ENDOGENOUS ROYALTY FACTOR IN A LICENSING CONTRACT

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Abstract: The owner of a well known fashion brand grants a manufacturer the rights to produce and sell a second-line brand against a percentage of the sales called royalty. To this end, the brand owner and the manufacturer sign a licensing contract which assigns the owner, who has already determined his advertising campaign, the right of determining the royalty factor. The manufacturer will plan her advertising campaign for the licenced product in order to maximize her profit. The brand owner's objective is twofold: on the one hand, he wants to maximize the profit coming from the contract, on the other hand, he wants to improve the value of the brand at the end of a given planning period. We model this interaction between the two agents using a Stackelberg game, where the brand owner is the leader and the manufacturer is the follower. We characterise the royalty percentage and the licensee's advertising effort which constitute the unique Stackelberg equilibrium of the game.

1. INTRODUCTION

Licensing contracts are widely used marketing tools that allow the involved firms to obtain a variety of benefits, such as increased revenues, new market penetration, and reposition of the brand. Raugust warns [19, p. 9] "... parties should also be aware of the risks and challenges." Kotler et al. [14, p. 577] write "... some companies license names or symbols previously created by other manufacturers, names of celebrities, and characters from popular movies and books, for a fee."

There exist several types of goods which can be licensed in different fields, such as sport, fashion, movie, technology, ... Here we focus on the fashion licensing, which represents one of the main areas of application, nowadays. In fact, as Raugust observes [19, p. 39], "most consumers are not aware that much of the fashion merchandise they buy is licensed" and "fashion licensors must maintain particular control over their licensing programs to ensure that licensed products do not harm the brand's positive image."

In this field, two main types of agreement can be considered: the first one is the so called "same business" licensing, where the licensor gives to another manufacturer the rights to produce and sell his second-line product. In the second type of licensing agreement (settled in a "complementary business"), the licensee produces and sells some accessories coordinated to a given fashion line. In both cases, the licensing contract is a sort of bilateral strategic alliance [8, p. 171] between firms of significantly different size.

Buratto and Zaccour [4] have analysed and modelled both types of licensing in the context of Stackelberg differential games, using the Nerlove and Arrow advertising model as the fundamental framework (see [9] for a recent survey on dynamic models in marketing where the Nerlove and Arrow approach is carefully described). In some other papers ([1], [3] and [2]) the complementary business licensing is analysed taking into account different aspects: production costs [1], brand sustainability [2], brand extension [3].

Royalties constitute the main fee of a licensing agreement. They may consist, in particular, either in a percentage of the sales or in a fixed amount of money [19, p. 135]. The choice of the type of royalty, as well as its actual amount, turns out to be a crucial decision. Moreover, it seems natural that the licensor, who has a dominant position in this kind of licensing agreement, should make such a choice. In [4], the royalties are assumed to be a percentage of the sales, considering the royalty factor as an exogenous parameter. This point of view is common in the literature on licensing. "Royalties, advances and guarantees vary depending on a number of factors and all are negotiable", as Raugust observes [19, p. 135]. Here we want to find the answer to the question: Is there an optimal value of the royalty factor from the licensor's viewpoint? Hence, in this paper we assume that the owner of a well known fashion brand considers the opportunity of granting to a manufacturer the rights to produce and sell a second-line brand against a percentage of the sales. The brand owner has already determined his advertising campaign, and he has to choose the royalty percentage. His objective is twofold:

on the one hand, he wants to maximize the profit coming from the contract, on the other hand, wants to improve the value of the brand at the end of the advertising campaign. The manufacturer has to plan the advertising campaign for the licensed product.

In [4] the licensee's advertising effort affects only the brand value evolution of the licensed product, whereas the licensor's advertising effort affects the evolution of both brand values. The effect of the latter effort is direct for the first–line brand, and indirect for the second–line, through a spillover mechanism.

In this work, we still adopt the Nerlove and Arrow concept of *goodwill*, as the variable which describes the effect of advertising on the demand [18]. In order to represent the synergy effect of the licensor's advertising activity on the licensee's goodwill, we introduce a term in the licensee's goodwill motion equation, proportional to the licensee's goodwill and to the licensor's advertising effort. In this way, we model the synergy effect in analogy with an interference model by Leitmann and Schmitendorf (see: [15], [10]), and we take care of the different importance of the two agents. A similar interaction is described also in [7] to model a negative interference. Moreover, we assume that the licensee's advertising effort has an effect on the licensor's goodwill evolution, consistently with the idea that the advertising message for a second-line product affects the value of the main brand too. We represent such an effect by means of an additive term in the licensor's goodwill motion equation.

We discuss the licensor–licensee relationship in the framework of a Stackelberg game, where the licensor is the leader and the licensee is the follower. The best response of the licensee, i.e. her advertising policy as a function of the royalty factor, is defined as the optimal solution of a dynamic optimization problem, whereas the choice of the equilibrium royalty factor by the licensor is defined as the optimal solution of a nonlinear programming problem.

