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NEW CONSTRUCTION OF MINIMAL $(\mathbf{v}, 3, 2)-$COVERINGS

Nebojša NIKOLIĆ
Faculty of Organizational Sciences, University of Belgrade, Serbia
sigma@fon.bg.ac.rs

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Abstract: A $(v, 3, 2)-$covering is a family of 3-subsets of a v-set, called blocks, such that any two elements of v-set appear in at least one of the blocks. In this paper, we propose new construction of $(v, 3, 2)-$coverings with the minimum number of blocks. This construction represents a generalization of Bose's and Skolem's constructions of Steiner systems $S(2, 3, 6n + 3)$ and $S(2, 3, 6n + 1)$. Unlike the existing constructions, our construction is direct and it uses the set of base blocks and permutation p, so by applying it to the remaining blocks of $(v, 3, 2)-$coverings are obtained.

Keywords: Covering design, Covering number, Steiner system.

MSC: 05B05, 05B07, 05B40.

1. INTRODUCTION

Let v, k, and t denote natural numbers where $v \geq k \geq t$. The family of k-subsets, called blocks, chosen from a v-set, such that each t-subset is contained in at least one of the blocks, is a (v, k, t) *covering design*, or $(v, k, t)-$covering. The number of blocks is the *size of the covering*. The *covering number* $C(v, k, t)$ is the minimum size of a $(v, k, t)-$covering. If each t-subset is contained in exactly one block, $(v, k, t)-$covering is Steiner system $S(t, k, v)$.

Covering designs and Steiner systems have application in statistical test creating, tournament scheduling, cryptography and coding, computer science, lottery systems creating etc.

Covering numbers have already been studied extensively, and numerous papers have been published for particular values of v, k, and t. Nevertheless, exact values of $C(v, k, t)$ are known only if v, k, and t are small, or in some special cases, such as $C(v, 3, 2)$. A large number of papers consider only lower and upper bounds on $C(v, k, t)$.

The best general lower bound on $C(v, k, t)$, according to Schönheim can be derived from the inequality $C(v, k, t) \geq \left\lceil \frac{v}{k} C(v-1, k-1, t-1) \right\rceil$, where $\lceil \cdot \rceil$ represents ceiling function, which iterated $t-1$ time gives the Schönheim bound [17]:

$$C(v, k, t) \geq L(v, k, t) = \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil. \tag{1}$$

Rödl gives the best upper bound [16]: $\lim_{v\to\infty} C(v, k, t) \cdot \binom{k}{t} \Big/ \binom{v}{t} = 1$. Erdős and Spencer give the bound [5]: $C(v, k, t) \cdot \binom{k}{t} \Big/ \binom{v}{t} \leq \left(1 + \ln \binom{k}{t}\right)$. Note that this bound is weaker than the Rödl bound. However, unlike Rödl's asymptotic bound, it can be applied to all v, k, and t.

Most of the best known lower and upper bounds, and exact values can be found at the site [7]. Numerous best known upper bounds can also be found in [4, 8, 12, 13, 14, 15].

Fort and Hedlund have proved that values $C(v, 3, 2)$ reach Schönheim lower bound, ie. for each $v \in \mathbb{N}$; $v \geqslant 3$, it holds [6]

$$C(v, 3, 2) = L(v, 3, 2). \tag{2}$$

Steiner systems $S(2, 3, v)$ exist for $v = 6n + 1$ and $v = 6n + 3$ [2], which implies the equality (2) for mentioned values of parameter v. In each of the remaining four cases, Fort and Hedlund give indirect construction $(v, 3, 2)-$covering with $L(v, 3, 2)$ blocks.

In this paper, we give new construction of the minimal $(v, 3, 2)-$coverings, which consequently proves the equality (2). This construction represents the generalization of the Bose's construction of the Steiner system $S(2, 3, 6n+3)$ [1, 11, 19] and Skolem's construction of the $S(2, 3, 6n+1)$ [11, 18]. Unlike the original Fort and Heldund construction, and the other indirect constructions, our construction belongs to the direct constructions. This construction is simple and do not require the construction of other covering designs such as pairwise balanced design (PBD) or group divisible design(GDD) [9].

