

MARKOVIAN INVENTORY MODEL WITH TWO PARALLEL QUEUES, JOCKEYING AND IMPATIENT CUSTOMERS

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Received: March 2015 / Accepted: July 2015

Abstract: This article presents a perishable stochastic inventory system under continuous review at a service facility consisting of two parallel queues with jockeying. Each server has its own queue, and jockeying among the queues is permitted. The capacity of each queue is of finite size L . The inventory is replenished according to an (s, S) inventory policy and the replenishing times are assumed to be exponentially distributed. The individual customer is issued a demanded item after a random service time, which is distributed as negative exponential. The life time of each item is assumed to be exponential. Customers arrive according to a Poisson process and on arrival; they join the shortest feasible queue. Moreover, if the inventory level is more than one and one queue is empty while in the other queue, more than one customer are waiting, then the customer who has to be received after the customer being served in that queue is transferred to the empty queue. This will prevent one server from being idle while the customers are waiting in the other queue. The waiting customer independently

reneges the system after an exponentially distributed amount of time. The joint probability distribution of the inventory level, the number of customers in both queues, and the status of the server are obtained in the steady state. Some important system performance measures in the steady state are derived, so as the long-run total expected cost rate.

Keywords: Markov process, Continuous review, Inventory with service time, Perishable commodity, Shortest queue, Jockeying and impatient.

MSC: 90B22, 60K25.

1. INTRODUCTION

Research on queueing systems with inventory control has captured much attention of researchers over the last decades. In this system, customers arrive at the service facility one by one and require service. In order to complete the customer service, an item from the inventory is needed. A served customer departs immediately from the system and the on - hand inventory decreases by one at the moment of service completion. This system is called a queueing - inventory system [11]. Berman and Kim [4] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [5] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [6] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [7] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [8] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure.

The study on multiserver queueing-inventory systems generally assumes the servers to be homogeneous in which the individual service rates are the same for all the servers in the system. This assumption may be valid only when the service process is mechanically or electronically controlled. The multiserver queueing-inventory systems with homogeneous servers are also widely studied. For a related bibliography see [14, 15]. In a queueing-inventory system with human servers, the above assumption can hardly be realized. It is common to observe server rendering service to identical jobs at different service rates. This reality leads to modelling such multiserver queueing-inventory systems with heterogeneous servers, i.e., the service time distributions may be different for different servers. In the case of perishable queueing-inventory system with two heterogeneous servers including one with unreliable server and repeated attempts, the

first paper was by Yadavalli et.al [16] who assumed the exponential life time for the items, exponential lead time for the supply of the ordered items and exponential retrial rate for the customers in the orbit.

In this paper, we consider a queueing-inventory system consisting of two parallel queues with jockeying and different server rates. The concept of jockeying is one of the important customer strategies. It refers to the movements of customers who have the option of switching from one queue to another when several servers, each having a separate and distinct queue, are available. The shortest queue problems with jockeying, but not assuming stochastic inventory management, have been widely studied by many researchers in the past. For the theory of shortest queueing problems with/ without jockeying, the often quoted articles are Haight [10], Zhao and Grassman [18], Adan et.al [1, 2, 3], Cohen [9], Van Houtum et.al [13], Yao and Knessl [17] and Tarabia [12].

The rest of this paper is organized as follows. In the next section, the mathematical model and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are obtained in section 3. Some key system performance measures are derived in section 4. In section 5, we calculate the total expected cost rate, and in the section 6, we present sensitivity analysis numerically. The last section is meant for conclusion.

2. MODEL DESCRIPTION

In this paper, stochastic queueing-inventory systems with the following assumptions are investigated.

Consider a continuous review perishable inventory system with two queues in parallel and jockeying. Maximum inventory level is denoted by S and the inventory is replenished according to (s, S) ordering policy. According to this policy, the reorder level is fixed as $s \geq 2$ and an order is placed when the inventory level reaches the reorder level. The ordering quantity is $Q(= S - s + 1)$ items. The condition $S - s > s + 1$ ensures that no perpetual shortage in the stock after replenishment. The lead time is assumed to be exponential with parameter $\beta(> 0)$. The life time of the commodity is assumed to be distributed as negative exponential with parameter $\gamma(> 0)$. We have assumed that an item of inventory that makes it into the service process cannot perish while in service. The queueing-inventory system consists of two parallel servers (server-1 and server-2) with different service rates μ_1 and μ_2 , respectively. The arrival of customers is assumed to form a Poisson process with parameter $\lambda(> 0)$. The capacity of each queue is restricted to L including the one being served.

An arriving customer joins the shortest queue, if both queues are equal, he chooses a first queue with probability p or second with q , where $p + q = 1$. The waiting customers receive their service one by one. The demand is for a single item per customer. The demanded item is delivered to the customer after a random time of service. The moment any server becomes idle, if the inventory level is more than one (including the servicing item) and if there is a customer waiting in the other queue, the customer immediately following the customer

who is receiving service at that counter is transferred to the idle server queue. An impatient customer leaves the system independently after a random time which is distributed as negative exponential with parameter $\alpha_1 (> 0)$ if the customer leaves from queue-1, and $\alpha_2 (> 0)$ if the customer leaves from queue-2. Note that in this model we have assumed that the servicing customer can not be impatient. Any arriving customer who finds that both queues are full is considered to be lost. Various stochastic processes involved in the system are independent of each other.