The paper is organized as follows. In Section 2 we introduce the model, defining the goodwill and sales dynamics and the objective functions. In Section 3 we discuss the follower's behavior. In Section 4 we analyse the choice of the royalty factor by the leader, resulting in the characterization of a Stackelberg equilibrium. In Section 5 we provide conclusions and ideas for further developments of the model. Finally, in the appendix, we discuss an instance of the licensing game with quadratic advertising cost and constant leader's advertising effort, which allows a particularly explicit description of the equilibrium.

2. LICENSING GAME

Two economic agents, a brand owner and a manufacturer, agree upon a licensing contract for the production and sale of a good with the owner name/mark. Such a good is referred to as the (owner's) second–line product. Once the manufacturer accepts the licensing contract, she plans her advertising policy $a_l(t)$ for the second-line product and agrees on paying a royalty R_L to the brand owner. The manufacturer's advertising policy affects the goodwill $G_l(t)$ of the second-line product, and hence the demand for it. The royalty is proportional to the

second-line revenue, with the proportionality factor *r* being chosen by the brand owner. The brand owner has his own first-line brand with a related goodwill $G_L(t)$. The licensor has chosen beforehand his advertising policy $a_L(t)$, on account of a variety of reasons concerning his businesses, and we assume that he does not alter it as a consequence of the licensing contract. Therefore, we consider the licensor advertising effort $a_L(t)$ as an exogenous information and assume it to be a continuous function. Moreover, we assume that the licensee's advertising effort affects positively the licensor goodwill $G_L(t)$ too. For the sake of shortness, in what follows, we will sometimes omit the terms brand/line and simply say licensor's/licensee's goodwill referring to the goodwill of the licensor's first-line product and of the licensee's second-line product, respectively.

Here, we focus on the strategic interactions between the two agents in terms of advertising campaign (licensee, renter) and royalty rate (licensor, brand owner). Observing a hierarchy between the agents' actions, we set the problem in the framework of a Stackelberg game, where the brand owner and the manufacturer act as the leader and the follower, respectively.

2.1. Second–line sales and royalty

Denote by $S(t)$ the cumulative sales, measured in quantity, of the licensed product. Let the product (market) price be $p > 0$ and let $c \in (0, p)$ be the constant unit production cost.

For granting the rights to produce and market his brand to the licensee, the licensor obtains a financial counterpart, called the *royalty*,

$$R_L = rpS(T), \tag{1}$$

where $r \in (0, 1)$ is the *royalty factor*, while the licensee keeps the part

$$R_l = (1 - r)pS(T) \tag{2}$$

as her revenue. As the production cost incurred by the licensee is $cS(T)$, the resulting licensee's profit, gross of advertising costs, is $((1 - r)p - c)S(T)$, it is non-negative if and only if $r \leq 1 - c/p$. For theoretical reasons, we admit also the two extreme situations where either $r = 0$ or $r = 1 - c/p$; in the former case the licensor does not require any royalty from the licensee, so that $R_L = 0$, in the latter case the licensor grabs all possible profit from the licensee, so that $R_L = (p - c)S(T)$. On the other hand, we do not admit that $r > 1 - c/p$, which would entail a negative licensee's profit, as this would be inconsistent with the assumption that the follower accepts the licensing contract, while she wants to maximize her profit.

Finally, we assume that the second-line product sales rate coincides with its brand value, or goodwill $G_l(t)$,

$$\dot{S}(t) = G_l(t), \tag{3}$$

where $G_l(t)$ is a result of the advertising activity of both players.

2.2. Goodwill dynamics

In order to represent the effects of the players' advertising we use two variables, the first-line goodwill, G_L, and the second-line goodwill, G_l, as done in [4].

We assume that the resulting goodwill levels are determined by the differential equations

$$\dot{G}_L(t) = \eta_L a_L(t) + \varepsilon_L a_l(t) - \delta G_L(t), \tag{4}$$
$$\dot{G}_l(t) = \eta_l a_l(t) - (\delta - \varepsilon_l a_L(t)) G_l(t), \tag{5}$$

and by the initial conditions

$$G_i(0) = G_i^0, \qquad i \in \{L, l\}, \tag{6}$$

where

- $\eta_i a_i(t)$ represents the effect of the advertising effort $a_i(t)$ of player i on his/her goodwill, $\eta_i > 0$, $i \in \{L, l\}$;

- $\varepsilon_L a_l(t)$, $\varepsilon_L \geq 0$, represents the synergy effect of the licensee advertising effort towards the licensor's brand goodwill, it adds to the effect of the advertising effort by the licensor;

- $\varepsilon_l a_L(t) G_l(t)$, $\varepsilon_l > 0$, represents the synergy effect of the licensor advertising effort towards the licensee's brand goodwill; it reduces the spontaneous decaying of $G_l(t)$ and is proportional to the follower's goodwill value;

- G_i^0, are the initial brand values of the line i of product, $G_i^0 > 0$, $i \in \{L, l\}$. We may think that G_l^0, the licensee's initial goodwill, is an increasing function of G_L^0, to account for the spillover effect of the licensor's initial goodwill on the initial licensee's goodwill, a start–up support by the licensor to the licensee.