We will construct minimal $(v, 3, 2)-$covering for each $v \pmod 6$ separately and present them in the respective subsections. In each of the 6 cases, we will construct $(v, 3, 2)-$coverings with $L(v, 3, 2)$ blocks, where (from (1)):

$$L(v, 3, 2) = \begin{cases} 6n^2 & , \text{ for } v = 6n, \\ 6n^2 + n & , \text{ for } v = 6n+1, \\ 6n^2 + 4n + 1 & , \text{ for } v = 6n+2, \\ 6n^2 + 5n + 1 & , \text{ for } v = 6n+3, \\ 6n^2 + 8n + 3 & , \text{ for } v = 6n+4, \\ 6n^2 + 9n + 4 & , \text{ for } v = 6n+5. \end{cases} \tag{3}$$

2. NEW CONSTRUCTION OF MINIMAL $(v,3,2)-$COVERINGS

During the construction of $(v,3,2)-$coverings, we will use certain permutations of a given set V; $|V| = v$. In the cycle notation, a permutation $p = (a_0\ a_1 \ldots a_{k-1})(b_0\ b_1 \ldots b_{l-1}) \ldots$ $(a_i, b_j \in V)$ represents the mapping $p : V \mapsto V$, defined by $p(a_i) = a_{i+1 \pmod k}$, $p(b_j) = b_{j+1 \pmod l}, \ldots$ The permutation $p^j : V \mapsto V$ is defined by $p^j(a_i) = \underbrace{p(p(\ldots p}_{j}(a_i)\ldots)) = a_{i+j \pmod k}$. For the block $\{p(a), p(b), p(c)\}$, we will say that it is obtained by applying the permutation p to block $\{a, b, c\}$; $a, b, c \in V$. The application of the permutations $p^0 = e, p^1, \ldots, p^{n-1}$ to block $\{a, b, c\}$ we will call applying the permutation p, n times to a block $\{a, b, c\}$. By applying the permutation p, n times to the block $\{a, b, c\}$, blocks $\{a, b, c\}, \{p(a), p(b), p(c)\}, \ldots, \{p^{n-1}(a), p^{n-1}(b), p^{n-1}(c)\}$, respectively, are obtained.

First, we give the known construction of the $(6n + 3, 3, 2)-$covering [3, 10].

2.1. Minimal $(6n + 3, 3, 2)-$covering

Theorem 2.1. *Let $v = 6n+3$ and $V = \{a_0, a_1, \ldots, a_{2n}\} \cup \{b_0, b_1, \ldots, b_{2n}\} \cup \{c_0, c_1, \ldots, c_{2n}\}$. Let $\mathcal{B}$ be the set of blocks obtained by applying the permutation*

$$p = (a_0\ a_1 \ldots a_{2n})(b_0\ b_1 \ldots b_{2n})(c_0\ c_1 \ldots c_{2n}), \tag{4}$$

$2n + 1$ *times to blocks*

$$\begin{array}{llll} \{a_0, b_1, b_{2n}\}, & \{a_0, b_2, b_{2n-1}\}, & \ldots & , \{a_0, b_n, b_{n+1}\}, \\ \{b_0, c_1, c_{2n}\}, & \{b_0, c_2, c_{2n-1}\}, & \ldots & , \{b_0, c_n, c_{n+1}\}, \\ \{c_0, a_1, a_{2n}\}, & \{c_0, a_2, a_{2n-1}\}, & \ldots & , \{c_0, a_n, a_{n+1}\}, \\ \{a_0, b_0, c_0\}. & & & \end{array} \tag{5}$$

Then, $(V, \mathcal{B})$ is one $(v, 3, 2)-$covering with $L(v, 3, 2)$ blocks.

Proof. By applying the permutation p, $2n + 1$ times to an arbitrary block from (5), $2n + 1$ blocks are obtained, which means that $\mathcal{B}$ contains $(3n + 1)(2n + 1) = 6n^2 + 5n + 1 = L(6n + 3, 3, 2)$ different blocks. Let us prove that $(V, \mathcal{B})$ is one $(v, 3, 2)-$covering, ie. that each pair of elements of the set V is contained in some block from $\mathcal{B}$.

Each pair $\{a_0, b_j\}$ $(0 \leqslant j \leqslant 2n)$ is contained in some block from (5): pair $\{a_0, b_0\}$ in the block $\{a_0, b_0, c_0\}$, and pair $\{a_0, b_j\}$ $(j \neq 0)$ in some block from the first row in (5). By applying the permutation p^i $(1 \leqslant i \leqslant 2n)$ to the blocks from (5), element a_0 is mapped into the element $a_i = p^i(a_0)$, while elements $b_0, b_1, \ldots, b_{2n}$ are mapped into $b_i, b_{i+1 \pmod{2n+1}}, \ldots, b_{i-1 \pmod{2n+1}}$, respectively. Hence, each pair $\{a_i, b_j\}$ $(0 \leqslant i, j \leqslant 2n)$ is contained in some block from $\mathcal{B}$. Due to the symmetry, the same holds for all pairs $\{b_i, c_j\}$ and $\{c_i, a_j\}$.

