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SOME NEW MODELS FOR MULTIPROCESSOR INTERCONNECTION NETWORKS

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Abstract: A multiprocessor system can be modeled by a graph G. The vertices of G correspond to processors while edges represent links between processors. To find suitable models for multiprocessor interconnection networks (briefly MINs), one can apply tools and techniques of spectral graph theory. In this paper, we extend some of the existing results and present several graphs which could serve as models for efficient MINs based on the small values of the previously introduced graph tightness. These examples of possible MINs arise as a result of some well-known and widely used graph operations. We also examine the suitability of strongly regular graphs (briefly SRGs) to model MINs, and prove the uniqueness of some of them.

Keywords: Spectra of graphs, Tightness, Interconnection networks, Graph operation.

MSC: 05C50, 68M10.

1. INTRODUCTION

Spectral graph theory is a mathematical theory in which linear algebra and graph theory meet. It is a very well developed mathematical field (see, for example, [7] or [8]), but also an engineering discipline [14]. Here it will be applied to the study of multiprocessor interconnection networks (briefly MINs).

Let G be a simple graph on n vertices, with the adjacency matrix $A = A(G)$. The *characteristic polynomial* $P_G(x) = \det(xI - A)$ of G is the characteristic polynomial of its adjacency matrix A. The eigenvalues of A, in non-increasing order, are denoted by $\lambda_1(G), \ldots, \lambda_n(G)$ and they form the *spectrum* of G. The multiplicity k of the eigenvalue λ_i in the spectrum of G will be denoted by $[\lambda_i]^k$. Since A is real and symmetric, the spectrum of G consists of reals. In particular, $\lambda_1(G)$, as the largest eigenvalue of G, is called the *index* of G, and it presents one of the graph invariants.

The *graph invariant* is a function of a graph G which does not depend on labeling of G's vertices or edges. The other examples of graph invariants are the minimum δ and the maximum Δ vertex degree, the diameter, the radius, the average distance, the independence number, the chromatic number, etc. In order to study the behavior of a property or invariant of graphs when the number of vertices varies, it is important that the property (invariant) is scalable. *Scalability* means that for each n there exists a graph with n vertices having that property (invariant) of certain value. A family of graphs is called *scalable* if for any n there exists an n-vertex graph in this family.

A path on n vertices is denoted by P_n. An n-vertex *cycle* is denoted by C_n, while for a *complete graph* with n vertices, in which any two vertices are connected by an edge, we use the label K_n. The set of vertices of the *complete bipartite graph* K_{n_1,n_2} is divided into two disjoint subsets of sizes n_1 and n_2 such that the edges are connecting each vertex from one subset to all vertices in the other subset. A complete multipartite graph with k parts and n_i $(1 \leq i \leq k)$ vertices in each part is denoted by $K_{n_1,n_2,\ldots,n_k}$. If $n_1 = n_2 = \cdots = n_k = m$, we use the short expression $K_{k\times m}$, while a graph of the form $K_{1,n}$ we call a *star*.

The *disjoint union* of graphs $G_1, G_2, \ldots, G_n$ is denoted by $G_1 \dot{+} G_2 \dot{+} \ldots \dot{+} G_n$, while in the case when $G_1 = G_2 = \ldots = G_n = G$, we say that the resulting graph is consisting of n copies of G, and we write $n\,G$. The *join* $G_1 \nabla G_2$ of disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \dot{+} G_2$ by joining each vertex of G_1 to each vertex of G_2. The graph $K_1 \nabla G_2$ is called the *cone* over G_2. The *coalescence* $G_1 \cdot G_2$ of two graphs G_1 and G_2, is the graph that is obtained from $G_1 \dot{+} G_2$ by identifying a vertex v of G_1 with a vertex u of G_2. The graph $G_1 vu G_2$ that is obtained from $G_1 \dot{+} G_2$ by adding an edge joining the vertex v of G_1 to the vertex u of G_2 is *a graph with a bridge*. If G_1 is a graph with n_1 vertices and G_2 is a graph with n_2 vertices, then the *corona* $G_1 \circ G_2$ is the graph with $n_1 + n_1\, n_2$ vertices obtained from G_1 and n_1 copies of G_2 by joining the i-th vertex of G_1 to each vertex in the i-th copy of G_2 $(i = 1, 2, \ldots, n_1)$. The *non-complete extended p-sum*, briefly NEPS, of graphs is a very general graph operation, and we give the definition as it is done in [8], p.44:

Definition 1.1. *Let B be a set of non-zero binary n-tuples, i.e. $B \subseteq \{0,1\}^n / \{(0,\ldots,0)\}$. The* NEPS *of graphs $G_1, \ldots, G_n$ with basis B is the graph with vertex set $V(G_1) \times \cdots \times V(G_n)$, in which two vertices, say $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, are adjacent if and only if there exists an n-tuple $\beta = (\beta_1, \ldots, \beta_n) \in B$ such that $x_i = y_i$ whenever $\beta_i = 0$, and x_i is* *adjacent to* y_i *(in* G_i*) whenever* $\beta_i = 1$.

In the special case, when $n = 2$, we have the following familiar operations:

(i) the *sum* $G_1 + G_2$, when $B = \{(0,1),(1,0)\}$;

(ii) the *product* $G_1 \times G_2$, when $B = \{(1,1)\}$;

(iii) the *strong product* $G_1 * G_2$, when $B = \{(0,1),(1,0),(1,1)\}$.

For the remaining notation and terminology we refer the reader to [7] and [8].

The paper is organized as follows. In Section 2 we present previously defined quantities and some of the previously obtained results related to the consideration of MINs by means of graph spectra. Some examples of widely used MINs are also given. In Section 3 we describe several graphs that could serve as models for efficient MINs. These examples of possible MINs arise as a result of some well-known and widely used graph operations. In Section 4 we examine the suitability of strongly regular graphs (briefly SRGs) to model MINs, and prove the uniqueness of some of them.