2.3. Payoffs

The brand owner (leader) is interested in maximising both the royalty to be obtained from the second-line product sales and the main brand value at the end of the planning period. The latter objective is consistent with the statement by Raugust that "Another benefit of licensing, especially for corporate trademarks owners, is its effectiveness in helping to relaunch or reposition a brand or property" [19, p. 13]. The objective function of the brand owner (licensor) is

$$\Pi_L(r) = rpS(T) + \sigma_L G_L(T), \tag{7}$$

where $\sigma_L > 0$ is the marginal value of the brand owner's first-line goodwill. We will refer to it as the licensor's utility.

On the other hand, the manufacturer (follower) wants to maximise her profit and may choose her advertising effort $a_l(t)$. The objective functional of the manufacturer (licensee) is

$$\Pi_l(a_l) = ((1-r)p - c)S(T) - \int_0^T C(a_l(t)) dt, \tag{8}$$

where $C(a_l)$ is the cost rate associated with the advertising effort a_l. We assume that $C(a_l)$ is a strictly increasing, convex, and continuously differentiable function, where $C(0) = 0$, $C'(0) = 0$, $C''(a_l) > 0$, $C'''(a_l) \geq 0$. It follows, in particular, that $\lim_{a_l \to +\infty} C'(a_l) = +\infty$. We will refer to (8) as the licensee's profit.

2.4. Comments on the players' interaction model

Note that the advertising for the second-line brand affects the first-line goodwill evolution adding a contribution to the effect of the advertising effort by the licensor. The first-line brand goodwill evolution is modelled essentially as in the Nerlove and Arrow model (see: [13, p. 54], [9]): it is a typical assumption to represent additively the joint effect of different advertising actions on the goodwill evolution (see e.g. [17, 12, 11] and [5, p. 304]).

On the other hand, the advertising for the first-line brand affects the second-line goodwill evolution modifying its decay attitude. The second-line brand goodwill evolution is formally modelled as in [7] by exploiting an original idea of [15] (see also [10]). Here, the multiplicative term $\varepsilon_l a_L(t) G_l(t)$ is meant to represent a positive interaction (synergy), whereas in [7] and [15], it describes a negative interference.

In the model of [4], a linear term (proportional to G_L) is present in place of $\varepsilon_l a_L(t) G_l(t)$, to represent a *spillover effect* of the licensor's advertising on the licensee's goodwill, whereas no effect of the licensee's advertising on the licensor's goodwill is assumed. Similarly as in [4], we want to represent a situation in which the effect of the licensee's advertising on the licensor's goodwill is weak (0 in [4], linear here), whereas the effect of the licensor's advertising on the licensee's goodwill is strong (linear in [4], non-linear here). In order to complete the comparison with [4], we observe that in [4] the licensor's advertising effort is a control function, i.e. a strategy to be chosen by the licensor who takes into account his relationship with the licensee, whereas here it is a strategy chosen in advance by the licensor, independently of the licensing agreement. Finally, an important distinction of the present model from that in [4] is that here the royalty factor r is the licensor's decision variable, whereas in [4], it is an exogenous parameter. In fact this feature is distinctive also with respect to the general literature on licensing, in which the royalty factor is dealt with mainly as an exogenous parameter (see e.g. [19]).

3. FOLLOWER'S BEST RESPONSE

We are assuming that the royalty factor $r \in [0, 1 - c/p]$ and the continuous function (licensor's advertising effort) $a_L(t)$ are known to the licensee. The licensee solves the problem of maximising the profit Π_l, defined by equation (8), subject to the motion equation (5), the initial condition (6), and the advertising effort positivity condition

$$a_l(t) \in [0, +\infty). \tag{9}$$

This is an optimal control problem with one state variable, G_l, and one control variable, a_l. After defining

$$\Delta(t) = \delta - \varepsilon_l a_L(t) \tag{10}$$

for notational convenience, the problem Hamiltonian is

$$H = [((1-r)p - c)G_l - C(a_l)] + \lambda [\eta_l a_l - \Delta(t)G_l] , \tag{11}$$

a continuously differentiable and concave function of (a_l, G_l). An optimal solution must satisfy the Pontryagin Maximum Principle conditions [20, p. 85] which give

$$C'(a_l(t)) = \eta_l \lambda(t), \ \text{if} \ \lambda(t) > 0 , \quad \text{or else} \ a_l(t) = 0 , \tag{12}$$

$$\dot{\lambda}(t) = -((1-r)p - c) + \Delta(t)\lambda(t) , \tag{13}$$

$$\lambda(T) = 0 . \tag{14}$$

The adjoint Cauchy problem (13-14) has the unique solution

$$\lambda(t) = ((1-r)p - c) \int_t^T e^{-\int_t^u \Delta(s)\,ds} \, du , \tag{15}$$

which has the same sign as the factor $(1-r)p - c$ for all $t < T$.