Let us now consider the pairs $\{a_i, a_j\}$. Each pair $\{a_i, a_j\}$, such that $i + j = 2n + 1$, is contained in some block from the third row in (5). For an arbitrary pair $\{a_i, a_j\}$ $(0 \leqslant i < j \leqslant 2n)$, it is sufficient to prove the existence of the pair $\{a_r, a_s\}$ $(0 \leqslant r, s \leqslant 2n,$

$r + s = 2n + 1$) and the permutation p^t ($0 \leqslant t \leqslant 2n$) by which the pair $\{a_r, a_s\}$ is mapped into the pair $\{a_i, a_j\}$, that is, it is sufficient to prove that the system of the equation

$$\begin{cases} r + s = 2n + 1, \\ r + t \pmod{2n+1} = i, \\ s + t \pmod{2n+1} = j, \end{cases}$$

has the solution on r, s and t. If i and j are of the same parity, the solution of the system is

$$r = 2n + 1 - \frac{j-i}{2},\ s = \frac{j-i}{2} \text{ and } t = \frac{i+j}{2},$$

and if i and j are with opposite parity, the solution of the system is

$$r = n - \frac{j-i-1}{2},\ s = n + \frac{j-i+1}{2} \text{ and } t = n + \frac{i+j+1}{2} \pmod{2n+1}.$$

Hence, each pair $\{a_i, a_j\}$ ($0 \leqslant i, j \leqslant 2n$, $i \neq j$) is contained in some block from $\mathcal{B}$. Due to the symmetry, the same holds for all pairs $\{b_i, b_j\}$ i $\{c_i, c_j\}$. This proves the theorem. □

Note: Obtained $(6n + 3, 3, 2)-$covering is Steiner system, because each pair of elements of the set V is contained in exactly one block from $\mathcal{B}$. Moreover, it can be shown that the previous construction is equivalent to Bose construction of the Steiner system $S(2, 3, 6n + 3)$.

In a similar way, we will construct $(6n + 4, 3, 2)-$covering.

2.2. *Minimal* $(6n + 4, 3, 2)-$*covering*

Theorem 2.2. *Let* $v = 6n+4$ *and* $V = \{a_0, a_1, \ldots, a_{2n}\} \cup \{b_0, b_1, \ldots, b_{2n}\} \cup \{c_0, c_1, \ldots, c_{2n}\} \cup \{\infty\}$. *Let* $\mathcal{B}$ *be the set of blocks obtained by applying the permutation*

$$p = (a_0\, a_1 \ldots a_{2n})(b_0\, b_1 \ldots b_{2n})(c_0\, c_1 \ldots c_{2n})(\infty), \tag{6}$$

$2n + 1$ *times to blocks*

$$\begin{array}{llll} \{a_0, b_1, b_{2n}\}, & \{a_0, b_2, b_{2n-1}\}, & \ldots & , \{a_0, b_n, b_{n+1}\}, \\ \{b_0, c_1, c_{2n}\}, & \{b_0, c_2, c_{2n-1}\}, & \ldots & , \{b_0, c_n, c_{n+1}\}, \\ \{c_0, a_1, a_{2n}\}, & \{c_0, a_2, a_{2n-1}\}, & \ldots & , \{c_0, a_n, a_{n+1}\}, \\ \{a_0, b_0, c_0\}, & \{a_0, b_0, \infty\}, & & \end{array} \tag{7}$$

including blocks obtained by applying the permutation p, $n + 1$ *times to the block*

$$\{c_0, c_n, \infty\}. \tag{8}$$

Then, $(V, \mathcal{B})$ *is one* $(v, 3, 2)-$*covering with* $L(v, 3, 2)$ *blocks.*

Proof. The set $\mathcal{B}$ contains $(3n + 2)(2n + 1) + (n + 1) = 6n^2 + 8n + 3 = L(6n + 4, 3, 2)$ different blocks. Let us prove that $(V, \mathcal{B})$ is one $(v, 3, 2)-$covering, ie. that each pair of elements of the set V is contained in some block from $\mathcal{B}$.

