{ \left\lfloor \frac{T_{s3}}{T_p} \right\rfloor \left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s} \right] Q_p^2 + \frac{1}{2}(Q_p + q_{31})(T_{s3} - t_n) \right\} \quad (8)$$

$$TC_n^{(4)}(Q_{s4}) = a\left\lceil \frac{T_{s4}}{T_p} \right\rceil + c(Q_{s4}+Q_p) + w(Q_{s4}+Q_p) + cb\left\{ \left\lceil \frac{T_{s4}}{T_p} \right\rceil \left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s} \right] Q_p^2 - \frac{q_{41}^2}{2\lambda} \right\} \quad (9)$$

Since $a = cb[(1-p)^2/2\lambda + p/s]Q_p^2$, $T_{sj} = (Q_{sj}+Q_p)(1-p)/\lambda$, $T_p = Q_p(1-p)/\lambda$, $t_n = \left\lfloor \frac{T_{sj}}{T_p} \right\rfloor T_p$ and $q_{j1} = \left\lceil \frac{T_{sj}}{T_p} \right\rceil Q_p - Q_p - Q_{sj}$, where $j = 3,4$, the saving cost of Case (3) and Case (4) is

$$D^{(3)}(Q_{s3}) = -(c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s} \right] Q_{s3}^2 + \left\{ k - (c-k)b\left[\frac{2pQ_p}{s} - \frac{p(2-p)Q_p}{\lambda} + \frac{q_{30}}{\lambda} \right] \right\} Q_{s3} - a + a\left\lfloor \frac{Q_{s3}}{Q_p} \right\rfloor + a\left\lceil \frac{Q_{s3}}{Q_p} \right\rceil + \frac{1-p}{2\lambda} cb\left(\left\lceil \frac{Q_{s3}}{Q_p} \right\rceil Q_p + Q_p - Q_{s3} \right)\left(Q_{s3} - \left\lfloor \frac{Q_{s3}}{Q_p} \right\rfloor Q_p \right) \quad (10)$$

$$D^{(4)}(Q_{s4}) = -(c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s} \right] Q_{s4}^2 + \left\{ k - (c-k)b\left[\frac{2pQ_p}{s} - \frac{p(2-p)Q_p}{\lambda} + \frac{q_{40}}{\lambda} \right] \right\} Q_{s4} - a + 2a\left\lceil \frac{Q_{s4}}{Q_p} \right\rceil - \frac{cb}{2\lambda}\left(\left\lceil \frac{Q_{s4}}{Q_p} \right\rceil Q_p - Q_{s4} \right)^2 \quad (11)$$

3.3. Model 3

According to the special period length ends before or after screening time of last regular EOQ period length, we have following two sub-cases to be discussed. (i) Case (5) : $t_n \le T_{s5} < t_n + Q_p/s$, as shown in Fig. 5. (ii) Case (6) : $t_n + Q_p/s < T_{s6} < t_{n+1}$, as shown in Fig. 6. Retailer places an economic order quantity Q_p at $t = 0$, the remnant stock level at $t = T$ is q_{j0}, $j = 5,6$. Because supplier offers a temporary discount at $t = T$, retailer additionally places a special order quantity Q_{sj}, $j = 5,6$. The procurement cost is $2a + cQ_p + (c-k)Q_{sj}$, the screening cost is $w(Q_p + Q_{sj})$, $j = 5,6$, and the holding cost is

$$cb\left[\frac{(1-p)^2}{\lambda}+\frac{p}{s}\right]Q_p^2+(c-k)b\left[Q_{sj}\,p\,\frac{Q_{sj}}{s}+\frac{(Q_{sj}+q_{j0}-Q_{sj}\,p)^2}{2\lambda}-\frac{q_{j0}^2}{2\lambda}\right]\quad j=5,6$$

The total cost corresponding to special order policy during $0\le t\le T_{sj}$, $j=5,6$, is

$$\begin{aligned}TC_s^{(j)}(Q_{sj})&=2a+cQ_p+(c-k)Q_{sj}+w(Q_p+Q_{sj})+cb\left[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}\right]Q_p^2\\&\quad+(c-k)b\left\{\left[\frac{pQ_{sj}^2}{s}+\frac{(Q_{sj}+q_{j0}-Q_{sj}\,p)^2}{2\lambda}\right]-\frac{q_{j0}^2}{2\lambda}\right\}\end{aligned}\quad j=5,6 \tag{12}$$

For Case (5) and Case (6), if there is no temporary price discount occurs, the total cost without special order during the identical period length T_{sj}, $j=5,6$, is

$$\begin{aligned}TC_n^{(5)}(Q_{s5})&=a\left\lceil\frac{T_{s5}}{T_p}\right\rceil+c(Q_{s5}+Q_p)+w(Q_{s5}+Q_p)\\&\quad+cb\left\{\left\lfloor\frac{T_{s5}}{T_p}\right\rfloor\left[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}\right]Q_p^2+\frac{1}{2}(Q_p+q_{51})(T_{s5}-t_n)\right\}\end{aligned} \tag{13}$$

$$\begin{aligned}TC_n^{(6)}(Q_{s6})&=a\left\lceil\frac{T_{s6}}{T_p}\right\rceil+c(Q_{s6}+Q_p)+w(Q_{s6}+Q_p)\\&\quad+cb\left\{\left\lceil\frac{T_{s6}}{T_p}\right\rceil\left[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}\right]Q_p^2-\frac{q_{61}^2}{2\lambda}\right\}\end{aligned} \tag{14}$$

Since $a=cb[(1-p)^2/2\lambda+p/s]Q_p^2$, $T_{sj}=(Q_{sj}+Q_p)(1-p)/\lambda$, $T_p=Q_p(1-p)/\lambda$, $t_n=\left\lfloor\frac{T_{sj}}{T_p}\right\rfloor T_p$ and $q_{j1}=\left\lceil\frac{T_{s5}}{T_p}\right\rceil Q_p-Q_p-Q_{sj}$, where $j=5,6$, the saving cost of Case (5) and Case (6) is

$$\begin{aligned}D^{(5)}(Q_{s5})&=-(c-k)b[\frac{(1-p)^2}{2\lambda}+\frac{p}{s}]Q_{s5}^2+\left[k-\frac{1-p}{\lambda}(c-k)bq_{50}\right]Q_{s5}-a+a\left\lceil\frac{Q_{s5}}{Q_p}\right\rceil\\&\quad+a\left\lfloor\frac{Q_{s5}}{Q_p}\right\rfloor+\frac{(1-p)cb}{2\lambda}\left(\left\lceil\frac{Q_{s5}}{Q_p}\right\rceil Q_p+Q_p-Q_{s5}\right)\left(Q_{s5}-\left\lfloor\frac{Q_{s5}}{Q_p}\right\rfloor Q_p\right)\end{aligned} \tag{15}$$

$$D^{(6)}(Q_{s6}) = -(c-k)b[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}]Q_{s6}^2 + \left[k - \frac{1-p}{\lambda}(c-k)bq_{60}\right]Q_{s6} - a + 2a\left[\frac{Q_{s6}}{Q_p}\right] - \frac{cb}{2\lambda}\left(\left[\frac{Q_{s6}}{Q_p}\right]Q_p - Q_{s6}\right)^2 \quad (16)$$

4. THEORETICAL RESULTS

In this section, we suggest properties of $D^{(j)}(Q_{sj})$, $j = 1, 2, \cdots, 6$, and give theorems to solve the proposed models.