2. PRELIMINARY RESULTS

The graph invariant obtained as the product of the number of distinct eigenvalues m and the maximum vertex degree Δ of G has been investigated in [11] related to the design of multiprocessor topologies. The main conclusion of [11] with respect to the multiprocessor design and, particulary to the load balancing within given multiprocessor systems was the following: if $m\Delta$ is small for a given graph G, the corresponding multiprocessor topology was expected to have good communication properties and has been called *well-suited*. It has also been pointed out that there exists an efficient algorithm which provides optimal load balancing within $m - 1$ computational steps. The graphs with large $m\Delta$ were called *ill-suited* and were not considered suitable for the design of multiprocessor networks.

On the other hand there are many known widely used multiprocessor topologies based on graphs which appear to be ill-suited, according to [11]. Some of the examples are star graphs, 1-dimensional processor arrays, 2-dimensional processor arrays (also known as mesh architectures), and processor rings (cycles).

In order to extend the application of the theory of graph spectra to the design of multiprocessor topologies, in [2, 3] some other similar graph invariants called *tightness* were defined and their suitability for describing the corresponding interconnection networks was investigated in [4, 6].

2.1. Basic definitions and properties of graph tightness

There are four types of graph tightness, two mixed and two homogeneous. Mixed tightness depends on one spectral and one structural invariant, while homogenous tightness is a product of two invariants of the same type.

Definition 2.1. *The* type one mixed tightness t_1 *of a graph G is defined as the product of the number of distinct eigenvalues m and the maximum vertex degree* Δ *of G, i.e.* $t_1(G) = m\Delta$.

Definition 2.2. *The* structural tightness $stt(G)$ *is product* $(D+1)\Delta$, *where D is diameter and* Δ *is the maximum vertex degree of a graph G.*

Definition 2.3. *The* spectral tightness $spt(G)$ *is product of the number of distinct eigenvalues m and the largest eigenvalue* λ_1 *of a graph G.*

Definition 2.4. *The* second type mixed tightness $t_2(G)$ *is defined as a product of the diameter D of G and the largest eigenvalue* λ_1, *i.e.* $t_2(G) = (D + 1)\lambda_1$.

It is easy to see [2] that there is a partial order between various types of tightness. The value for $t_1(G)$ is always greater than or equal to any other tightness. The two homogeneous tightness appear to be incomparable, while for $t_2(G)$ holds $t_2(G) \leq spt(G)$ and $t_2(G) \leq stt(G)$.

In [2, 4, 5] it was suggested that $t_2(G)$ is more appropriate parameter for selecting well-suited interconnection topologies than $t_1(G)$. The star graph and some other examples suggest that the classification based on the tightness t_2 seems to be more adequate.

The obvious conclusion is that the well-suited interconnection network according to the value for t_1 remain well-suited also when t_2 is taken into consideration. In this way, some new graphs become suitable for modelling multiprocessor interconnection networks. Some of these ”new” types of graphs are already recognized by multiprocessor system designers (like stars and bipartite graphs). A new family of suitable graphs is described in [5]; it is the family of quasi-regular trees.

The defined graph invariants can be applied to the investigation of MINs in three ways. The tightness values of widely used multiprocessor architectures should be calculated in order to confirm/refute that they are well-suited. The second possibility is to propose new MINs that are well-suited according to t_2. Finally, the theoretically challenging task is to search for extremal graphs according to tightness values in some (scalable) classes of graphs. Here, we first analyze some frequently used MINs, and then we propose some new well-suited graphs for modelling MINs.

We first present a theorem and its corollaries from [2] that seem to be of fundamental importance in studying the tightness of a graph.

Theorem 2.5. *For any kind of tightness, the number of connected graphs with a bounded tightness is finite.*

Corollary 1. *The tightness of graphs in any scalable family of graphs is unbounded.*
Corollary 2. *Any scalable family of graphs contains a sequence of graphs, not necessarily scalable, with increasing tightness diverging to* $+\infty$.

Having these facts in mind, the asymptotic behavior of the tightness, when n tends to $+\infty$, is of particular interest in the analysis of MINs. Typically, in suitable (scalable) families of graphs the tightness values have asymptotic behavior, for example, $O(\log n)$ or $O(\sqrt{n})$ [4]. Several cases are studied in [2] and reviewed also here in the rest of this section.

2.2. Examples of widely used interconnection networks

Here we present some graphs that are often used to model MINs and compare the corresponding tightness values.

1. The *d-dimensional hypercube* $Q(d)$. It consists of $n = 2^d$ vertices, each of them connected to d neighbors. For these graphs it holds $m = d + 1,\ D = d,\ \Delta = d,\ \lambda_1 = d$ and therefore

$$t_1(Q(d)) = stt(Q(d)) = spt(Q(d)) = t_2(Q(d)) = (d + 1)d = O((\log n)^2).$$

2. The *butterfly* graph $B(k)$ is containing $n = 2^k(k + 1)$ vertices. The vertices of this graph are organized in $k+1$ levels (columns) each containing 2^k vertices. In each column, vertices are labelled in the same way (from 0 to $2^k - 1$). An edge is connecting two vertices if and only if they are in the consecutive columns i and $i + 1$ and their numbers are the same or they differ only in the bit at the i-th position. For these graphs we have $\Delta = 4$, $D = 2k$, $\lambda_1 = 4\cos(\pi/(k + 1))$ [11]. However, it is not obvious how to determine parameter m. Therefore,