If $r < 1 - c/p$, then, in view of condition (12), the unique candidate optimal control is

$$a_l^*(t) = A(\eta_l \lambda(t)) , \tag{16}$$

where $A(\cdot)$ is the inverse function of the marginal cost $C'(\cdot)$. As the marginal cost $C'(a_l)$ is a strictly increasing function and $C'(0) = 0$, then its inverse function $A(\cdot)$ is strictly increasing and $A(0) = 0$. It follows that $a_l^*(t) > 0$ at all times $t < T$, as $\lambda(t) > 0$ (the integral in (15) operates on a positive function), and it vanishes at $t = T$.

If $r = 1 - c/p$, then we observe that $\lambda(t) = 0$, and consequently

$$a_l^*(t) \equiv 0 , \tag{17}$$

because of (12). This result was expected, as the licensee's marginal profit is negative. In this case, the licensor grabs all the profit from the second-line production.

For all r, we observe that, after integrating the motion equation using the control $a_l^*(t)$, we obtain the goodwill function $G_l^*(t)$. The unique solution $(a_l^*(t), G_l^*(t))$ to the necessary conditions is optimal, because the Hamiltonian (11) is concave in (a_l, G_l) (see e.g. [20, The Mangasarian sufficiency theorem, p. 105]). Finally, the value of total sales associated with the optimal solution $(a_l^*(t), G_l^*(t))$ is

$$S^*(T) = \int_0^T G_l^*(t)\,dt > 0 . \tag{18}$$

The licensee advertises her product if and only if the royalty factor r is less than $1 - c/p = (p - c)/p$, which represents the second-line maximum profit/revenue ratio.

3.1. Sensitivity to exogenous information

From our assumptions, the initial value of goodwill for the second-line product, G_l^0, and the advertising effort $a_L(t)$ of the licensor are exogenously given and are part of the features of the licensing agreement. Moreover, the licensee solves her optimal control problem while knowing the value of r chosen by the licensor.

The first result, concerning the sensitivity to G_l^0 and $a_L(t)$ is stated in the following proposition.

Proposition 3.1. *The optimal sales $S^*(T)$ from the second-line business and the licensee's optimal profit $\Pi_l(a_l^*)$ are monotonically increasing functions of G_l^0, $a_L(t)$.*

Proof. Using the equation (18), it is easy to prove that $S^*(T)$ is strictly increasing in G_l^0. Moreover, the advertising cost $\int_0^T C(a_l^*(t))dt$ does not depend on G_l^0, hence $\Pi_l(a_l^*)$ is an increasing function of G_l^0.

We need to introduce some notation for the sake of clarity in the second part of the proof and later. Let $a_l^*(t; a_L)$ be the licensee's optimal advertising effort associated with the licensor's advertising policy $a_L(t)$, and $S(T; a_l, a_L)$ be the optimal second-line sales associated with the licensor and licensee's advertising policies $a_L(t)$ and $a_l(t)$, respectively. Finally, let $\Pi_l(a_l^*(t; a_L); a_L)$ be the licensee's optimal profit associated with the licensor's advertising policy $a_L(t)$, and let us denote by $w = ((1 - r)p - c)$ the marginal profit gross of advertising costs, which is a positive constant.

To prove the monotonicity with respect to $a_L(t)$, let

$$a_L^1(t) \leq a_L^2(t), \qquad t \in [0, T]. \tag{19}$$

Using the equations (5), (19), and (3), it is easy to prove that
$S(T; a_l^*(t; a_L^1), a_L^1) \leq S(T; a_l^*(t; a_L^2), a_L^2)$. Moreover, we have that

$$\Pi_l\left(a_l^*(t; a_L^1); a_L^1\right) = w \cdot S(T; a_l^*(t; a_L^1), a_L^1) - \int_0^T C(a_l^*(t, a_L^1(t)))dt \leq$$

$$\leq w \cdot S(T; a_l^*(t; a_L^1), a_L^2) - \int_0^T C(a_l^*(t; a_L^1))dt \leq$$

$$\leq w \cdot S(T; a_l^*(t; a_L^2), a_L^2) - \int_0^T C(a_l^*(t; a_L^2))dt = \Pi_l\left(a_l^*(t; a_L^2); a_L^2\right),$$

where the second inequality follows from the fact that $a_l^*(t; a_L^1)$ is an admissible control of the follower's problem when $a_L^2(t)$ is the advertising policy of the leader, whereas $a_l^*(t; a_L^2)$ is an optimal control of that problem. $\square$

In order to obtain some information on the sensitivity to the royalty factor r, we examine the differentiability of optimal control and state functions $a_l^*(t)$ and $G_l^*(t)$ w.r.t. r, and hence the differentiability of optimal sales $S^*(T)$ w.r.t. r. The following results are relevant because r is the decision variable of the leader. We will use the notation $a_l^*(t; r)$, $G_l^*(t; r)$ and $S^*(T; r)$ for the licensee's optimal advertising effort

and goodwill, and the optimal second-line sales, respectively, associated with the royalty factor r.