Property 4-1

$D^{(j)}(Q_{sj})$ is a piecewise continous function in which jump values at $Q_{sj} = mQ_p$ are

$$\lim_{\alpha \to 0^+} D^{(j)}((m+\alpha)Q_p) - \lim_{\alpha \to 0^+} D^{(j)}((m-\alpha)Q_p) = \begin{cases} 2a - (1-p)cbQ_p^2 / 2\lambda & j = 1,3,5 \\ 2a - cbQ_p^2 / 2\lambda & j = 2,4,6 \end{cases} \quad (17)$$

where m is a non-negative integer.

Proof of Property 4-1 is given in appendix.

Property 4-2

Let m is a non-negative integer, and

$$\omega_i = \begin{cases} 0 & i = 1,2 \\ \sigma & i = 3,4 \\ \tau & i = 5,6 \end{cases} \quad (18)$$

$$\sigma = (c-k)b\left[\frac{2pQ_p}{s} - \frac{p(2-p)Q_p}{\lambda} + \frac{q_{i0}}{\lambda}\right] \quad i = 3,4 \quad (19)$$

$$\tau = \frac{(1-p)(c-k)bq_{i0}}{\lambda} \quad i = 5,6 \quad (20)$$

$$m_{iL} = \frac{c(k-\omega_i)Q_p}{2(c-k)a} - 1 \quad i = 1,2,\cdots,6 \quad (21)$$

$$m_{iR} = \frac{c[k-\omega_i + (1-p)cbQ_p / \lambda]Q_p}{2(c-k)a} \quad i = 1,3,5 \quad (22)$$

$$m_{iR} = \frac{c(k-\omega_i + cbQ_p / \lambda)Q_p}{2(c-k)a} \quad i = 2,4,6 \quad (23)$$

(a) $D^{(i)}(Q_{si})$ is an increasing function of Q_{si} between mQ_p and $(m+1)Q_p$ when $m < \lceil m_{iL} \rceil$, where $i = 1, 2, \cdots, 6$.

(b) $D^{(i)}(Q_{si})$ is a decreasing function of Q_{si} between mQ_p and $(m+1)Q_p$ when $m > \lceil m_{iR} \rceil$, where $i = 1, 2, \cdots, 6$.

(c) $D^{(i)}(Q_{si})$ is a concave function of Q_{si} between mQ_p and $(m+1)Q_p$ when $\lceil m_{iL} \rceil < m < \lceil m_{iR} \rceil$, where $i = 1, 2, \cdots, 6$.

Proof of Property 4-2 is given in appendix.

Theorem 1

Let $$\Delta_1 = (c-k)\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right] + \frac{1-p}{2\lambda}c \tag{24}$$

$$Q_{si}(m) = \frac{k - \omega_i + (1-p)cbQ_p(m+1)/\lambda}{2\Delta_1 b} \quad i = 1,3,5 \tag{25}$$

$$\begin{aligned} DM_1(m) = {} & \frac{-(1-p)(c-k)a}{2\lambda\Delta_1}(m+1)^2 + \left[\frac{(1-p)kcQ_p}{2\lambda\Delta_1} + 2a\right](m+1) \\ & + \frac{k^2}{4\Delta_1 b} - 2a + \frac{1-p}{2\lambda}cbQ_p^2 - k(Q_p + \frac{a}{c}) \end{aligned} \tag{26}$$

$$\begin{aligned} DM_i(m) = {} & \frac{-(1-p)(c-k)a}{2\lambda\Delta_1}(m+1)^2 + \left[\frac{(1-p)(k-\omega_i)cQ_p}{2\lambda\Delta_1} + 2a\right](m+1) \\ & + \frac{(k-\omega_i)^2}{4\Delta_1 b} - 2a + \frac{1-p}{2\lambda}cbQ_p^2 \end{aligned} \quad i = 3,5 \tag{27}$$

$$h_{1R}(m) = \lim_{\alpha \to 0^+} D^{(1)}\left((m+\alpha)Q_p\right) = -a(1-\frac{k}{c})m^2 + (kQ_p + 2a)m - k(Q_p + \frac{a}{c}) \tag{28}$$

$$h_{iR}(m) = \lim_{\alpha \to 0^+} D^{(i)}\left((m+\alpha)Q_p\right) = -a(1-\frac{k}{c})m^2 + [(k-\omega_i)Q_p + 2a)]m \quad i = 3,5 \tag{29}$$

$$z_i = \frac{[(k-\omega_i)cQ_p - ca + 3ka](1-p) + 4a\lambda(c-k)\left[(1-p)^2/2\lambda + p/s\right]}{2(1-p)(c-k)a} \quad i = 1,3,5 \tag{30}$$

For $i = 1,3,5$, if z_i is not an integer, let $m_{ei} = \lceil z_i \rceil$. If z_i is an integer, let $m_{ei} = \lceil z_i \rceil$ and $m_{ei} = \lfloor z_i \rfloor + 1$. The special order quantity Q_{si} and maximum value of $D^{(i)}(Q_{si})$ can be found in the following:

(a) When $m_{ei} \le \lceil m_{iL} \rceil$

$$Q_{si} = \begin{cases} Q_{si}(\lceil m_{iL} \rceil) & \text{if } T_{si} - t_n < Q_p / s \\ \lfloor Q_{si}(\lceil m_{iL} \rceil) / Q_p \rfloor Q_p + \lambda Q_p / s(1-p) & \text{if } T_{si} - t_n \ge Q_p / s \end{cases} \tag{31}$$

$$D^{(i)}(Q_{si}^*) = \begin{cases} DM_i(\lceil m_{iL} \rceil) & \text{if } T_{si} - t_n < Q_p / s \\ \max\left\{ D^{(i)}(Q_{si}), DM_i(\lceil m_{iL} \rceil + 1) \right\} & \text{if } T_{si} - t_n \ge Q_p / s \end{cases} \tag{32}$$