$$t_1(B(k)) \geq stt(B(k)) = 4(2k + 1) = O(\log n)$$

and

$$spt(B(k)) \geq t_2(B(k)) = 4(2k + 1)\cos(\pi/(k + 1)) = O(k) = O(\log n).$$

3. The *d-ary de Bruijn directed graph* $BD(d, n)$, is a graph on $N = d^n$ vertices labelled with all n-tuples over the alphabet $\{0, 1, \ldots, d - 1\}$, such that there is a directed edge from a vertex $(a_1, a_2, \ldots, a_n)$ to a vertex $(b_1, b_2, \ldots, b_n)$, whenever $b_i = a_{i+1}$ for all i in the range $1 \leq i \leq n - 1$ [10]. By replacing each directed edge by an undirected edge, one can obtain the undirected de Bruijn graph $B(d, n)$. Graph $B(d, n)$ contains loops and double edges and it is regular of degree $2d$. More precisely this $B(d, n)$ has exactly d loops (one for each vertex labelled with $(\alpha, \alpha, \ldots, \alpha)$), and there are $d(d - 1)/2$ double edges between the vertices labelled with $(\alpha, \beta, \alpha, \beta, \ldots)$ and $(\beta, \alpha, \beta, \alpha, \ldots)$ (with $\alpha \neq \beta$). For MINs, loops and double edges are not relevant and they can be easily deleted from $B(d, n)$. The resulting graph will have smaller λ_1 value. The original graph $B(d, n)$ has $\lambda_1 = 2d$, $\Delta = 2d$, $D = n = \log N$ [15]. Therefore,

$$t_1(B(d, n)) \geq stt(B(d, n)) = 2d(n + 1)$$

and

$$spt(B(d, n)) \geq t_2(B(d, n)) = 2d(n + 1).$$

Consecutively, $stt(B(d,n))$, $t_2(B(d,n)) = O(\log N)$, where N represents the number of vertices.

4. The *stars* $S_n = K_{1,n-1}$ have the following invariants $m = 3$, $\Delta = n-1$, $D = 2$, $\lambda_1 = \sqrt{n-1}$, and therefore,

$$t_1(S_n) = 3(n-1), \quad stt(S_n) = 3(n-1), \quad spt(S_n) = 3\sqrt{n-1}, \quad t_2(S_n) = 3\sqrt{n-1}.$$

5. The *cycles* C_n are used to model processor ring interconnection networks. Their characteristics are $m = \lfloor n/2 \rfloor + 1, \Delta = 2, D = \lfloor n/2 \rfloor, \lambda_1 = 2$. Therefore,

$$t_1(C_n) = stt(C_n) = stt(C_n) = t_2(C_n) = 2(\lfloor n/2 \rfloor + 1).$$

6. For K_{n_1,n_2} we have $m = 3$, $\Delta = \max\{n_1, n_2\}$, $D = 2$, $\lambda_1 = \sqrt{n_1 n_2}$ and hence

$$t_1(K_{n_1,n_2}) = stt(K_{n_1,n_2}) = 3\max\{n_1, n_2\}, \quad spt(K_{n_1,n_2}) = t_2(K_{n_1,n_2}) = 3\sqrt{n_1 n_2}.$$

In the case $n_1 = n_2 = n/2$ all tightness values are of order $O(n)$. However, for the star S_n we have $t_2(S_n) = O(\sqrt{n})$. This may be the indication that complete bipartite graphs are suitable for modelling multiprocessor interconnection networks only in some special cases.

7. *Mesh* (or *greed*) $M_{n_1,n_2} = P_{n_1} + P_{n_2}$ consists of $n = n_1 n_2$ vertices organized within layers. For these graphs $\Delta = 4$, $D = n_1 + n_2 - 2$, $\lambda_1 = 2\cos(\pi/(n_1 + 1)) + 2\cos(\pi/(n_2 + 1))$ [7] and therefore,

$$t_1(M_{n_1,n_2}) \geq stt(M_{n_1,n_2}) = 4(n_1 + n_2 - 1)$$

and

$$spt(M_{n_1,n_2}) \geq t_2(M_{n_1,n_2}) = (n_1 + n_2 - 1)(2\cos(\pi/(n_1 + 1)) + 2\cos(\pi/(n_2 + 1))).$$

Hence, $t_2 = O(\sqrt{n})$ if $n_1 \approx n_2$.

8. *Torus* $T_{n_1,n_2} = C_{n_1} + C_{n_2}$ is obtained if mesh architecture is closed among both dimensions. The characteristics of a torus are $\Delta = 4$, $D = [n_1/2] + [n_2/2]$, $\lambda_1 = 2\cos(2\pi/n_1) + 2\cos(2\pi/n_2)$ [7], and thus

$$t_1(T_{n_1,n_2}) \geq stt(T_{n_1,n_2}) = 4([n_1/2] + [n_2/2] + 1)$$

and

$$spt(T_{n_1,n_2}) \geq t_2(T_{n_1,n_2}) = ([n_1/2] + [n_2/2] + 1)(2\cos(2\pi/n_1) + 2\cos(2\pi/n_2)).$$

As in the previous case, we have $t_2 = O(\sqrt{n})$ if $n_1 \approx n_2$.

Simple graphs with relatively good characteristics were used in practice to model MINs based on intuitive knowledge of their creators. For example, cycles, meshes, toruses, arrays may be considered ill-suited even according to t_2, however, in some cases they perform very well in practice [12]. In the rest of this paper we propose new graphs that should be considered as models for MINs with the expectations to show good performance with respect to efficiency of communication and/or fault tolerance.

3. A FAMILY OF POSSIBLE INTERCONNECTION NETWORKS

In this section we propose several new families of t_2-based well-suited graphs. We give six examples of graphs with the good tightness values. Since the values of *spt* and t_2 of all of these graphs are at most $O(\sqrt{N})$, where N is the number of vertices of a graph under consideration, we propose them as models for MINs. Although star graphs are previously [11] classified to be ill-suited, from the exposed examples it appears that the topologies obtained as the result of some graph operations with the star graph as one of the operands, are well-suited. All of the Figures given in the subsequent examples are obtained by using *AutoGraphiX* system, briefly AGX (see, for example, [1]).