The first result concerns the dependence of the licensee's optimal advertising effort and of the second-line brand goodwill on r: both of them are lower at higher royalty factor values. More precisely, they are decreasing and concave functions of r.

Proposition 3.2. *If $r < 1 - c/p$, then*

$$\frac{\partial}{\partial r} a_l^*(t;r) < 0, \qquad \frac{\partial^2}{\partial r^2} a_l^*(t;r) \leq 0, \tag{20}$$

and

$$\frac{\partial}{\partial r} G_l^*(t;r) < 0, \qquad \frac{\partial^2}{\partial r^2} G_l^*(t;r) \leq 0. \tag{21}$$

Proof. We notice that the marginal cost $C'(a_l)$ is a continuously differentiable function, with $C'(0) = 0$, $C''(a_l) > 0$, $C'''(a_l) \geq 0$, hence its inverse function $A(\cdot)$ is continuously differentiable, with $\lim_{z \to 0} A'(z) = +\infty$, $A'(z) > 0$, $A''(z) \leq 0$. We have that

$$\frac{\partial^k}{\partial r^k} a_l^*(t;r) = A^{(k)}\left(\eta_l \lambda(t)\right) \left(-p \eta_l \int_t^T e^{-\int_t^u \Delta(s)\,ds}\,du\right)^k, \tag{22}$$

where $A^{(k)}(\cdot)$ is the kth derivative of $A(\cdot)$, so that the signs in (20) hold. Moreover, we have that

$$\frac{\partial^k}{\partial r^k} G_l^*(t;r) = \eta_l \int_0^t e^{-\int_u^t \Delta(s)\,ds} \frac{\partial^k}{\partial r^k} a_l^*(u)\,du, \tag{23}$$

so that the signs in (21) hold. □

Hence, the licensee should not invest in advertising significantly if she has to pay a high royalty percentage. Also, the optimal cumulative sales of the second-line brand are lower at higher royalty factor values, as stated below.

Proposition 3.3. *As far as $r < 1 - c/p$, the optimal sales $S^*(T;r)$ from the second-line business and the licensee's optimal profit $\Pi_l(a_l^*)$ are strictly decreasing functions of r. Moreover $S^*(T;r)$ is concave and we have that*

$$S^*(T;r) > 0, \qquad \frac{\partial}{\partial r} S^*(T;r) < 0, \qquad \frac{\partial^2}{\partial r^2} S^*(T;r) \leq 0. \tag{24}$$

Proof. From equation (18) we know the first inequality of the thesis, moreover we have that

$$\frac{\partial^k}{\partial r^k} S^*(T;r) = \int_0^T \frac{\partial^k}{\partial r^k} G_l^*(t;r)\,dt, \tag{25}$$

and, for $k = 1, 2$, using Proposition 3.2 we prove equation (24).

Let $r_1 < r_2$; we need to prove that $\Pi_l\left(a_l^*(t; r_1)\right) > \Pi_l\left(a_l^*(t; r_2)\right)$. By contradiction, let us assume that $\Pi_l\left(a_l^*(t; r_1)\right) \leq \Pi_l\left(a_l^*(t; r_2)\right)$. Then

$$
\begin{aligned}
\Pi_l\left(a_l^*(t; r_1)\right) = & \;\; ((1 - r_1)p - c)S^*(T; r_1) - \int_0^T C(a_l^*(t; r_1))dt \;\; \leq \\
\leq & \;\; ((1 - r_2)p - c)S^*(T; r_2) - \int_0^T C(a_l^*(t; r_2))dt \;\; < \\
< & \;\; ((1 - r_1)p - c)S^*(T; r_2) - \int_0^T C(a_l^*(t; r_2))dt \,,
\end{aligned}
$$

contradicting the optimality of $a_l^*(t; r_1)$ when the royalty factor is r_1. $\quad\square$

The influence of the royalty factor r on the profit of the brand owner is not trivial. This justifies the importance of a formulation in the terms of an optimization problem in order to determine the optimal royalty the licensor should impose to the licensee.

4. LEADER'S CHOICE OF ROYALTY FACTOR

The leader wants to maximize his utility, given by equation (7), once the behaviour of the follower is known.