(b) When $\lceil m_{iL} \rceil < m_{ei} < \lceil m_{iR} \rceil$

$$Q_{si} = \begin{cases} Q_{si}(m_{ei}) & \text{if } T_{si} - t_n < Q_p / s \\ \lfloor Q_{si}(m_{ei}) / Q_p \rfloor Q_p + \lambda Q_p / s(1-p) & \text{if } T_{si} - t_n \ge Q_p / s \end{cases} \tag{33}$$

$$D^{(i)}(Q_{si}^*) = \begin{cases} DM_i(m_{ei}) & \text{if } T_{si} - t_n < Q_p / s \\ \max\left\{ DM_i(m_{ei} - 1), D^{(i)}(Q_{si}), DM_i(m_{ei} + 1) \right\} & \text{if } T_{si} - t_n \ge Q_p / s \end{cases} \tag{34}$$

(c) When $m_{ei} \ge \lceil m_{iR} \rceil$

$$Q_{si} = \begin{cases} Q_{si}(\lfloor m_{iR} \rfloor) & \text{if } T_{si} - t_n < Q_p / s \\ \lfloor Q_{si}(\lfloor m_{iR} \rfloor) / Q_p \rfloor Q_p + \lambda Q_p / s(1-p) & \text{if } T_{si} - t_n \ge Q_p / s \end{cases} \tag{35}$$

$$D^{(i)}(Q_{si}^*) = \begin{cases} \max\left\{ DM_i(\lfloor m_{iR} \rfloor), h_{iR}(\lfloor m_{iR} \rfloor + 1) \right\} & \text{if } T_{si} - t_n < Q_p / s \\ \max\{ DM_i(\lfloor m_{iR} \rfloor - 1), D^{(i)}(Q_{si}), h_{iR}(\lfloor m_{iR} \rfloor + 1) \} & \text{if } T_{si} - t_n \ge Q_p / s \end{cases} \tag{36}$$

Proof of Theorem 1 is given in the appendix.

Theorem 2

$$\text{Let } \Delta_2 = (c-k)\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s} \right] + \frac{1}{2\lambda} c \tag{37}$$

$$Q_{si}(m) = \frac{k - \omega_i + cbQ_p(m+1)/\lambda}{2\Delta_2 b} \qquad i = 2,4,6 \tag{38}$$

$$DM_2(m) = \frac{-(c-k)a}{2\lambda\Delta_2}(m+1)^2 + \left(\frac{kcQ_p}{2\lambda\Delta_2} + 2a\right)(m+1) + \frac{k^2}{4\Delta_2 b} - a - k(Q_p + \frac{a}{c}) \quad (39)$$

$$DM_i(m) = \frac{-(c-k)a}{2\lambda\Delta_2}(m+1)^2 + \left(\frac{(k-\omega_i)cQ_p}{2\lambda\Delta_2} + 2a\right)(m+1) + \frac{(k-\omega_i)^2}{4\Delta_2 b} - a \quad i = 4,6 \quad (40)$$

$$z_i = \frac{(k-\omega_i)cQ_p - ca + 3ka + 4a\lambda(c-k)\left[(1-p)^2/2\lambda + p/s\right]}{2(c-k)a} \quad i = 2,4,6 \quad (41)$$

For $i = 2,4,6$, if z_i is not an integer, let $m_{ei} = \lceil z_i \rceil$. If z_i is an integer, let $m_{ei} = \lceil z_i \rceil$ and $m_{ei} = \lfloor z_i \rfloor + 1$. The special order quantity Q_{si} and maximum value of $D^{(i)}(Q_{si})$ can be found in the following:

(a) When $m_{ei} \le \lceil m_{iL} \rceil$

$$Q_{si} = \begin{cases} Q_{si}(\lceil m_{iL} \rceil) & \text{if } Q_p/s < T_{si} - t_n < T_p \\ \lfloor Q_{si}(\lceil m_{iL} \rceil)/Q_p \rfloor Q_p + \lambda Q_p/s(1-p) & \text{if } Q_p/s \ge T_{si} - t_n \end{cases} \quad (42)$$

$$D^{(i)}(Q_{si}^*) = \begin{cases} DM_i(\lceil m_{iL} \rceil) & \text{if } Q_p/s < T_{si} - t_n < T_p \\ \max\left\{D^{(i)}(Q_{si}), DM_i(\lceil m_{iL} \rceil + 1)\right\} & \text{if } Q_p/s \ge T_{si} - t_n \end{cases} \quad (43)$$

(b) When $\lceil m_{iL} \rceil < m_{ei} < \lceil m_{iR} \rceil$

$$Q_{si} = \begin{cases} Q_{si}(m_{ei}) & \text{if } Q_p/s < T_{si} - t_n < T_p \\ \lfloor Q_{si}(m_{ei})/Q_p \rfloor Q_p + \lambda Q_p/s(1-p) & \text{if } Q_p/s \ge T_{si} - t_n \end{cases} \quad (44)$$

$$D^{(i)}(Q_{si}^*) = \begin{cases} DM_i(m_{ei}) & \text{if } Q_p/s < T_{si} - t_n < T_p \\ \max\left\{DM_i(m_{ei}-1), D^{(i)}(Q_{si}), DM_i(m_{ei}+1)\right\} & \text{if } Q_p/s \ge T_{si} - t_n \end{cases} \quad (45)$$

(c) When $m_{ei} \ge \lceil m_{iR} \rceil$

$$Q_{si} = \begin{cases} Q_{si}(\lfloor m_{iR} \rfloor) & \text{if } Q_p/s < T_{si} - t_n < T_p \\ \lfloor Q_{si}(\lfloor m_{iR} \rfloor)/Q_p \rfloor Q_p + \lambda Q_p/s(1-p) & \text{if } Q_p/s \ge T_{si} - t_n \end{cases} \quad (46)$$

$$D^{(i)}(Q_{si}^*) = \begin{cases} \max\left\{DM_i(\lfloor m_{iR} \rfloor), D^{(i)}\left(\lfloor Q_{si}(\lfloor m_{iR} \rfloor+1)/Q_p \rfloor Q_p + \lambda Q_p / s(1-p)\right)\right\} \\ \max\left\{DM_i(\lfloor m_{iR} \rfloor-1), D^{(i)}\left(\lfloor Q_{si}(\lfloor m_{iR} \rfloor)/Q_p \rfloor Q_p + \lambda Q_p / s(1-p)\right), \right. \\ \left. \qquad D^{(i)}\left(\lfloor Q_{si}(\lfloor m_{iR} \rfloor+1)/Q_p \rfloor Q_p + \lambda Q_p / s(1-p)\right)\right\} \end{cases}$$

$$\text{if } Q_p / s < T_{si} - t_n < T_p \qquad \text{if } Q_p / s \geq T_{si} - t_n \tag{47}$$

Proof of Theorem 2 is the same as Theorem 1. For $j = 1, 2, \cdots, 6$, comparing $D^{(j)}(Q_{sj}^*)$ each other in Model 1 to Model 3, we can find maximum saving cost $D^{(j)}(Q_{sj}^*)$ and special order quantity Q_{sj}^* in each Model.