Example 3.1. *Let us consider the graph G_1 that is the join of an isolated vertex and $p \geq 1$ copies of complete graph with $n \geq 1$ vertices, i.e. $G_1 = K_1 \nabla pK_n$. With another words, G_1 is a cone over pK_n. The example of G_1 for $p = 4$ and $n = 4$ is given on Figure 1.*

The characteristic polynomials of the two graphs that form G_1 are:

$$P_{K_1}(x) = x \quad \text{and} \quad P_{pK_n}(x) = (x - n + 1)^p \, (x + 1)^{p(n-1)},$$

so according to Theorem 2.1.8 from [8], one finds that the characteristic polynomial of G_1 is:

$$P_{G_1}(x) = (x - n + 1)^{p-1} \, (x + 1)^{p(n-1)} \, (x^2 - (n - 1)x - np),$$

and that the spectrum of G_1 consists of the following eigenvalues $[n - 1]^{p-1}$, $[-1]^{p(n-1)}$, $\frac{1}{2}(n - 1) + \frac{1}{2}\sqrt{(n - 1)^2 + 4np}$, and $\frac{1}{2}(n - 1) - \frac{1}{2}\sqrt{(n - 1)^2 + 4np}$.

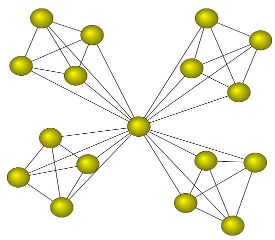

Figure 1: Graph G_1 for $p = 4$ and $n = 4$

The largest eigenvalue of G_1 is $\lambda_1 = \frac{1}{2}(n - 1) + \frac{1}{2}\sqrt{(n - 1)^2 + 4np}$, the number of distinct eigenvalues is $m \leq 4$, the diameter is $D \leq 2$ and the maximum vertex degree is

$\Delta = np$, so one can calculate:

$$t_1(G_1) \leq 4np;$$
$$stt(G_1) \leq 3np;$$
$$spt(G_1) \leq 4\lambda_1 = 2(n-1) + 2\sqrt{(n-1)^2 + 4np};$$
$$t_2(G_1) \leq 3\lambda_1 = \frac{3}{2}(n-1) + \frac{3}{2}\sqrt{(n-1)^2 + 4np}.$$

If we denote with $N = np + 1$ the number of vertices of G_1, then for $n = p$ we find $t_1(G_1) = O(N)$, $stt(G_1) = O(N)$ and $spt(G_1) = O(\sqrt{N})$, $t_2(G_1) = O(\sqrt{N})$, so it seems that G_1, in certain cases, can be used as a model for MINs.

Example 3.2. *In this example we analyze the graph G_2 that is the result of corona of a star $K_{1,n-1}$ with $n > 1$ vertices and a complete graph K_p with $p \geq 1$ vertices, i.e. $G_2 = K_{1,n-1} \circ K_p$. Graph G_2 for $n = 5$ and $p = 4$ is depicted on Figure 2.*

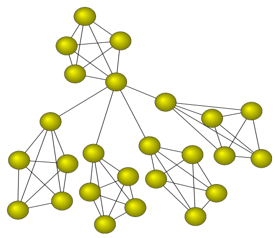

Figure 2: Graph G_2 for $n = 5$ and $p = 4$

The number of vertices of G_2 is equal to $N = n + np$. Since the characteristic polynomials of $K_{1,n-1}$ and K_p are

$$P_{K_{1,n-1}}(x) = (x^2 - (n-1))\, x^{n-2} \quad \text{and} \quad P_{K_p}(x) = (x - p + 1)(x+1)^{p-1}, \tag{1}$$

respectively, then, according to Theorem 2.2.5 from [8], we find that the characteristic polynomial of G_2 is:

$$P_{G_2}(x) = (x+1)^{n(p-1)} \times (x - (p-1))^n \times \left(x - \frac{p}{x-p+1}\right)^{n-2} \times \\ \times \left(1 - n + \left(x - \frac{p}{x-p+1}\right)^2\right). \tag{2}$$

Transforming relation (2) for $x \neq p - 1$ we obtain:

$$P_{G_2}(x) = (x+1)^{n(p-1)} \cdot (x^2 + (1-p)x - p)^{n-2} \cdot Q(x),$$

where

$$Q(x) = x^4 - 2(p-1)x^3 + (p^2 - 4p + 2 - n)x^2 + (2p^2 - 4p + 2pn - 2n + 2)x + p^2 - (n-1)(p-1)^2,$$

and wherefrom we conclude that the number of distinct eigenvalues m of G_2 cannot be greater then 7. By using the software package named Mathematica *we found that $m \leq 6$ and that the index of G_2 is equal to:*

$$\lambda_1(G_2) = \frac{\sqrt{n-1}}{2} + \frac{p-1}{2} + \frac{1}{2}\sqrt{\frac{-2+2n+2p-2np}{\sqrt{n-1}} + n + 2p + p^2},$$

which means that if, for example, $n = p$, we have $\lambda_1 = O(\sqrt{N})$. The maximum vertex degree of G_2 is $\Delta = n - 1 + p$, while its diameter is $D \leq 4$. So, for $n = p$ we have: $t_1(G_2) = O(\sqrt{N})$, $stt(G_2) = O(\sqrt{N})$, $spt(G_2) = O(\sqrt{N})$ and $t_2(G_2) = O(\sqrt{N})$.