Let $G_L(t; a_l)$ be the licensor's goodwill function when the licensee adopts the advertising effort $a_l(t)$. From the knowledge of the follower's best response, we obtain that the leader's utility is

$$
\Pi_L(r) = rpS^*(T; r) + \sigma_L G_L(T; a_l^*(t; r)) \,, \tag{26}
$$

and we observe that

$$
\Pi_L(r) > \sigma_L G_L(T; a_l^*(t; r)) \geq \sigma_L G_L(T; 0) \,, \tag{27}
$$

because $S^*(T; r) > 0$ and $\dot{G}_L(t; a_l(t)) \geq \dot{G}_L(t; 0)$ for all t, and all admissible $a_l(t)$.

Lemma 4.1. *The leader's utility $\Pi_L(r)$ is a twice continuously differentiable and concave function in $[0, 1 - c/p]$.*

Proof. For any given $r \leq 1 - c/p$ the follower's advertising effort is $a_l^*(t; r)$, hence

$$
\Pi_L(r) = rpS^*(T; r) + \sigma_L G_L(T; a_l^*(t; r)) \,. \tag{28}
$$

From Proposition 3.3 we know that $S^*(T; r)$ is concave in r, moreover

$$
\frac{\partial^k}{\partial r^k} G_L(T; a_l^*(t; r)) = \varepsilon_L \int_0^T e^{-\delta(T-t)} \frac{\partial^k}{\partial r^k} a_l^*(t; r) \, dt \,, \tag{29}
$$

so that

$$\frac{\partial^2}{\partial r^2} G_L(T; a_l^*(t; r)) \leq 0 \,, \tag{30}$$

because, from Proposition 3.2, we have that $\partial^2 a_l^*(t; r)/\partial r^2 \leq 0$ for all t. $\square$

The information on the leader's utility enables us to discuss equilibria. The leader wants to determine the values of the royalty factor $r \in [0, 1 - c/p]$ which maximise his utility. Now, as the interval $[0, 1 - c/p]$ is compact and the leader's utility is continuous, there exists a value of r which maximizes the licensor's utility.

Theorem 4.2. *There exists a unique $r^* \in [0, 1 - c/p]$, such that*

$$\Pi_L(r^*) = \max_{r \in [0, 1 - c/p]} \Pi_L(r) \,; \tag{31}$$

then the pair

$$(r, a_l(t)) = \left(r^*, a_l^*(t; r^*) \right) \tag{32}$$

is the unique Stackelberg equilibrium of the game.
If r^* is an internal point, $r^* \in (0, 1 - c/p)$, then r^* is the solution of the following equilibrium royalty factor equation

$$pS^*(T; r) + rp \frac{\partial S^*(T; r)}{\partial r} + \sigma_L \frac{\partial G_L(T; a_l^*(t; r))}{\partial r} = 0 \,. \tag{33}$$

Proof. The first part follows from Lemma 4.1. The second part follows from the definition of Stackelberg equilibrium (see [16]). The equation (33) is the first order necessary condition $d\Pi_L(r)/dr = 0$. $\square$

We observe that this result depends critically on the value of the parameter σ_L, and that the equilibrium royalty factor may well be 0. In the following we try to gain some insight from such result.

4.1. Sensitivity to σ_L

The parameter σ_L is fundamental to establish the relevance for the leader of his final goodwill value with respect to the royalty obtainable from the licensing contract. We want to understand to what extent such a parameter affects the equilibrium of the game.

To this purpose, let us consider the left hand side of equation (33) as a function of r and σ_L,

$$\varphi(r, \sigma_L) = pS^*(T; r) + rp \frac{\partial S^*(T; r)}{\partial r} + \sigma_L \frac{\partial G_L(T; a_l^*(t; r))}{\partial r} \,. \tag{34}$$

Theorem 4.3. *Let $\sigma_L > 0$; if*

$$\varphi(0, \sigma_L) > 0, \qquad \varphi(1 - c/p, \sigma_L) < 0, \tag{35}$$

then there exists a unique $r^* \in (0, 1 - c/p)$ which solves the equilibrium royalty factor equation (33), i.e. such that $\varphi(r^*, \sigma_L) = 0$.
Otherwise, if

$$\varphi(0, \sigma_L) \leq 0, \quad or \quad \varphi(1 - c/p, \sigma_L) \geq 0, \tag{36}$$

then $\varphi(r, \sigma_L) \neq 0$ for all $r \in (0, 1 - c/p)$.
If $\varphi(r^*, \sigma_L) = 0$, and $r^* \in (0, 1 - c/p)$, then there exist a neighborhood of σ_L, $U = (\sigma_L - \varepsilon, \sigma_L + \varepsilon) \subset (0, +\infty)$, and a unique differentiable function $r : U \to (0, 1 - c/p)$, such that

$$r(\sigma_L) = r^*, \qquad \varphi(r(\sigma), \sigma) = 0, \quad for\ all\ \sigma \in U, \tag{37}$$

and

$$r'(\sigma_L) < 0. \tag{38}$$

Proof. Let conditions (35) hold. The function $\varphi(r, \sigma_L)$, $r \in [0, 1 - c/p]$, for a given σ_L is continuous and monotonically strictly decreasing, by Lemma 4.1. Moreover, from the theorem hypotheses, we have that such a function takes opposite sign values at $r = 0$ and $r = 1 - c/p$ respectively. Therefore, there exists a unique $r^* \in (0, 1 - c/p)$ at which the function vanishes.