5. NUMERICAL EXAMPLES

In this section, we use the same cost parameters of Hsu and Yu (2009) to illustrate the theorems proposed. The sensitivity analysis of major parameters on the optimal solutions will also be carried out.

Example 1. *Given* $a = \$80/\text{order}$, $b = 0.1$, $c = \$12/\text{unit}$, $s = \$24000$ units/yr, $\lambda = \$8000$ units/yr, $q_{30} = q_{40} = 900$ units, $q_{50} = q_{60} = 200$ units, $w = \$2/\text{unit}$, $p = 0.1$ *and* $k = \$4/\text{unit}$ *in Model 1. In Case (1), we find* $\lceil m_{1L} \rceil = 41$, $\lfloor m_{1R} \rfloor = 42$, $\lceil m_{1R} \rceil = 43$ *and* $m_{e1} = 43$, *then* $m_{e1} \geq \lceil m_{1R} \rceil$. *Because* $Q_{s1} = Q_{s1}(\lfloor m_{1R} \rfloor) = 46721$ *satisfies* $0 \leq T_{s1} - t_n < Q_p / s$, *the maximum saving cost of Case (1) is* $D^{(1)}(Q_{s1}^*) = \max\left\{DM_1(\lfloor m_{1R} \rfloor), h_{1R}(\lfloor m_{1R} \rfloor+1)\right\} = 93553.2$, *then the special order quantity is* $Q_{s1}^* = (\lfloor m_{1R} \rfloor+1)Q_p = 47431$ units. *The result is shown in Fig. 7. In Case (2), we find* $\lceil m_{2L} \rceil = 41$, $\lfloor m_{2R} \rfloor = 43$, $\lceil m_{2R} \rceil = 44$ *and* $m_{e2} = 43$, *then* $\lceil m_{2L} \rceil < m_{e2} < \lceil m_{2R} \rceil$. *Owing to* $Q_{s2} = Q_{s2}(m_{e2}) = 47462$ *does not satisfy* $Q_p / s < T_{s2} - t_n < T_p$, *we take* $Q_{s2} = \lfloor Q_{s2}(43)/Q_p \rfloor Q_p + (Q_p / s)(\lambda / 1 - p) = 47840$ *into Eq.(6) and obtain* $D^{(2)}(47840) = 93525.1$. *The maximum saving cost of Case (2) is* $D^{(2)}(Q_{s2}^*) = \max\left\{DM_2(42), D^{(2)}(47840), D^{(2)}(48943)\right\} = 93525.8$, *then the special order quantity is* $Q_{s2}^* = Q_{s2}(42) = 46766$ units. *The result is shown in Fig. 8. Comparing* $D^{(1)}(Q_{s1}^*)$ *with* $D^{(2)}(Q_{s2}^*)$ *in Model 1, we can find maximum saving cost of Model 1 is* $D^{(1)}(Q_{s1}^*) = 93553.2$ *and special order quantity is* $Q_{s1}^* = 47431$ units. *The optimal ordering policies for Model 1 to Model 3 under different discounts are represented in Table 1.*

From Table 1, we can obtain following results: (a) Ordering quantity and saving cost increase as discount price increases. This implies that when supplier offers more temporary discount, retailers will order more quantity to save cost. (b) The rankings of special order quantity $Q_{s1}^* \ge Q_{s5}^* > Q_{s3}^*$ are not consistent with saving cost $D^{(5)}(Q_{s5}^*) > D^{(3)}(Q_{s3}^*) > D^{(1)}(Q_{s1}^*)$ for the same discount. The reason is the purchasing cost for the first economic order quantity in Model 1 is $(c-k)Q_p$, but the purchasing cost in Model 2 and Model 3 are cQ_p. The difference kQ_p influences the ranking of saving cost. The largest saving cost in three models is $D^{(5)}(Q_{s5}^*)$. The reason is the defective items are withdrawn from inventory before special order occurs. The holding cost does not involve defective items.

Example 2. The sensitivity analysis is performed to study the effects of changes of major parameters on the optimal solutions. All the parameters are identical to Example 1 except the given parameter. The following inferences can be made based on Table 2.

(a) Higher values of screening rate s cause a higher value of special order quantity Q_{si}^* and maximum saving cost $D^{(i)}(Q_{si}^*)$, $i = 1,3,5$. It implies that the retailer should take some actions to increase the item's screening rate in order to save more cost.

(b) Higher values of holding cost rate b and purchasing cost c cause a lower value of special order quantity Q_{si}^* and maximum saving cost $D^{(i)}(Q_{si}^*)$, $i = 1,3,5$. Hence, in order to increase saving cost, the retailer should have low holding cost rate and purchasing cost.

(c) Higher values of remnant stock level q_{i0} cause a lower value of special order quantity Q_{si}^* and maximum saving cost $D^{(i)}(Q_{si}^*)$, $i = 3,5$. It implies when remnant stock level is high, it don't need to orders more special order quantity. It induces low saving cost.

6. CONCLUSION

In this article, we developed an inventory model to determine the optimal special order and maximum saving cost of imperfective items for retailers who use economic order quantity model and are faced with a temporary discount. According to the time that supplier offers a temporary reduction to retailers, we discuss three models in this article. Each model has two sub-cases to be discussed. In temporary discount problems, the ordering number is an integer variable, there are integer operators in objective function. It is hard to find closed-form solutions of their extreme values. A distinguishing feature of the proposed theorems is that they can easily apply to find closed-form solutions of temporary discount problems. The results in numerical examples and sensitivity analysis of key model parameters indicate following insights: (a) Both ordering quantity and saving cost increase as discount price increases; (b) For the same discount, Case (5) has larger saving cost than others. This means, in Case (5), retailers earn maximum saving cost; (c) Higher values of screening rate induce special order quantity and higher saving cost; (d) Higher values of holding cost rate and purchasing cost cause a lower value of special order quantity and saving cost.

The further advanced research will extend the proposed models in several ways. For example, we can extend the imperfect model by allowing shortages the horizon may be finite. Also we can consider the demand rate as not been constant.

REFERENCES

[1] Abad, P.L., "Optimal price and lot size when the supplier offers a temporary price reduction over an interval", *Computers and Operations Research*, 30 (1) (2003) 63-74.

[2] Andriolo, A., Battini, D., Grubbström, R.W., Persona, A., and Sgarbossa, F., "A century of evolution from Harris′s basic lot size model: Survey and research agenda", *International Journal of Production Economics*, In Press, 2014, Available online.

[3] Ardalan, A., "Combined optimal price and optimal inventory replenishment policies when a sale results in increase in demand", *Computers and Operations Research*, 18 (8) (1991) 721–730.