Example 3.3. *We consider the graph G_3 that is the result of NEPS with the basis $\mathfrak{B} = \{(0,1),(1,0)\}$ of a star $K_{1,n-1}$ with $n > 1$ vertices, and a complete graph K_p with $p \geq 1$ vertices, i.e. $G = K_{1,n-1} + K_p$. This graph for $n = 3$ and $p = 4$ is depicted on Figure 3.*

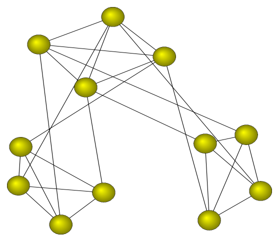

Figure 3: Graph G_3 for $n = 3$ and $p = 4$

Graph G_3 has $N = n \cdot p$ vertices. The diameter of such a graph is $D \leq 3$, while the maximum vertex degree is $\Delta = n + p - 2$. Since the spectrum of $K_{1,n-1}$ is $\sqrt{n-1}, [0]^{n-2}, -\sqrt{n-1}$, and the spectrum of K_p is $p-1, [-1]^{p-1}$, from Theorem 2.5.4 from [8] one can find that the index of G_3 is $\lambda_1 = \sqrt{n-1} + p - 1$, while the number of distinct eigenvalues is $m \leq 6$. Now, for $n = p$ it is easy to check that:

$$\begin{aligned} t_1(G_3) &\leq 6(2n-2) = O(\sqrt{N}); \\ stt(G_3) &\leq 4(2n-2) = O(\sqrt{N}); \\ spt(G_3) &\leq 6(\sqrt{n-1} + n - 1) = O(\sqrt{N}); \\ t_2(G_3) &\leq 4(\sqrt{n-1} + n - 1) = O(\sqrt{N}). \end{aligned}$$

Example 3.4. *Now, we analyze the tightness values of the graph G_4 that is the result of NEPS with the basis $\mathfrak{B} = \{(0,1),(1,0)\}$ of two stars, $K_{1,n}$ and $K_{1,p}$, $n, p \geq 1$. Graph G_4 for $n = 3$ and $p = 4$ is depicted on Figure 4.*

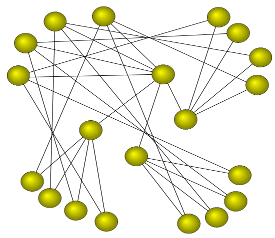

Figure 4: Graph G_4 for $n = 3$ and $p = 4$

Graph G_4 has $N = np + n + p + 1$ vertices. We have that the spectrum of $K_{1,n}$ is $\sqrt{n}, [0]^{n-1}, -\sqrt{n}$, so by applying Theorem 2.5.4 from [8], we get that the index of G_4 is $\lambda_1 = \sqrt{n} + \sqrt{p}$, while the number of distinct eigenvalues is $m \leq 9$. The maximum vertex degree of G_4 is $\Delta = n + p$, and its diameter is $D \leq 4$. So, for $n = p$ it follows:

$$\begin{aligned} t_1(G_4) &\leq 18\,n = O(\sqrt{N}); \\ stt(G_4) &\leq 10\,n = O(\sqrt{N}); \\ spt(G_4) &\leq 18\,\sqrt{n} = O(\sqrt[4]{N}); \\ t_2(G_4) &\leq 10\,\sqrt{n} = O(\sqrt[4]{N}). \end{aligned}$$

Example 3.5. *Let us consider the graph with a bridge G_5, that is obtained from $K_{1,n} \dot{+} K_{1,p}$, $n, p \geq 1$ by adding an edge joining the vertex of the maximum degree in $K_{1,n}$ to the vertex of the maximum degree in $K_{1,p}$. Graph G_5 for $n = 5$ and $p = 4$ is depicted on Figure 5.*

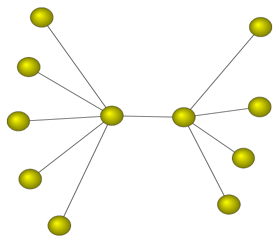

Figure 5: Graph G_5 for $n = 5$ and $p = 4$

This graph has $N = n + p + 2$ vertices. Since the characteristic polynomial of a star is known (see relation (1)), then the characteristic polynomial of G_5, according to Theorem 2.2.4 from [8], is:

$$P_{G_5}(x) = x^{n+p-2}\,(x^4 - (n+p+1)x^2 + np).$$

Therefore, the number of its distinct eigenvalues is $m \leq 5$, while its index is equal to

$$\lambda_1 = \sqrt{\frac{n+p+1+\sqrt{(n+p+1)^2 - 4np}}{2}}.$$

The diameter of G_5 is $D = 3$, while its maximum vertex degree is $\Delta = \max\{n, p\} + 1$. Now, it is easy to verify that for $n = p$, the tightness values of this graph are:

$$t_1(G_5) \leq 5(n+1) = O(N);$$
$$stt(G_5) = 4(n+1) = O(N);$$
$$spt(G_5) \leq 5\lambda_1 = O(\sqrt{N});$$
$$t_2(G_5) = 4\lambda_1 = O(\sqrt{N}).$$

Example 3.6. *Graph G_6 is formed from $(nK_2\nabla K_1)+K_{1,p}$ by identifying the vertex of the maximum degree in the graph $nK_2\nabla K_1$, $n \geq 1$ with the vertex of the maximum degree in the star graph $K_{1,p}$, $p \geq 1$, i.e. $G_6 = (nK_2\nabla K_1) \cdot K_{1,p}$ is one kind of the coalescence of the two mentioned graphs. Graph G_6 for $n = 4$ and $p = 3$ is depicted on Figure 6.*