On the other hand, if $\varphi(0, \sigma_L) \leq 0$, then $\varphi(r, \sigma_L) < 0$, $r > 0$, whereas if $\varphi(1 - c/p, \sigma_L) \geq 0$, then $\varphi(r, \sigma_L) > 0$, $r > 0$.

We can use the *implicit function theorem* (see e.g. [21, p. 339]), with the equation $\varphi(r, \sigma_L) = 0$, We notice that $\varphi(r, \sigma_L)$ is a continuously differentiable function and has negative partial derivatives, $\partial \varphi(r, \sigma_L)/\partial r < 0$, $\partial \varphi(r, \sigma_L)/\partial \sigma_L < 0$, as we can obtain from the inequalities (30), (24), and from the fact that $\partial G_L(T; a_l^*(t; r))/\partial r < 0$, which follows from equations (29) and (20). □

We can notice that the first condition in (35) is equivalent to

$$\sigma_L < \bar{\sigma}_L = -\frac{pS^*(T; 0)}{\partial G_L(T; a_l^*(t; 0))/\partial r}, \tag{39}$$

with $\bar{\sigma}_L > 0$, whereas the second condition in (35) is equivalent to

$$\sigma_L > \underline{\sigma}_L = -\frac{pS^*(T; 1 - c/p) + (1 - c/p)p\partial S^*(T; 1 - c/p)/\partial r}{\partial G_L(T; a_l^*(t; 1 - c/p))/\partial r}, \tag{40}$$

where $\underline{\sigma}_L$ may even be negative.

If the condition (39) does not hold, i.e. if $\sigma_L \geq \bar{\sigma}_L$, then the leader's utility reaches its maximum at $r = 0$, so that we can extend the definition of $r(\sigma)$ by setting $r(\sigma) = 0$, as $\sigma \geq \bar{\sigma}_L$. On the other hand, if the condition (40) does not hold, i.e. if $\sigma_L \leq \underline{\sigma}_L$, then the leader's utility reaches its maximum at $r = 1 - c/p$.

We can summarize the analysis with respect to the marginal utility of the final licensor's brand value σ_L as follows.

Corollary 4.4. *Let the thresholds $\bar{\sigma}_L$ and $\underline{\sigma}_L$ be defined as in equations (39) and (40);*

- *if $\sigma_L \geq \bar{\sigma}_L$, then $r(\sigma_L) = 0$, and the game has a Stackelberg equilibrium with zero royalty factor;*

- *if $\underline{\sigma}_L < \sigma_L < \bar{\sigma}_L$, then $r(\sigma_L) > 0$, and the game has a Stackelberg equilibrium with positive royalty factor, which is as larger as σ_L is smaller ($r(\sigma_L) = 0$ is characterized by (33));*

- *if $\underline{\sigma}_L > 0$ and $\sigma_L \leq \underline{\sigma}_L$, then the game has a Stackelberg equilibrium with $1 - c/p$ royalty factor, in this extreme case we observe the manufacturer producing without any profit.*

The first of the three cases occurs if the leader wants to use the licensing contract only to improve his brand value, and the royalty is irrelevant to him. The third case may occur if the leader wants only to make profit from the royalty, whereas he foregoes any possible positive contribution from the licensing contract to his brand value. The intermediate case is the one where the two leader's objectives are balanced.

The higher the marginal value of the licensor's first-line brand, the more likely there exists an optimal royalty factor for the leader, and in case it exists, the lower its value. As σ_L is higher, the leader is more interested in obtaining a higher value of his final goodwill than in getting a larger royalty. Brand devaluation is, in fact, one of the licensor's risks and if the licensor does not want to dilute his brand image and value, he has to maintain control over his licensee and his licensing strategy (see [19, p. 17]).

5. CONCLUSION

In this paper we have studied the strategic interaction between two economic agents involved in a licensing contract. We use a Stackelberg differential game to describe the strategies of the two players. We assume that the owner of the brand is the leader, that he has already determined his advertising policy, and that he exploits the licensing contract in order to increase his revenues and to raise his brand value. The follower is a manufacturer who has to plan the advertising campaign for the licensed product. There are two main innovations introduced with our work. First, we modify the model proposed by [4], assuming that the owner of the brand has to decide the value of the royalty factor, whereas he has already decided his advertising strategy. Then, the second-line goodwill dynamics is inspired by the Leitmann-Schmitendorf motion equation [15].