[4] Cárdenas-Barrón, L.E., "Optimal ordering policies in response to a discount offer: Extensions", *International Journal of Production Economics*, 122 (2) (2009a) 774-782.

[5] Cárdenas-Barrón, L.E., "Optimal ordering policies in response to a discount offer: Corrections", *International Journal of Production Economics*, 122 (2) (2009b) 783-789.

[6] Cárdenas-Barrón, L.E., "Observation on: Economic production quantity model for items with imperfect quality", *International Journal of Production Economics*, 67 (2) (2000) 201.

[7] Cárdenas-Barrón, L.E., Smith, N.R., and Goyal, S.K., "Optimal order size to take advantage of a one-time discount offer with allowed backorders", *Applied Mathematical Modelling*, 34 (6) (2010) 1642-1652.

[8] Chang, H.C., "An application of fuzzy sets theory to the EOQ model with imperfect quality items", *Computers and Operations Research*, 31 (12) (2004) 2079–2092.

[9] Chang, H.C., "A comprehensive note on: An economic order quantity with imperfect quality and quantity discounts, *Applied Mathematical Modelling*, 35 (10) (2011) 5208–5216.

[10] Chang, H.C., and Ho, C.H., "Exact closed-form solutions for optimal inventory model for items with imperfect quality and shortage backordering", *Omega*, 38 (3) (2010) 233–237.

[11] Chang, H.J., and Lin, W.F., "A simple solution method for the finite horizon EOQ model for deteriorating items with cost changes", *Asia-Pacific Journal of Operational Research*, 28(6) (2011) 689-704.

[12] Chang, H.J., Lin, W.F., and Ho, J.F., "Closed-form solutions for Wee's and Martin's EOQ models with a temporary price discount", *International Journal of Production Economics*, 131(2) (2011) 528-534.

[13] Chung, K.J., and Huang, Y.F, "Retailer's optimal cycle times in the EOQ model with imperfect quality and permissible credit period", *Quality and Quantity*, 40 (1) (2006) 59-77.

[14] Eroglu, A., and Ozdemir, G., "An economic order quantity model with defective items and shortages", *International Journal of Production Economics*, 106 (2) (2007) 544–549.

[15] García-Laguna, J., San-José, L.A., Cárdenas-Barrón, L.E., and Sicilia, J., "The integrality of the lot size in the basic EOQ and EPQ models: Applications to other production-inventory models", *Applied Mathematics and Computation*, 216 (5) (2010) 1660-1672.

[16] Goyal, S.K., and Cárdenas-Barrón, L.E., "Note on: Economic production quantity model for items with imperfect quality - a practical approach", *International Journal of Production Economics*, 77 (1) (2002) 85–87.

[17] Hsu, W.K., and Yu, H.F., "EOQ model for imperfective items under a one-time-only discount", *Omega*, 37 (5) (2009) 1018-1026.

[18] Khan, M., Jaber, M.Y., and Wahab, M.I.M., "Economic order quantity model for items with imperfect quality with learning in inspection", *International Journal of Production Economics*, 124 (1) (2010) 87–96.

[19] Khouja, M., and Park, S., "Optimal lot sizing under continuous price decrease", *Omega*, 31 (6) (2003) 539–545.

[20] Kovalev, A., and Ng, C.T., "A discrete EOQ problem is solvable in O(log n) time", *European Journal of Operational Research*, 189 (3) (2008) 914-919.

[21] Lev, B., and Weiss, H. J., "Inventory models with cost changes", *Operations Research*, 38(1) (1990) 53-63.

[22] Li, C.L., "A new solution method for the finite-horizon discrete-time EOQ problem", *European Journal of Operational Research*, 197 (1) (2009) 412–414.
[23] Lin, T.Y., "An economic order quantity with imperfect quality and quantity discounts", *Applied Mathematical Modelling*, 34 (10) (2000) 3158-3165.
[24] Maddah, B., and Jaber, M.Y., "Economic order quantity model for items with imperfect quality: revisited", *International Journal of Production Economics*, 112(2) (2008) 808–815.
[25] Martin, G.E., "Note on an EOQ model with a temporary sale price", *International Journal of Production Economics*, 37 (2) (1994) 241-243.
[26] Ouyang, L.Y., Chang, C.T., and Shum, P., "The EOQ with defective items and partially permissible delay in payments linked to order quantity derived algebraically", *Central European Journal of Operations Research*, 20 (1) (2012) 141-160.
[27] Ouyang, L.Y., Wu, K.S., and Ho, C.H., "Analysis of optimal vendor-buyer integrated inventory policy involving defective items", *International Journal of Advanced Manufacturing Technology*, 29 (2006) 1232-1245.
[28] Papachristos, S., and Konstantaras, I., "Economic ordering quantity models for items with imperfect quality", *International Journal of Production Economics*, 100 (1) (2006) 148–154.
[29] Salameh, M.K., and Jaber, M.Y., "Economic production quantity model for items with imperfect quality", *International Journal of Production Economics*, 64 (1) (2000) 59–64.
[30] Sarker, B.R. and Kindi, M.A., "Optimal ordering policies in response to a discount offer", *International Journal of Production Economics*, 100 (2) (2006) 195–211.
[31] Schwarz, L.B., "Economic order quantities for products with finite demand horizon", *AIIE Transactions*, 4 (1972) 234-236.
[32] Taleizadeh, A. A., Mohammadi, B. Cárdenas-Barrón and Samimi, H., "An EOQ model for perishable product with special sale and shortage", *International Journal of Production Economics*, 145 (1) (2013) 318-338.
[33] Tersine, R.J., *Principles of Inventory and Materials Management*, 4th ed, Prentice-Hall, Englewood Cliffs, New Jork, 1994.
[34] Tersine, R.J., and Barman, S., "Lot size optimization with quantity and freight rate discounts", *Logistics and Transportation Review*, 27 (4) (1991) 319-332.
[35] Wahab, M.I.M., and Jaber, M.Y., "Economic order quantity model for items with imperfect quality, different holding costs, and learning effects: A note", *Computers and Industrial Engineering*, 58 (1) (2010) 186–190.
[36] Wee, H.M., Chung, S.L. and Yang, P.C., "Technical Note - A modified EOQ model with temporary sale price derived without derivatives", *The Engineering Economist*, 48 (2) (2003) 190-195.
[37] Wee, H.M., and Yu, J., "A deteriorating inventory model with a temporary price discount", *International Journal of Production Economics*, 53 (1) (1997) 81-90.
[38] Wee, H.M., Yu, J. and Chen, M.C., "Optimal inventory model for items with imperfect quality and shortage backordering", *Omega*, 35(1) (2007) 7–11.