As in theorem 2.1, we prove that each of the pairs $\{a_i, b_j\}$, $\{b_i, c_j\}$ i $\{c_i, a_j\}$ $(0 \leqslant i, j \leqslant 2n)$, as well as each of the pairs $\{a_i, a_j\}$, $\{b_i, b_j\}$ i $\{c_i, c_j\}$ $(0 \leqslant i, j \leqslant 2n,\ i \neq j)$, is contained in some block from $\mathcal{B}$.

It remains to prove that each pair containing the element ∞ is contained in some block from $\mathcal{B}$. By applying the permutation p^i to the block $\{a_0, b_0, \infty\}$, block $\{a_i, b_i, \infty\}$ is obtained, and each of the pairs $\{a_i, \infty\}$ and $\{b_i, \infty\}$ $(0 \leqslant i \leqslant 2n)$ is contained in some block from $\mathcal{B}$. By applying the permutation p^i to block $\{c_0, c_n, \infty\}$, block $\{c_i, c_{n+i}, \infty\}$ $(0 \leqslant i \leqslant n)$ is obtained. Hence, each pair $\{c_i, \infty\}$ $(0 \leqslant i \leqslant 2n)$ is also contained in some block from $\mathcal{B}$. This proves the theorem. □

Note: Obtained $(6n + 4, 3, 2)-$covering is not Steiner system because each of the pairs $\{a_i, b_i\}$ $(0 \leqslant i \leqslant 2n)$, $\{c_i, c_{n+i}\}$ $(0 \leqslant i \leqslant n)$ and $\{c_n, \infty\}$ is contained in two different blocks from $\mathcal{B}$.

In a similar way, we will construct $(6n + 5, 3, 2)-$covering.

2.3. *Minimal* $(6n + 5, 3, 2)-$*covering*

Theorem 2.3. *Let* $v = 6n+5$ *and* $V = \{a_0, a_1, \ldots, a_{2n}\} \cup \{b_0, b_1, \ldots, b_{2n}\} \cup \{c_0, c_1, \ldots, c_{2n}\} \cup \{\infty_0, \infty_1\}$. *Let* $\mathcal{B}$ *be the set of blocks obtained by applying the permutation*

$$p = (a_0\, a_1 \ldots a_{2n})(b_0\, b_1 \ldots b_{2n})(c_0\, c_1 \ldots c_{2n})(\infty_0\, \infty_1), \tag{9}$$

$2n + 1$ *times to blocks*

$$\begin{array}{llll} \{a_0, b_1, b_{2n}\}, & \{a_0, b_2, b_{2n-1}\}, & \ldots & , \{a_0, b_n, b_{n+1}\}, \\ \{b_0, c_1, c_{2n}\}, & \{b_0, c_2, c_{2n-1}\}, & \ldots & , \{b_0, c_n, c_{n+1}\}, \\ \{c_1, a_1, a_{2n}\}, & \{c_1, a_2, a_{2n-1}\}, & \ldots & , \{c_1, a_n, a_{n+1}\}, \\ \{a_0, b_0, \infty_0\}, & \{b_0, c_0, \infty_1\}, & \{c_1, a_0, \infty_1\}, & \end{array} \tag{10}$$

including the block $\{c_0, \infty_0, \infty_1\}$. *Then,* $(V, \mathcal{B})$ *is one* $(v, 3, 2)-$*covering with* $L(v, 3, 2)$ *blocks.*

Before proving the theorem, note that blocks in the third row and the last block in (10) contain the element c_1 instead the "expected" element c_0. Also, the element ∞_1 is contained in two, and ∞_0 in just one block from (10). Thereby, the symmetry is lost, and therefore the proof requires considering a larger number of cases.

Proof. The set $\mathcal{B}$ contains $(3n+3)(2n+1)+1 = 6n^2+9n+4 = L(6n+5, 3, 2)$ different blocks. Let us prove that $(V, \mathcal{B})$ is one $(v, 3, 2)-$covering, ie. that each pair of elements of the set V is contained in some block from $\mathcal{B}$.