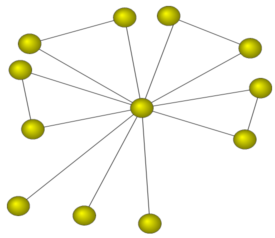

Figure 6: Graph G_6 for $n = 4$ and $p = 3$

The number of vertices of G_6 is $N = 2n + p + 1$. Since the characteristic polynomial of a star graph is known (see relation (1)), and the characteristic polynomial of graph $nK_2\nabla K_1$ can be obtained by using Theorem 2.1.8 from [8], and is of the following form:

$$P_{nK_2\nabla K_1}(x) = (x-1)^{n-1}(x+1)^n(x^2 - x - 2n),$$

according to Theorem 2.2.3 from [8] we find that the characteristic polynomial of G_6 is:

$$P_{G_6}(x) = (x-1)^{n-1}(x+1)^n x^{p-1}(x^3 - x^2 - (2n+p)x + p).$$

Therefore the number of its distinct eigenvalues is $m \leq 6$. The adjacency matrix $A(G_6)$ of G_6 is of the following form:

$$A(G_6) = A(K_{1,p} \dot{+} (2n)K_1) + A((nK_2\nabla K_1) \dot{+} pK_1),$$

where $A(G)$ is the adjacency matrix of a graph G. So, by use of Courant-Weyl inequalities (see Theorem 1.3.15 from [8]) one can find that

$$\lambda_1(G_6) \leq \sqrt{p} + \frac{1}{2}(1 + \sqrt{1+8n}).$$

The diameter of G_6 is $D = 2$, while the maximum vertex degree is $\Delta = 2n + p$. Now, it is a matter of routine to check that for $n = p$ it holds $t_1(G_6) = O(N)$, $stt(G_6) = O(N)$,

$spt(G_6) = O(\sqrt{N})$ *and* $t_2(G_6) = O(\sqrt{N})$.

4. STRONGLY REGULAR GRAPHS

A strongly regular graph with parameters (n, r, λ, μ), i.e. $SRG(n, r, \lambda, \mu)$ for short, is an r-regular graph on n vertices in which any two adjacent vertices have exactly λ common neighbours and any two non-adjacent vertices have exactly μ common neighbours. The largest eigenvalue λ_1 is equal to the degree r of the graph. A strongly regular graph G has diameter 2, and it is called *primitive* if both G and its complement $\overline{G}$ are connected. For a sketch of the theory of strongly regular graphs see, for example, [7], Section 7.2 or [8], Section 3.6.

It is difficult to construct scalable families of SRGs which would be of interest to the context of this paper. However, there are sporadic examples of SRGs which are suitable for MINs.

In strongly regular graphs all four types of tightness are mutually equal with the common value $3r$. It seems to be reasonable for a given tightness (i.e. for the given degree r) to look for graphs with as many vertices as possible.

Looking at a vertex of a strongly regular graph, there are exactly r vertices at distance 1 and at most $r(r-1)$ vertices at distance 2. Hence, for the number n of vertices we have

$$n \leq 1 + r + r(r-1) = 1 + r^2. \tag{3}$$

Strongly regular graphs that attain this bound are known as *Moore graphs* of diameter 2. Such graphs exist for $r = 2, 3, 7$ and possibly for $r = 57$ (cf. [7], p. 165). For $r = 2$ we have the pentagon C_5 (5 vertices) and for $r = 3$ the Petersen graph (10 vertices). If $r = 7$ there is a unique graph known as the Hoffman-Singleton graph. It has 50 vertices. It is not known whether the Moore graph of degree $r = 57$ exists. It would have $1 + 57^2 = 3250$ vertices.

The Petersen graph has been considered as interconnection network in the literature. It has also the property that all its eigenvalues are integral. Such graphs are called *integral graphs*. It is claimed in [6] that integral graphs are suitable as network models since the load balancing in such networks can be performed in integer arithmetics.

Concerning strongly regular graphs which do not attain the bound (3), we point out to the case $n = 16, r = 5$. In fact we have the complement of a strongly regular graph know as the *Clebsh graph*. The Clebsh graph is described, for example, in [9], p. 9 and Table A3. It has degree 10 and eigenvalues $10, [2]^5, [-2]^{10}$ (multiplicities of eigenvalues being presented as exponents). Let us denote the complement of the Clebsh graph by CC. It has degree 5 and eigenvalues $5, [1]^{10}, [-3]^5$, hence it is integral.

The graph CC can be described as a modified 4-cube. In a k-cube for each vertex there is a unique vertex at distance k called *antipode*. If we connect in a 4-cube each vertex to its antipode by additional edges, we get the graph CC.

We shall prove that CC is the unique strongly regular graph of degree $r = 5$. In the proof of the uniqueness of CC we are looking for all possible nonnegative solutions of the corresponding Diophantine equations obtained from the following proposition:

Proposition 4.1. (relation (3.14), p.72 from [8]) *The parameters* (n, r, λ, μ) *of a strongly regular graph satisfy the equation:*

$$r(r - \lambda - 1) = (n - r - 1)\mu.$$

We also manage some conclusions by use of the following well-known facts about strongly regular graphs:

Theorem 4.2. (Theorem 3.6.5 from [8]) *The distinct eigenvalues of a connected strongly regular graph with parameters* (n, r, λ, μ) *are* r, s *and* t *where* $s, t = \frac{1}{2}(\lambda - \mu) \pm \sqrt{\omega}$ *and* $\omega = (\lambda - \mu)^2 + 4(r - \mu)$. *Their respective multiplicities are* 1, f *and* g, *where*

$$f, g = \frac{1}{2}\left\{n - 1 \mp \frac{2r + (n-1)(\lambda - \mu)}{\sqrt{\omega}}\right\}.$$

As a consequence of the statement given in Theorem 4.2, it follows that parameters of a strongly regular graph must be such that f and g are positive integers.

Theorem 4.3. (Theorem 3.6.7 from [8]) *Let G be a primitive strongly regular graph on n vertices with eigenvalue multiplicities* $1, f, g$. *Then*

$$n \le \min\left\{\frac{1}{2}f(f+3), \frac{1}{2}g(g+3)\right\}.$$

The bound for n given by Theorem 4.3 is known as the *absolute bound* for strongly regular graphs. The following statement is also well-known:

Proposition 4.4. *A strongly regular graph is disconnected if and only if it is isomorphic to* pK_l *(i.e. the disjoint union of p copies of K_l) for some positive integers p and l. This occurs if and only if* $\mu = 0$.

Propositions 4.5 and 4.6 follow from the existing databases of small strongly regular graphs. Here we offer theoretical self-contained proofs. First, we have the following statement:

Proposition 4.5. *The graph CC is the unique strongly regular graph of degree* $r = 5$.