APPENDICES

Proof of Property 4-1：

We prove property of $D^{(1)}(Q_{s1})$ only, others are similar to the proof of $D^{(1)}(Q_{s1})$. Let $\alpha \to 0^+$

$$\lim_{\alpha \to 0^+} D^{(1)}\left((m+\alpha)Q_p\right) = -a(1-\frac{k}{c})m^2 + (kQ_p + 2a)m - k(Q_p + \frac{a}{c}) \tag{A1}$$

$$\lim_{\alpha\to 0^+} D^{(1)}\left((m-\alpha)Q_p\right) = -a(1-\frac{k}{c})m^2 + (kQ_p + 2a)m - k(Q_p + \frac{a}{c}) - 2a + \frac{1-p}{2\lambda}cbQ_p^2 \quad \text{(A2)}$$

$$\lim_{\alpha\to 0^+} D^{(1)}\left((m+\alpha)Q_p\right) - \lim_{\alpha\to 0^+} D^{(1)}\left((m-\alpha)Q_p\right) = 2a - \frac{1-p}{2\lambda}cbQ_p^2 \quad \text{(A3)}$$

This implies $D^{(1)}(Q_{s1})$ is a piecewise continous function in which jump values at $Q_{s1} = mQ_p$ are $2a-(1-p)cbQ_p^2/2\lambda$. □

Proof of Property 4-2：

We prove property of $D^{(1)}(Q_{s1})$ only, others are similar to the proof of $D^{(1)}(Q_{s1})$. During $mQ_p < Q_{s1} < (m+1)Q_p$

$$D^{(1)}(Q_{s1}) = -\left\{(c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right] + \frac{1-p}{2\lambda}cb\right\}Q_{s1}^2 + \left[k + \frac{1-p}{\lambda}cbQ_p(m+1)\right]Q_{s1}$$
$$-\frac{1-p}{2\lambda}cbQ_p^2(m+1)^2 + 2a(m+1) - 2a + \frac{1-p}{2\lambda}cbQ_p^2 - k(Q_p + \frac{a}{c}) \quad \text{(A4)}$$

$$\frac{dD^{(1)}(Q_{s1})}{dQ_{s1}} = -\left\{2(c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right] + \frac{1-p}{\lambda}cb\right\}Q_{s1} + k + \frac{1-p}{\lambda}cbQ_p(m+1) \quad \text{(A5)}$$

$$\frac{d^2D^{(1)}(Q_{s1})}{dQ_{s1}^2} = -\left\{2(c-k)b\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right] + \frac{1-p}{\lambda}cb\right\} < 0 \quad \text{(A6)}$$

This implies $D^{(1)}(Q_{s1})$ is a concave function during $mQ_p < Q_{s1} < (m+1)Q_p$. From Eq. (A5), if $D^{(1)}(Q_{s1})$ has $dD^{(1)}(Q_{s1})/dQ_{s1} = 0$ property during $mQ_p < Q_{s1} < (m+1)Q_p$, it will be happened at

$$Q_{s1}(m) = \frac{k + (1-p)cbQ_p(m+1)/\lambda}{2(c-k)b[(1-p)^2/2\lambda + p/s] + (1-p)cb/\lambda} \quad \text{(A7)}$$

Since $Q_{s1}(m)$ should satisfy $mQ_p < Q_{s1} < (m+1)Q_p$, we have

$$m_{1L} = \frac{ckQ_p}{2(c-k)a} - 1 < m < \frac{c[k + (1-p)cbQ_p/\lambda]Q_p}{2(c-k)a} = m_{1R} \quad \text{(A8)}$$

Owing to m is an integer, the region of m should change to $\lceil m_{1L}\rceil < m < \lceil m_{1R}\rceil$. According to concavity of $D^{(1)}(Q_{s1})$ during $mQ_p < Q_{s1} < (m+1)Q_p$, $D^{(1)}(Q_{s1})$ is an increasing function of Q_{s1} for $m < \lceil m_{1L}\rceil$ and a decreasing function of Q_{s1} for $m > \lceil m_{11R}\rceil$. □

Proof of Theorem 1

We prove property of $D^{(1)}(Q_{s1}^*)$ only, others are similar to the proof of $D^{(1)}(Q_{s1}^*)$. Taking $Q_{s1}(m)$ into Eq. (A4) and let

$$\Delta_1 = (c-k)\left[\frac{(1-p)^2}{2\lambda} + \frac{p}{s}\right] + \frac{1-p}{2\lambda}c$$

We have

$$\begin{aligned} DM_1(m) &= \frac{-(1-p)(c-k)a}{2\lambda\Delta_1}(m+1)^2 + \left[\frac{(1-p)kcQ_p}{2\lambda\Delta_1} + 2a\right](m+1) \\ &+ \frac{k^2}{4\Delta_1 b} - 2a + \frac{1-p}{2\lambda}cbQ_p^2 - k(Q_p + \frac{a}{c}) \end{aligned} \tag{A9}$$

The firsr amd second derivatives of $DM_1(m)$ respect to m are respectively

$$\frac{dDM_1(m)}{dm} = \frac{-(1-p)(c-k)a}{\lambda\Delta_1}(m+1) + \frac{(1-p)kcQ_p}{2\lambda\Delta_1} + 2a \tag{A10}$$

$$\frac{d^2DM_1(m)}{dm^2} = \frac{-(1-p)(c-k)a}{\lambda\Delta_1} < 0 \tag{A11}$$

It means that $DM_1(m)$ is a concave function of m. Owing to m is an integer, by $DM_1(m) - DM_1(m+1) \ge 0$ and let

$$z_1 = \frac{(kcQ_p - ca + 3ka)(1-p) + 4a\lambda(c-k)\left[(1-p)^2/2\lambda + p/s\right]}{2(1-p)(c-k)a}$$

then $m = \lceil z_1 \rceil$ is the value that maximizes $DM_1(m)$. By $DM_1(m) - DM_1(m-1) \ge 0$, then $m = \lfloor z_1 + 1 \rfloor$ is the value that maximizes $DM_1(m)$. To sum up, if z_1 is not an integer, let $m_{e1} = \lceil z_1 \rceil = \lfloor z_1 + 1 \rfloor$; otherwise, let $m_{e1} = \lceil z_1 \rceil$ and $m_{e1} = \lfloor z_1 \rfloor + 1$. The maximum value of $DM_1(m)$ is $DM_1(m_{e1})$.