As in theorem 2.1, we prove that each of the pairs $\{a_i, b_j\}$ and $\{b_i, c_j\}$ $(0 \leqslant i, j \leqslant 2n)$ is contained in some block from $\mathcal{B}$. Also, each pair $\{c_1, a_j\}$ $(0 \leqslant j \leqslant 2n)$ is contained in some block from (10): pair $\{c_1, a_0\}$ in the last block, and pair $\{c_1, a_j\}$ $(j \neq 0)$ in some block in the third row in (10). By applying the permutation p^i $(1 \leqslant i \leqslant 2n)$ to the blocks from (10), element c_1 is mapped into $c_{i+1} = p^i(c_1)$ (into c_0, when $i = 2n$), while the elements $a_0, a_1, \ldots, a_{2n}$ are mapped into $a_i, a_{i+1 \pmod{2n+1}}, \ldots, a_{i-1 \pmod{2n+1}}$, respectively. Hence, each pair $\{c_i, a_j\}$ $(0 \leqslant i, j \leqslant 2n)$ is also contained in some block from $\mathcal{B}$.

As in theorem 2.1, we prove that each of the pairs $\{a_i, a_j\}$, $\{b_i, b_j\}$ and $\{c_i, c_j\}$ $(0 \leqslant i, j \leqslant 2n, i \neq j)$ is contained in some block from $\mathcal{B}$.

It remains to prove that each pair containing elements ∞_0 or ∞_1 is contained in some block from $\mathcal{B}$. By applying the permutation p, $2n+1$ times to the last three blocks from (10), we obtain, respectively, the blocks:

$$\begin{array}{llll}
\{a_0, b_0, \infty_0\}, & \{a_1, b_1, \infty_1\}, & \ldots & , \{a_{2n}, b_{2n}, \infty_0\}, \\
\{b_0, c_0, \infty_1\}, & \{b_1, c_1, \infty_0\}, & \ldots & , \{b_{2n}, c_{2n}, \infty_1\}, \\
\{c_1, a_0, \infty_1\}, & \{c_2, a_1, \infty_0\}, & \ldots & , \{c_0, a_{2n}, \infty_1\}.
\end{array}$$

By direct verification, we establish that each of the pairs $\{a_i, \infty_j\}$, $\{b_i, \infty_j\}$ i $\{c_i, \infty_j\}$ $(0 \leqslant i \leqslant 2n, j \in \{0,1\})$, except the pair $\{c_0, \infty_0\}$, is contained in some of the specified blocks. The pair $\{c_0, \infty_0\}$, as well as the pair $\{\infty_0, \infty_1\}$, is contained in additional block $\{c_0, \infty_0, \infty_1\}$. This proves the theorem. □

Note: Obtained $(6n+5,3,2)-$covering is not Steiner system because the pair $\{c_0, \infty_1\}$ is contained in three different blocks from $\mathcal{B}$.

2.4. *Minimal* $(6n+1,3,2)-$*covering*

The construction of the Steiner system $STS(6n+1)$ differs somewhat from three previous constructions.

Theorem 2.4. *Let* $v = 6n+1$ *and* $V = \{a_0, a_1, \ldots, a_{2n-1}\} \cup \{b_0, b_1, \ldots, b_{2n-1}\} \cup \{c_0, c_1, \ldots, c_{2n-1}\} \cup \{\infty\}$. *Let* $\mathcal{B}$ *be the set obtained by allying the permutation*

$$p = (a_0\, a_1 \ldots a_{2n-1})(b_0\, b_1 \ldots b_{2n-1})(c_0\, c_1 \ldots c_{2n-1})(\infty), \tag{11}$$

n times to blocks

$$\begin{array}{llll}
\{a_0, b_0, b_{2n-1}\}, & \{a_0, b_1, b_{2n-2}\}, & \ldots & , \{a_0, b_{n-1}, b_n\}, \\
\{b_0, c_0, c_{2n-1}\}, & \{b_0, c_1, c_{2n-2}\}, & \ldots & , \{b_0, c_{n-1}, c_n\}, \\
\{c_0, a_0, a_{2n-1}\}, & \{c_0, a_1, a_{2n-2}\}, & \ldots & , \{c_0, a_{n-1}, a_n\}, \\
\{a_n, b_0, \infty\}, & \{a_n, b_1, b_{2n-1}\}, & \ldots & , \{a_n, b_{n-1}, b_{n+1}\}, \\
\{b_n, c_0, \infty\}, & \{b_n, c_1, c_{2n-1}\}, & \ldots & , \{b_n, c_{n-1}, c_{n+1}\}, \\
\{c_n, a_0, \infty\}, & \{c_n, a_1, a_{2n-1}\}, & \ldots & , \{c_n, a_{n-1}, a_{n+1}\}, \\
\{a_n, b_n, c_n\}. & & &
\end{array} \tag{12}$$

Then, $(V, \mathcal{B})$ *is one* $(v,3,2)-$*covering with* $L(v,3,2)$ *blocks.*

Proof. The set $\mathcal{B}$ contains $n(6n+1) = 6n^2 + n = L(6n+1,3,2)$ different blocks. Let us prove that $(V, \mathcal{B})$ is one $(v,3,2)-$covering, ie. that each pair of elements of the set V is contained in some block from $\mathcal{B}$.