Proof. Let us suppose that $G = SRG(n, 5, \lambda, \mu)$ is a connected strongly regular graph of degree $r = 5$. According to (3), we find $6 \le n \le 26$. We will consider $G = SRG(n, 5, \lambda, \mu)$ for all possible values of n.

If $G = SRG(6, 5, \lambda, \mu)$, then G is the complete graph with 6 vertices, and thus its spectrum consists of just two distinct eigenvalues, which means that G is not strongly regular graph.

If $G = SRG(7, 5, \lambda, \mu)$, then the equality of Proposition 4.1 yields the following Diophantine equation $5\lambda + \mu = 20$. All possible nonnegative solutions of this equation are $(\lambda, \mu) \in \{(0, 20), (1, 15), (2, 10), (3, 5), (4, 0)\}$. Since G is a graph with 7 vertices, we need to consider only the cases when $(\lambda, \mu) = (3, 5)$ and $(\lambda, \mu) = (4, 0)$. For $G = SRG(7, 5, 3, 5)$ we calculate that the multiplicities of the corresponding eigenvalues are $\frac{1}{2}(6 \pm 1)$, which is in the contradiction with the fact that they should be integers. If $G = SRG(7, 5, 4, 0)$, then according to Proposition 4.4, G is disconnected, which is in the contradiction with our assumption.

We can analyze all the remaining cases in the similar way. Namely, for $G = SRG(8, 5, \lambda, \mu)$ the corresponding Diophantine equation is: $5\lambda + 2\mu = 20$, and its nonnegative solutions are $(\lambda, \mu) \in \{(4, 0), (2, 5), (0, 10)\}$. For $(\lambda, \mu) = (4, 0)$, G is disconnected. For $(\lambda, \mu) = (2, 5)$ (i.e. $(\lambda, \mu) = (0, 10)$) the values of the corresponding eigenvalue multiplicities are not integers, i.e. $\frac{1}{2}\left(\frac{21}{3} \pm \frac{11}{3}\right) \left(\frac{1}{2}\left(7 \pm \frac{60}{4\sqrt{5}}\right)\right)$.

If $G = SRG(9, 5, \lambda, \mu)$, the corresponding Diophantine equation is of the form: $5\lambda + 3\mu = 20$, and its nonnegative solutions are $(\lambda, \mu) = (1, 5)$ and $(\lambda, \mu) = (4, 0)$. In the former case, the values of the corresponding eigenvalue multiplicities are $\frac{32 \pm 22}{8}$, while in the later case G is not connected.

In the case $G = SRG(10, 5, \lambda, \mu)$, the corresponding Diophantine equation is: $5\lambda + 4\mu = 20$. Its nonnegative solutions are $(\lambda, \mu) = (0, 5)$ and $(\lambda, \mu) = (4, 0)$. In the former case, we find that the spectrum of $G = SRG(10, 5, 0, 5)$ is 5, $[0]^8$, -5, and that the *absolute bound* (see Theorem 4.3) is not satisfied i.e. we have $10 = n \le 1 \cdot (1 + 3)/2 = 2$. In the later case G is not connected.

If $G = SRG(11, 5, \lambda, \mu)$, the corresponding Diophantine equation is: $5\lambda + 5\mu = 20$, and its nonnegative solutions are $(\lambda, \mu) \in \{(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)\}$. For $(\lambda, \mu) = (4, 0)$, G is disconnected. It can be checked that in all remaining cases the values of the corresponding eigenvalue multiplicities are not integers. They are equal to: $\frac{1}{2}\left(10 \pm \frac{30}{2\sqrt{5}}\right)$ if $(\lambda, \mu) = (3, 1)$, $\frac{1}{2}\left(10 \pm \frac{10}{2\sqrt{3}}\right)$ if $(\lambda, \mu) = (2, 2)$, $\frac{1}{2}\left(10 \pm \frac{10}{2\sqrt{3}}\right)$ if $(\lambda, \mu) = (1, 3)$ and $\frac{1}{2}\left(10 \pm \frac{30}{2\sqrt{5}}\right)$ if $(\lambda, \mu) = (0, 4)$.

For $n \in \{12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 25\}$ we find that there is only one nonnegative solution $(\lambda, \mu) = (4, 0)$ of the corresponding Diophantine equation: $5\lambda + (n - 6)\mu = 20$. Since $\mu = 0$, according to Proposition 4.4, G is disconnected.

If $G = SRG(16, 5, \lambda, \mu)$ the corresponding Diophantine equation is: $5\lambda + 10\mu = 20$, so the set of its nonnegative solutions is $(\lambda, \mu) \in \{(4, 0), (2, 1), (0, 2)\}$. For $(\lambda, \mu) = (4, 0)$ G is disconnected. For $(\lambda, \mu) = (2, 1)$ the corresponding eigenvalue multiplicities are not integers, i.e. $\frac{1}{2}\left(15 \pm \frac{-25}{\sqrt{17}}\right)$. For $(\lambda, \mu) = (0, 2)$, we have $G = SRG(16, 5, 0, 2) = CC$.

For $G = SRG(21, 5, \lambda, \mu)$ the corresponding Diophantine equation is: $5\lambda + 15\mu = 20$, and its nonnegative solutions are $(\lambda, \mu) = (4, 0)$ and $(\lambda, \mu) = (1, 1)$. In the first case, G is disconnected, while in the second case, the values of the eigenvalue multiplicities are $\frac{40 \pm 5}{4}$.

And, in the last case, if $G = SRG(26, 5, \lambda, \mu)$, the nonnegative solutions of the corresponding Diophantine equation: $5\lambda + 20\mu = 20$ are $(\lambda, \mu) = (4, 0)$ and $(\lambda, \mu) = (0, 1)$. In the former case, G is disconnected, while in the later case the values of the corresponding eigenvalue multiplicities are $\frac{1}{2}\left(25 \pm \frac{15}{\sqrt{17}}\right)$.