(a) When $m_{e1} \le \lceil m_{1L} \rceil$, it means $DM_1(m)$ is a decreasing function of m during $\lceil m_{1L} \rceil \le m \le \lfloor m_{1R} \rfloor$. Hence, the maximum value of $DM_1(m)$ during $\lceil m_{1L} \rceil \le m \le \lfloor m_{1R} \rfloor$ is $DM_1(\lceil m_{1L} \rceil)$ and the special order quantity is

$Q_{s1} = Q_{s1}(\lceil m_{1L} \rceil)$. Case (1) is justified only in the condition $0 \le T_{s1} - t_n < Q_p / s$. If Q_{s1} is not satisfied $0 \le T_{s1} - t_n < Q_p / s$, the maximum value of $D^{(1)}(Q_{s1})$ will happened at $T_{s1} = t_n + Q_p / s$, i.e., $Q_{s1} = \lfloor Q_{s1}(\lceil m_{1L} \rceil) / Q_p \rfloor Q_p + \lambda Q_p / s(1-p)$. Because $D^{(1)}(Q_{s1})$ has positive jumps at break points, $DM_1(\lceil m_{1L} \rceil + 1)$ maybe greater than $D^{(1)}(Q_{s1})$. So both $D^{(1)}(Q_{s1})$ and $DM_1(\lceil m_{1L} \rceil + 1)$ should be compared to determine the global maxima.

(b) When $\lceil m_{1L} \rceil < m_{e1} < \lceil m_{1R} \rceil$, it means $DM_1(m)$ is a concave function of m during $\lceil m_{1L} \rceil \le m \le \lfloor m_{1R} \rfloor$. Hence, the maximum value of $DM_1(m)$ during $\lceil m_{1L} \rceil \le m \le \lfloor m_{1R} \rfloor$ is $DM_1(m_{e1})$ and the special ordering quantity is $Q_{s1} = Q_{s1}(m_{e1})$. If Q_{s1} is not satisfied $0 \le T_{s1} - t_n < Q_p / s$, the maximum value of $D^{(1)}(Q_{s1})$ will happened at $T_{s1} = t_n + Q_p / s$, i.e., $Q_{s1} = \lfloor Q_{s1}(m_{e1}) / Q_p \rfloor Q_p + \lambda Q_p / s(1-p)$. In this time, $D^{(1)}(Q_{s1})$ may be smaller than $DM_1(m_{e1} - 1)$ or $DM_1(m_{e1} + 1)$. So $DM_1(m_{e1} - 1)$ 、 $D^{(1)}(Q_{s1})$ and $DM_1(m_{e1} + 1)$ should be compared to determine the global maxima.

(c) When $m_{e1} \ge \lceil m_{1R} \rceil$, $DM_1(m)$ is an increasing function of m during $\lceil m_{1L} \rceil \le m \le \lfloor m_{1R} \rfloor$. Because $D^{(1)}(Q_{s1})$ has positive jumps at break points, $h_{1R}(\lfloor m_{1R} \rfloor + 1)$ maybe greater than $DM_1(\lfloor m_{1R} \rfloor)$. We need to check whether $Q_{s1} = Q_{s1}(\lfloor m_{1R} \rfloor)$ is satisfied $0 \le T_{s1} - t_n < Q_p / s$ or not. If Q_{s1} is not satisfied the condition, the maximum value of $D^{(1)}(Q_{s1})$ will happened at $T_{s1} = t_n + Q_p / s$, i.e., $Q_{s1} = \lfloor Q_{s1}(\lfloor m_{1R} \rfloor) / Q_p \rfloor Q_p + \lambda Q_p / s(1-p)$. In this time, $D^{(1)}(Q_{s1})$ may be smaller than $DM_1(\lfloor m_{1R} \rfloor - 1)$ or $h_{1R}(\lfloor m_{1R} \rfloor + 1)$. So $DM_1(\lfloor m_{1R} \rfloor - 1)$ 、 $D^{(1)}(Q_{s1})$ and $h_{1R}(\lfloor m_{1R} \rfloor + 1)$ should be compared to determine the global maxima. □

Table 1: The optimal ordering policies for three Models under different discounts

discount	Model 1		Model 2		Model 3	
k	Q_{s1}^*	$D^{(1)}(Q_{s1}^*)$	Q_{s3}^*	$D^{(3)}(Q_{s3}^*)$	Q_{s5}^*	$D^{(5)}(Q_{s5}^*)$
5	67286	166996.0	66208	168092.0	67286	171485.0
4	47431	93553.2	46378	94414.2	47431	97138.3
3	31989	46816.6	30958	47442.8	31989	49498.0
2	19855	18770.5	18752	19166.5	9044	20545.7
1	9928	4317.8	8824	4484.2	9061	5199.9

Table 2: Sensitivity analysis of some parameters on the optimal solutions

parameter		Model 1		Model 2		Model 3	
		Q_{s1}^*	$D^{(1)}(Q_{s1}^*)$	Q_{s3}^*	$D^{(3)}(Q_{s3}^*)$	Q_{s5}^*	$D^{(5)}(Q_{s5}^*)$
b	0.20	24179	47277.3	22808	46635.7	23561	49560.7
	0.15	31633	62729.2	30733	62652.1	31553	65505.4
	0.10	47431	93553.2	46378	94414.2	47431	97138.3
	0.05	93621	185723.0	92357	188685.0	93597	191147.0
	0.01	460913	919615.0	460702	931429.0	460832	932764.0
c	18	27124	53258.5	26223	53181.4	27035	56025.8
	15	34545	67889.9	335991	68209.8	34531	71003.0
	12	47431	93553.2	46378	94414.2	47431	97138.3
	9	75222	150192.0	74098	151841.0	75150	154476.0
	6	187224	377685.0	185848	380670.0	187195	383136.0
s	36000	48252	95952.3	48044	96900.8	48170	99580.4
	30000	47903	94980.0	46887	95890.8	47822	98591.3
	24000	47431	93553.2	46378	94414.2	47431	97138.3
	18000	45905	91266.9	44900	92036.2	45825	94825.3
	12000	43732	87020.2	42740	87637.9	43654	90513.9
q_{30}	1100			46328	93487.4		
	1000			46333	93950.7		
	900			46378	94414.2		
	800			46423	94878.2		
	700			46468	95342.7		
q_{50}	600					46479	95454.0
	500					46519	95872.5
	400					46559	96291.3
	300					47431	96711.5
	200					47431	97138.3