2.1. Notations:

- \mathbf{e} : A column vector of appropriate dimension containing all ones,
- $\mathbf{0}$: Zero matrix of appropriate dimension,
- $[A]_{ij}$: Entry at $(i, j)^{th}$ position of a matrix A,
- δ_{ij} : $\begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$
- $\bar{\delta}_{ij}$: $1 - \delta_{ij}$,
- $k \in V_i^j$: $k = i, i + 1, \dots, j$,
- $H(x)$: $\begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$
- I : Identity matrix,
- I_k : An Identity matrix of order k.

3. ANALYSIS

Let $L(t)$, $Y(t)$, $X_1(t)$ and $X_2(t)$, respectively, denote the inventory level, the server status, the number of customers in queue-1, and the number of customers in queue-2 at time t .

Further, let the status of the server $Y(t)$ be defined as follows:

$$Y(t) = \begin{cases} S_{00}, & \text{if both the servers are idle at time } t, \\ S_{10}, & \text{if server-1 is busy and server-2 is idle at time } t, \\ S_{01}, & \text{if server-1 is idle and server-2 is busy at time } t, \\ S_{11}, & \text{if both the servers are busy at time } t. \end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the quadruplet $\{(L(t), Y(t), X_1(t), X_2(t)), t \geq 0\}$ is a continuous time Markov chain with discrete state space given by

$$E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7,$$

where,

- $E_1 : \{(0, S_{00}, i_3, i_4) \mid i_3 = 0, 1, 2, \dots, L; i_4 = 0, 1, 2, \dots, L\},$
- $E_2 : \{(i_1, S_{00}, 0, 0) \mid i_1 = 1, 2, \dots, S\},$
- $E_3 : \{(1, S_{10}, i_3, i_4) \mid i_3 = 1, 2, \dots, L; i_4 = 0, 1, 2, \dots, L\},$
- $E_4 : \{(1, S_{01}, i_3, i_4) \mid i_3 = 0, 1, 2, \dots, L; i_4 = 1, 2, \dots, L\},$
- $E_5 : \{(i_1, S_{10}, 1, 0) \mid i_1 = 2, 3, \dots, S\},$
- $E_6 : \{(i_1, S_{01}, 0, 1) \mid i_1 = 2, 3, \dots, S\},$
- $E_7 : \{(i_1, S_{11}, i_3, i_4) \mid i_1 = 2, 3, \dots, S; i_3 = 1, 2, \dots, L; i_4 = 1, 2, \dots, L\}.$

Define the following ordered sets:

$$\langle i_1, i_2, i_3 \rangle = \begin{cases} ((i_1, S_{00}, i_3, 0), (i_1, S_{00}, i_3, 1), \dots, (i_1, S_{00}, i_3, L)), i_1 = 0; i_3 = 0, 1, \dots, L; \\ ((i_1, S_{00}, i_3, 0)), i_1 = 1, 2, \dots, S; i_3 = 0; \\ ((i_1, S_{10}, i_3, 0), (i_1, S_{10}, i_3, 1), \dots, (i_1, S_{10}, i_3, L)), i_1 = 1; i_3 = 1, \dots, L; \\ ((i_1, S_{01}, i_3, 1), (i_1, S_{01}, i_3, 2), \dots, (i_1, S_{01}, i_3, L)), i_1 = 1; i_3 = 0, 1, \dots, L; \\ ((i_1, S_{10}, i_3, 0)), i_1 = 2, 3, \dots, S; i_3 = 1; \\ ((i_1, S_{01}, i_3, 1)), i_1 = 2, 3, \dots, S; i_3 = 0; \\ ((i_1, S_{11}, i_3, 1), (i_1, S_{11}, i_3, 2), \dots, (i_1, S_{11}, i_3, L)), i_1 = 2, 3, \dots, S; i_3 = 1, 2, \dots, L; \end{cases}$$

$$\ll i_1, i_2 \gg = \begin{cases} \langle i_1, S_{00}, 0 \rangle, \langle i_1, S_{00}, 1 \rangle, \dots, \langle i_1, S_{00}, L \rangle, i_1 = 0; \\ \langle i_1, S_{00}, 0 \rangle, i_1 = 1, 2, \dots, S; \\ \langle i_1, S_{10}, 1 \rangle, \langle i_1, S_{10}, 2 \rangle, \dots, \langle i_1, S_{10}, L \rangle, i_1 = 1; \\ \langle i_1, S_{01}, 0 \rangle, \langle i_1, S_{01}, 1 \rangle, \dots, \langle i_1, S_{01}, L \rangle, i_1 = 1; \\ \langle i_1, S_{10}, 1 \rangle, \langle i_1, S_{01}, 0 \rangle, i_1 = 2, 3, \dots, S; \\ \langle i_1, S_{11}, 1 \rangle, \langle i_1, S_{11}, 2 \rangle, \dots, \langle i_1, S_{11}, L \rangle, i_1 = 2, 3, \dots, S; \end{cases}$$

$$\lll i_1 \ggg = \begin{cases} \ll i_1, S_{00} \gg, i_1 = 0; \\ \ll i_1, S_{00} \gg, i_1 = 1, 2, \dots, S; \\ \ll i_1, S_{10} \gg, \ll i_1, S_{01} \gg, i_1 = 1; \\ \ll i_1, S_{10} \gg, \ll i_1, S_{01} \gg, \ll i_1, S_{11} \gg, i_1 = 2, 3, \dots, S; \end{cases}$$