From the considered cases, we conclude that only $SRG(16, 5, 0, 2)$ can be a strongly regular graph with the degree equal to 5. One such graph is CC. From the theory of graphs with the least eigenvalue at least -2 (see [9], Theorem 4.3.6) we know that the Clebsh graph has no cospectral mates. Hence, CC also has no cospectral mates. ■

It would be interesting to analyze for CC the load balancing algorithm described in [6].

Recall that $L(G)$ is the *line graph* of a graph G (for details see [7] or [8]). The following statement can be proved in the similar fashion like the previous one:

Proposition 4.6. *The graph $L(K_{3,3})$ is the unique strongly regular graph of degree $r = 4$.*

Proof. Let us suppose that $G = SRG(n, 4, \lambda, \mu)$ is a connected strongly regular graph of degree $r = 4$. According to (3), we find $5 \leq n \leq 17$. We will analyze $G = SRG(n, 4, \lambda, \mu)$ for all possible values of n.

If $G = SRG(5, 4, \lambda, \mu)$, then G is the complete graph with 5 vertices, and thus its spectrum consists of just two distinct eigenvalues, which means that G is not strongly regular graph.

If $G = SRG(6, 4, \lambda, \mu)$, then we have the following Diophantine equation: $4\lambda + \mu = 12$. All possible nonnegative solutions of this equation are $(\lambda, \mu) \in \{(0, 12), (1, 8), (2, 4), (3, 0)\}$. Since G is a graph with 6 vertices, we need to consider only the cases when $(\lambda, \mu) = (2, 4)$ and $(\lambda, \mu) = (3, 0)$. For $G = SRG(6, 4, 2, 4)$ we calculate that the multiplicities of the corresponding eigenvalues are equal to 3 and 2, so the *absolute bound* is not satisfied, i.e. we have $6 \leq 2 \cdot (2 + 3)/2 = 5$. If $G = SRG(6, 4, 3, 0)$, G is disconnected, which is in the contradiction with our assumption.

We can analyze all the remain cases in the similar way. Namely, for $G = SRG(7, 4, \lambda, \mu)$ the corresponding Diophantine equation is: $4\lambda + 2\mu = 12$, i.e. $2\lambda + \mu = 6$, and its nonnegative solutions are $(\lambda, \mu) \in \{(0, 6), (1, 4), (2, 2), (3, 0)\}$. For $(\lambda, \mu) = (3, 0)$, G is disconnected. For $(\lambda, \mu) = (0, 6)$ the corresponding eigenvalue multiplicities are not integers, i.e. they are equal to $3 \pm \sqrt{7}$. The similar situation is in the case when $(\lambda, \mu) = (1, 4)$ and $(\lambda, \mu) = (2, 2)$, when the values of the corresponding eigenvalue multiplicities are $3 \pm \frac{5}{3}$ and $3 \pm \sqrt{2}$, respectively.

In the case when $n \in \{8, 11, 17\}$ we find that the corresponding Diophantine equation $4\lambda + (n - 5)\mu = 12$ has just two nonnegative solutions, one of which is equal to $(\lambda, \mu) = (3, 0)$, which means that $G = SRG(n, 4, \lambda, \mu)$ is disconnected.

The second solution is of the form $(0, n - i^2)$ for $(n, i) \in \{(8,2),(11,3),(17,4)\}$. For $G = SRG(8,4,0,4)$, the values of the corresponding eigenvalue multiplicities are 6 and 1, so the *absolute bound* is not satisfied, i.e. we have $8 \le 1 \cdot (1+3)/2 = 2$. For $G = SRG(11,4,0,2)$ and $G = SRG(17,4,0,1)$ the corresponding eigenvalue multiplicities are not integers, i.e. they are equal to $5 \pm \sqrt{3}$ and $8 \pm \frac{4}{\sqrt{13}}$, respectively.

If $G = SRG(9,4,\lambda,\mu)$, the corresponding Diophantine equation is: $\lambda + \mu = 3$. Its nonnegative solutions are: $(\lambda,\mu) \in \{(3,0),(2,1),(1,2),(0,3)\}$. For $(\lambda,\mu) = (3,0)$, G is disconnected. In the cases $G = SRG(9,4,2,1)$ and $G = SRG(9,4,0,3)$, the values of the corresponding eigenvalue multiplicities are $4 \pm \frac{8}{\sqrt{13}}$. And, for $G = SRG(9,4,1,2)$ we have that $G = L(K_{3,3})$.

For $n \in \{10,12,14,15,16\}$, we find that the corresponding Diophantine equation $4\lambda + (n-5)\mu = 12$ has unique nonnegative solution of the form $(\lambda,\mu) = (3,0)$, which means that G is disconnected.

And finally, in the case when $G = SRG(13,4,\lambda,\mu)$, the corresponding Diophantine equation is: $\lambda + 2\mu = 6$, and its nonnegative solutions are $(\lambda,\mu) \in \{(6,0),(4,1),(2,2),(0,3)\}$. In the case $G = SRG(13,4,6,0)$, G is disconnected. For $G = SRG(13,4,4,1)$, $G = SRG(13,4,2,2)$ and $G = SRG(13,4,0,3)$, the values of the corresponding eigenvalue multiplicities are $6 \pm \frac{22}{\sqrt{21}}$, $6 \pm \sqrt{2}$ and $6 \pm \frac{14}{\sqrt{13}}$, respectively.

From the considered cases, we conclude that only $SRG(9,4,1,2)$ can be a the strongly regular graph with the degree equal to 4. One such graph is $L(K_{3,3})$. From the theory of graphs with the least eigenvalue at least −2 (see [9], Theorem 2.6.2) we know that $L(K_{3,3})$ has no cospectral mates. ■

Note that $L(K_{3,3})$ is self-complementary.

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