Each pair $\{a_0, b_j\}$ $(0 \leqslant j \leqslant 2n-1)$ is contained in some block from the first row in (12). Also, each pair $\{a_n, b_j\}$ $(0 \leqslant j \leqslant 2n-1)$ is contained in some block from (12): pair $\{a_n, b_n\}$ in the block $\{a_n, b_n, c_n\}$, and the pair $\{a_n, b_j\}$ $(j \neq n)$ in some block from the fourth row in (12). By applying the permutation p^i $(1 \leqslant i \leqslant n-1)$ to blocks from (12), element a_0 is mapped into a_i, a_n is mapped into a_{n+i}, while the elements $b_0, b_1, \ldots, b_{2n-1}$ are mapped into $b_i, b_{i+1 \pmod{2n}}, \ldots, b_{i-1 \pmod{2n}}$, respectively. Hence, each of the pairs $\{a_i, b_j\}$ and $\{a_{n+i}, b_j\}$ $(0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant 2n-1)$ is contained in some block from $\mathcal{B}$. To put it more simply, each pair $\{a_i, b_j\}$ $(0 \leqslant i, j \leqslant 2n-1)$ is contained in some block from $\mathcal{B}$. Due to the symmetry, the same holds for all pairs $\{b_i, c_j\}$ and $\{c_i, a_j\}$.

In a similar way, we prove that each of the pairs $\{a_i, \infty\}$ and $\{a_{n+i}, \infty\}$ $(0 \leqslant i \leqslant n-1)$, that is $\{a_i, \infty\}$ $(0 \leqslant i \leqslant 2n-1)$, is contained in some block from $\mathcal{B}$. Due to the symmetry, the same holds for all pairs $\{b_i, \infty\}$ and $\{c_i, \infty\}$.

Let us now consider the pairs $\{a_i, a_j\}$. Each pair $\{a_i, a_j\}$ such that $i + j = 2n - 1$ is contained in some block from the third row, while each pair $\{a_i, a_j\}$ such that $i + j = 2n$ is contained in some block from the sixth row in (12). For an arbitrary pair $\{a_i, a_j\}$ $(0 \leqslant i < j \leqslant 2n-1)$, it is sufficient to prove the existence of the pair $\{a_r, a_s\}$ $(0 \leqslant r, s \leqslant 2n-1,\ r+s = 2n-1$ or $r+s = 2n)$ and the permutation p^t $(0 \leqslant t \leqslant n-1)$ by which the pair $\{a_r, a_s\}$ is mapped into the pair $\{a_i, a_j\}$, that is, it is sufficient to prove that at least one of the systems

$$I: \begin{cases} r+s = 2n-1, \\ r+t \pmod{2n} = i, \\ s+t \pmod{2n} = j, \end{cases} \qquad II: \begin{cases} r+s = 2n, \\ r+t \pmod{2n} = i, \\ s+t \pmod{2n} = j, \end{cases}$$

has the solution on r, s and t. If $1 \leqslant i+j \leqslant 2n-2$, the solution is

$$r = 2n - \frac{j-i+\delta}{2},\ s = \frac{j-i-\delta}{2} \text{ and } t = \frac{i+j+\delta}{2},$$

and if $2n-1 \leqslant i+j \leqslant 4n-3$, the solution is

$$r = n - \frac{j-i+\delta}{2},\ s = n + \frac{j-i-\delta}{2} \text{ and } t = \frac{i+j+\delta}{2} - n,$$

where

$$\delta = \begin{cases} 0\,, & \text{if } i \text{ and } j \text{ are of the same parity (solution of the system } II), \\ 1\,, & \text{if } i \text{ and } j \text{ are with opposite parity (solution of the system } I). \end{cases}$$

Hence, each pair $\{a_i, a_j\}$ $(0 \leqslant i, j \leqslant 2n-1,\ i \neq j)$ is contained in some block from $\mathcal{B}$. Due to the symmetry, the same holds for all pairs $\{b_i, b_j\}$ i $\{c_i, c_j\}$. This proves the theorem. □

Note: Obtained $(6n+1, 3, 2)-$covering is Steiner system, because each pair of elements of the set V is contained in exactly one block from $\mathcal{B}$. Moreover, it can be proved that the previous construction is equivalent to Skolem construction of the Steiner system $S(2, 3, 6n+1)$

In a similar way, we will construct $(6n, 3, 2)-$covering.