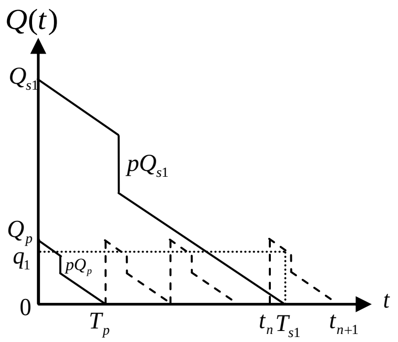

Figure 1: Case (1) diagram

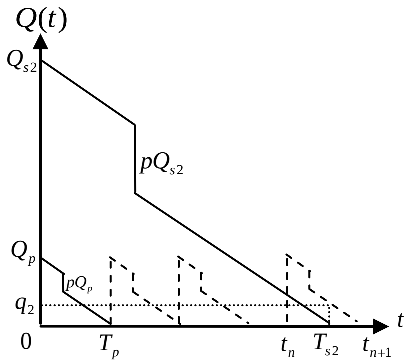

Figure 2: Case (2) diagram

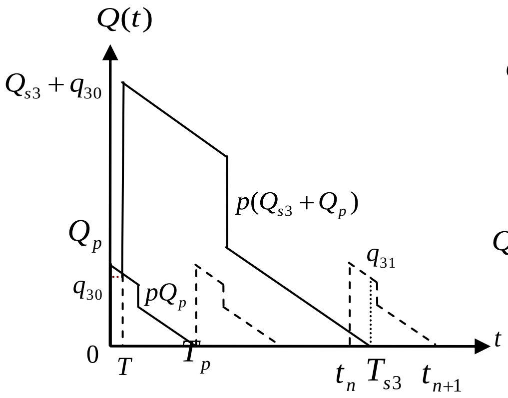

Figure 3: Case (3) diagram

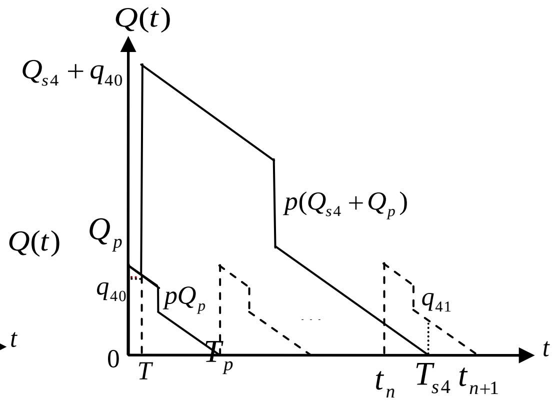

Figure 4: Case (4) diagram

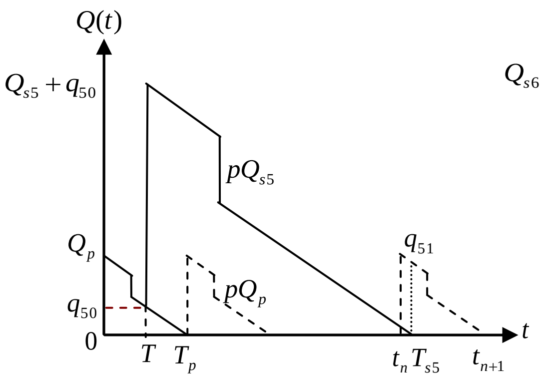

Figure 5: Case (5) diagram

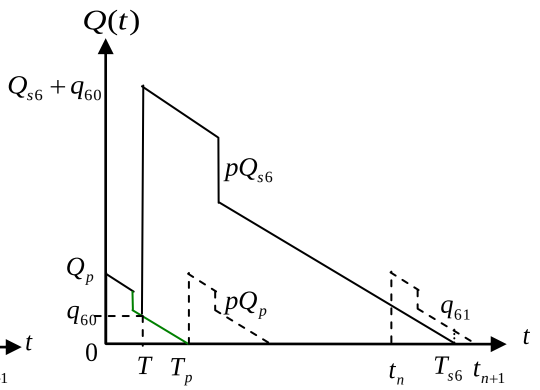

Figure 6: Case (6) diagram

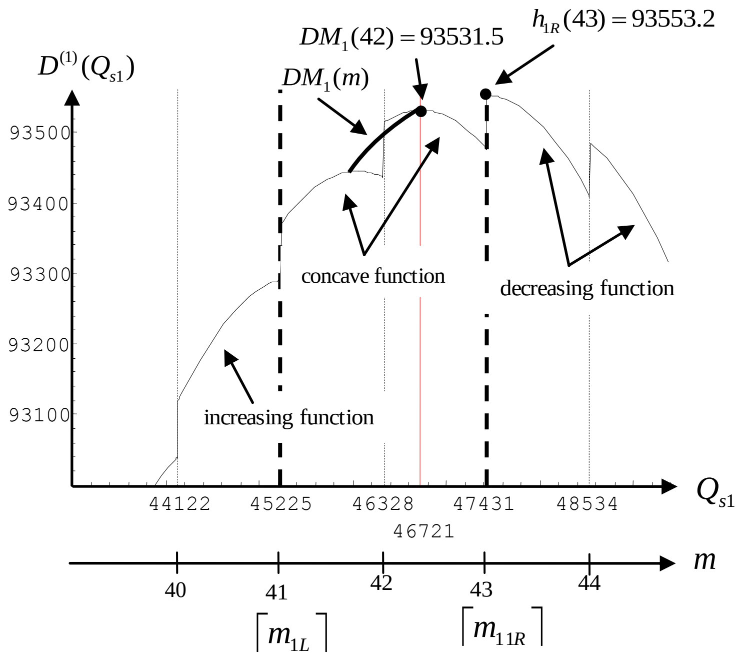

Figure 7: The saving cost $D^{(1)}(Q_{s1})$ of example 1

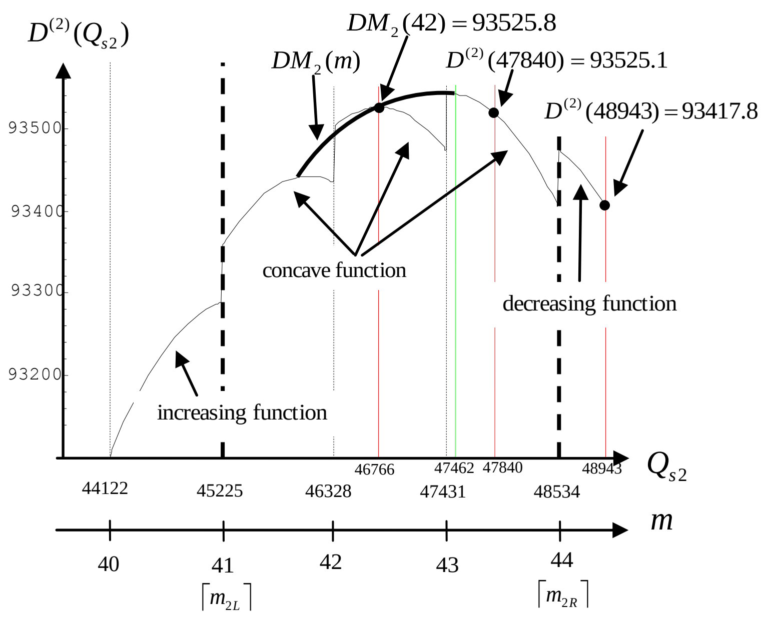

Figure 8: The saving cost $D^{(2)}(Q_{s2})$ of example 1