This kind of commercial agreement has a twofold objective. On the one hand, the licensing contract should improve the brand value, on the other hand, it should give an extra profit to the licensor [19, p. 9]. In our model, this trade off is well described in terms of the licensor's marginal utility with respect to the final brand value. If this quantity is too small, then the leader is just interested in the extra profit and the royalty factor will be large. If this quantity is large, then the royalty factor will be small.

A natural extension of our study may consider the possibility that the brand owner proposes several licensing contracts to different manufacturers. If a strategic interaction among different followers is taken into account, then the model is substantially different from the one presented in this paper and further analysis is needed to formalize and characterize the strategies of the players.

The model discussed here applies also to licensing contracts used as internationalization strategies by retail firms. In that case, licensing, like franchising, is considered as an opportunistic behaviour, a lower risk entry mode in international markets [6], and exhibits similar features as the same business licensing in the fashion goods industry.

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Appendix A. Example

In the case of a quadratic advertising cost,

$$C(a_l) = \frac{\kappa}{2}a_l^2 , \tag{A.1}$$

with $\kappa > 0$, which is a customary assumption (see e.g. [13, p. 103]), we have that the inverse function of the marginal cost is $A(z) = z/\kappa$ and the licensee's optimal advertising effort (16) is *proportional* to the adjoint function

$$a_l^*(t;r) = \frac{\eta_l}{\kappa}\lambda(t) . \tag{A.2}$$

Let us assume further that the licensor's advertising effort is constant

$$a_L(t) \equiv a_L , \tag{A.3}$$

so that the decay parameter (10) reduces to a constant too, $\Delta(t) = \Delta = \delta - \varepsilon_l a_L$.

For simplicity of exposition, we restrict the attention to the case $\Delta > 0$, but the qualitative results in the opposite case are similar. Assuming $\Delta > 0$, i.e. $\varepsilon_l a_L < \delta$, means that the synergy effect of the licensor advertising effort towards the licensee is relatively small. From (15) and (16) we obtain the licensee's optimal advertising effort

$$a_l^*(t;r) = \eta_l \frac{(1-r)p - c}{\kappa\Delta}\left(1 - e^{-\Delta(T-t)}\right) . \tag{A.4}$$

The associated optimal goodwill function is

$$G_l^*(t;r) = e^{-\Delta t}G_l^0 + \eta_l^2\frac{(1-r)p - c}{\kappa\Delta^2}\left(1 - e^{-\Delta t}\left(1 - \frac{e^{-\Delta T} - e^{-\Delta(T-2t)}}{2}\right)\right), \tag{A.5}$$

with sales

$$S^*(T;r) = \frac{\kappa G_l^0 D + \eta_l^2((1-r)p - c)E}{2\kappa\Delta^3} , \tag{A.6}$$

where

$$D = 2\Delta^2\left(1 - e^{-\Delta T}\right) , \qquad E = -3 + 2\Delta T + 4e^{-\Delta T} - e^{-2\Delta T} . \tag{A.7}$$

In order to determine the bounds $\overline{\sigma}_L$ and $\underline{\sigma}_L$, as in (39) and (40), we need to know the first derivatives of total sales $S^*(T; r)$ and goodwill $G_L(T; a_l^*(t; r))$ w.r.t. r. From the equations (29), (23), (25) and (22), we obtain

$$\frac{\partial S^*(T; r)}{\partial r} = -\frac{\eta_l^2 pE}{2\kappa\Delta^3} < 0, \tag{A.8}$$

and

$$\frac{\partial}{\partial r} G_L(T; a_l^*(t; r)) = -\frac{p}{2\kappa\Delta^3} < 0, \tag{A.9}$$

where

$$M = 2\Delta^2 \varepsilon_L \eta_l \frac{\Delta - (\delta + \Delta)e^{-\delta T} + \delta e^{-(\delta+\Delta)T}}{\delta(\delta + \Delta)} > 0. \tag{A.10}$$

We observe that both $\partial S^*(T; r)/\partial r$ and $\partial G_L(T; a_l^*(t; r))/\partial r$ are constant w.r.t. r.

Therefore we have the bounds

$$\overline{\sigma}_L = M^{-1} \left\{ \kappa G_l^0 D + \eta_l^2((p - c)E \right\}, \tag{A.11}$$

and

$$\underline{\sigma}_L = M^{-1} \left\{ \left[\kappa G_l^0 D + \eta_l^2(p - c)E \right] + (1 - c/p)\eta_l^2 pE \right\}, \tag{A.12}$$

with $\underline{\sigma}_L < \overline{\sigma}_L$. Furthermore, from (33), the equilibrium royalty factor is

$$r(\sigma_L) = \frac{\kappa G_l^0 D + \eta_l^2((p - c)E}{2\eta_l^2 pE} - \frac{M}{2\eta_l^2 pE}\sigma_L, \quad \underline{\sigma}_L \leq \sigma_L \leq \overline{\sigma}_L, \tag{A.13}$$

a (decreasing) linear affine function of σ_L.