2.5. Minimal $(6n,3,2)-$*covering*

Theorem 2.5. *Let* $v = 6n$ *and* $V = \{a_0, a_1, \ldots, a_{2n-1}\} \cup \{b_0, b_1, \ldots, b_{2n-1}\} \cup \{c_0, c_1, \ldots, c_{2n-1}\}$. *Let* $\mathcal{B}$ *be the set of blocks obtained by applying the permutation*

$$p = (a_0\, a_1 \ldots a_{2n-1})(b_0\, b_1 \ldots b_{2n-1})(c_0\, c_1 \ldots c_{2n-1}), \tag{13}$$

n times to blocks

$$\begin{array}{llll}
\{a_0,b_0,b_{2n-1}\}, & \{a_0,b_1,b_{2n-2}\}, & \ldots & ,\{a_0,b_{n-1},b_n\},\\
\{b_0,c_0,c_{2n-1}\}, & \{b_0,c_1,c_{2n-2}\}, & \ldots & ,\{b_0,c_{n-1},c_n\},\\
\{c_0,a_0,a_{2n-1}\}, & \{c_0,a_1,a_{2n-2}\}, & \ldots & ,\{c_0,a_{n-1},a_n\},\\
\{a_n,b_0,b_n\}, & \{a_n,b_1,b_{2n-1}\}, & \ldots & ,\{a_n,b_{n-1},b_{n+1}\},\\
\{b_n,c_0,c_n\}, & \{b_n,c_1,c_{2n-1}\}, & \ldots & ,\{b_n,c_{n-1},c_{n+1}\},\\
\{c_n,a_0,a_n\}, & \{c_n,a_1,a_{2n-1}\}, & \ldots & ,\{c_n,a_{n-1},a_{n+1}\}.
\end{array} \tag{14}$$

Then, $(V,\mathcal{B})$ *is one* $(v,3,2)-$*covering with* $L(v,3,2)$ *blocks.*

Proof. The proof is completely analogous to the proof of the previous theorem. The only diference is that now pairs $\{a_n,b_n\}$, $\{b_n,c_n\}$ and $\{c_n,a_n\}$ are contained in blocks $\{a_n,b_0,b_n\}$, $\{b_n,c_0,c_n\}$ and $\{c_n,a_0,a_n\}$ from (14), respectively, instead in block $\{a_n,b_n,c_n\}$. Hence, each pair of the elements of the set V is contained in some block from $\mathcal{B}$, that is $(V,\mathcal{B})$ is $(v,3,2)-$covering.

$\mathcal{B}$ contains $6n \cdot n = 6n^2 = L(6n,3,2)$ different blocks, which proves the theorem. □

Note: The obtained $(6n,3,2)-$covering is not Steiner system because each of the pairs $\{a_i,a_{n+i}\}$, $\{b_i,b_{n+i}\}$ and $\{c_i,c_{n+i}\}$ $(0 \leqslant i \leqslant n-1)$ is contained in two different blocks from $\mathcal{B}$.

Finally, we give the construction of $(6n+2,3,2)-$covering.

2.6. Minimal $(6n+2,3,2)-$*covering*

Theorem 2.6. *Let* $v = 6n+2$ *and* $V = \{a_0, a_1, \ldots, a_{2n-1}\} \cup \{b_0, b_1, \ldots, b_{2n-1}\} \cup \{c_0, c_1, \ldots, c_{2n-1}\} \cup \{\infty_0, \infty_1\}$. *Let* $\mathcal{B}$ *be the set of blocks obtained by applying the permutation*

$$p = (a_0\, a_1 \ldots a_{2n-1})(b_0\, b_1 \ldots b_{2n-1})(c_0\, c_1 \ldots c_{2n-1})(\infty_0)(\infty_1), \tag{15}$$

n times to blocks

$$\begin{array}{lllll}
 & \{a_0,b_0,b_{2n-1}\}, & \{a_0,b_1,b_{2n-2}\}, & \ldots & ,\{a_0,b_{n-1},b_n\},\\
 & \{b_0,c_0,c_{2n-1}\}, & \{b_0,c_1,c_{2n-2}\}, & \ldots & ,\{b_0,c_{n-1},c_n\},\\
 & \{c_0,a_0,a_{2n-1}\}, & \{c_0,a_1,a_{2n-2}\}, & \ldots & ,\{c_0,a_{n-1},a_n\},\\
\{a_n,b_0,\infty_0\}, & \{a_n,b_0,\infty_1\}, & \{a_n,b_1,b_{2n-1}\}, & \ldots & ,\{a_n,b_{n-1},b_{n+1}\},\\
\{b_n,c_0,\infty_0\}, & \{b_n,c_0,\infty_1\}, & \{b_n,c_1,c_{2n-1}\}, & \ldots & ,\{b_n,c_{n-1},c_{n+1}\},\\
\{c_n,a_0,\infty_0\}, & \{c_n,a_0,\infty_1\}, & \{c_n,a_1,a_{2n-1}\}, & \ldots & ,\{c_n,a_{n-1},a_{n+1}\},\\
 & \{a_n,b_n,c_n\}, & & &
\end{array} \tag{16}$$

including block $\{a_0,\infty_0,\infty_1\}$. *Then,* $(V,\mathcal{B})$ *is one* $(v,3,2)-$*covering with* $L(v,3,2)$ *blocks.*

Proof. The set $\mathcal{B}$ contains $n(6n+4)+1 = 6n^2+4n+1 = L(6n+1,3,2)$ different blocks. Let us prove that $(V,\mathcal{B})$ is one $(v,3,2)-$covering, ie. that each pair of elements of the set V is contained in some block from $\mathcal{B}$.

As in theorem 2.4, we prove that each of the pairs $\{a_i,b_j\}$, $\{b_i,c_j\}$ and $\{c_i,a_j\}$ $(0 \leqslant i,j \leqslant 2n-1)$, as well as each of the pairs $\{a_i,a_j\}$, $\{b_i,b_j\}$ and $\{c_i,c_j\}$ $(0 \leqslant i,j \leqslant 2n-1$, $i \neq j)$, is contained in some block from $\mathcal{B}$.

It remains to prove that each pair containing elements ∞_0 or ∞_1, is contained in some block from $\mathcal{B}$. By applying the permutation p, n times to blocks $\{a_n,b_0,\infty_0\}$, $\{b_n,c_0,\infty_0\}$, $\{c_n,a_0,\infty_0\}$, we obtain, respectively, the blocks:

$$\begin{array}{llcl}\{a_n,b_0,\infty_0\}, & \{a_{n+1},b_1,\infty_0\}, & \ldots & ,\{a_{2n-1},b_{n-1},\infty_0\},\\ \{b_n,c_0,\infty_0\}, & \{b_{n+1},c_1,\infty_0\}, & \ldots & ,\{b_{2n-1},c_{n-1},\infty_0\},\\ \{c_n,a_0,\infty_0\}, & \{c_{n+1},a_1,\infty_0\}, & \ldots & ,\{c_{2n-1},a_{n-1},\infty_0\}.\end{array}$$

By direct verification, we establish that each of the pairs $\{a_i,\infty_0\}$, $\{b_i,\infty_0\}$ and $\{c_i,\infty_0\}$ $(0 \leqslant i \leqslant 2n-1)$ is contained in some of the specified blocks. In a similar way, each of the pairs $\{a_i,\infty_1\}$, $\{b_i,\infty_1\}$ and $\{c_i,\infty_1\}$ $(0 \leqslant i \leqslant 2n-1)$ is contained in some block from $\mathcal{B}$. The pair $\{\infty_0,\infty_1\}$ is contained in additional block $\{a_0,\infty_0,\infty_1\}$. This proves the theorem. □

Note: Obtained $(6n+2,3,2)-$covering is not Steiner system because each of the pairs $\{a_{n+i},b_i\}$, $\{b_{n+i},c_i\}$ and $\{c_{n+i},a_i\}$ $(0 \leqslant i \leqslant n-1)$, as well as the pairs $\{a_0,\infty_0\}$ i $\{a_0,\infty_1\}$, is contained in two different blocks from $\mathcal{B}$. In the additional block $\{a_0,\infty_0,\infty_1\}$, we could use an arbitrary element of the set V instead of the element a_0.

3. CONCLUSION

In this paper, we consider the $(v,k,t)-$coverings and give a new construction of the minimal $(v,3,2)-$coverings. We have constructed minimal $(v,3,2)-$covering for each v (mod 6) separately. In each of six cases, the construction apply permutation p to the base blocks in order to obtain the remaining blocks of $(v,3,2)-$covering. Consequently, the equality $C(v,3,2) = L(v,3,2)$ is proved.

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