By ordering the state space ($\lll 0 \ggg, \lll 1 \ggg, \dots, \lll S \ggg$), the infinitesimal generator Θ can be conveniently written in a block partitioned matrix with entries

$$\Theta = \begin{matrix} & \lll 0 \ggg & \lll 1 \ggg & \lll 2 \ggg & \dots & \lll S-1 \ggg & \lll S \ggg \\ \lll 0 \ggg & \left(\begin{matrix} A_{0,0} & A_{0,1} & A_{0,2} & \dots & A_{0,S-1} & A_{0,S} \\ \lll 1 \ggg & A_{1,0} & A_{1,1} & A_{1,2} & \dots & A_{1,S-1} & A_{1,S} \\ \lll 2 \ggg & A_{2,0} & A_{2,1} & A_{2,2} & \dots & A_{2,S-1} & A_{2,S} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lll S-1 \ggg & A_{S-1,0} & A_{S-1,1} & A_{S-1,2} & \dots & A_{S-1,S-1} & A_{S-1,S} \\ \lll S \ggg & A_{S,0} & A_{S,1} & A_{S,2} & \dots & A_{S,S-1} & A_{S,S} \end{matrix} \right) \end{matrix}$$

More explicitly, due to the assumptions made on the demand and replenishment processes, we note that

$$A_{i_1, j_1} = 0, \quad \text{for } j_1 \neq i_1, i_1 - 1, i_1 + Q.$$

We first consider the case A_{i_1, i_1+Q} . This will occur only when the inventory level is replenished.

Case (1) First we consider the inventory level to be zero, that is $A_{0,Q}$. For this

Case (1a) Let $i_2 = S_{00}$, $i_3 = 0$ and $i_4 = 0$.

At the time of replenishment, the state of the system changes from $(0, S_{00}, 0, 0)$ to $(Q, S_{00}, 0, 0)$, with intensity of transition β . The sub matrix of the transition rates from $\ll 0, S_{00} \gg$ to $\ll Q, S_{00} \gg$, is given by

$$[C_0^{(1)}]_{i_3 j_3} = \begin{cases} C_0^{(11)}, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_0^{(11)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (1b) Let $i_2 = S_{00}$, $i_3 = 0$ and $i_4 = 1$.

Replenishment of inventory takes the system state from $(0, S_{00}, 0, 1)$ to $(Q, S_{01}, 0, 1)$, with intensity of transition β . The sub matrix of the transition rates from $\ll 0, S_{00} \gg$ to $\ll Q, S_{01} \gg$ is given by

$$[C_0^{(2)}]_{i_3 j_3} = \begin{cases} C_0^{(21)}, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_0^{(21)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

Case (1c) Let $i_2 = S_{00}$, $i_3 = 1$ and $i_4 = 0$.

When a replenishment takes place at $(0, S_{00}, 1, 0)$, the inventory level reaches to $(Q, S_{10}, 1, 0)$, with intensity of transition β . The sub matrix of the transition rates from $\ll 0, S_{00} \gg$ to $\ll Q, S_{10} \gg$ is given by

$$[C_0^{(3)}]_{i_3 j_3} = \begin{cases} C_0^{(31)}, & j_3 = i_3, & i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_0^{(31)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (1d) • Let $i_2 = S_{00}$, $1 \leq i_3 \leq L$ and $i_4 = 0$.

When the inventory level is replenished, the state of the system changes from $(0, S_{00}, i_3, i_4)$ to (Q, S_{11}, i_3, i_4) , $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity of transition β .

• Let $i_2 = S_{00}$, $i_3 = 0$ and $2 \leq i_4 \leq L$.

Replenishment changes the system state from $(0, S_{00}, 0, i_4)$ to $(Q, S_{11}, 1, i_4 - 1)$, $i_4 \in V_2^L$, with intensity of transition β .

• Let $i_2 = S_{00}$, $2 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

A transition from $(0, S_{00}, i_3, 0)$ to $(Q, S_{11}, i_3 - 1, 1)$, $Q = S - s$ for $i_3 \in V_2^L$, takes place with intensity β when a replenishment for Q items occur.

The sub matrix of this transition rates from $\ll 0, S_{00}, \gg$ to $\ll Q, S_{11}, \gg$ is given by

$$[C_0^{(4)}]_{i_3 j_3} = \begin{cases} C_0^{(41)}, & j_3 = 1, & i_3 = 0, \\ C_0^{(42)}, & j_3 = i_3, & i_3 \in V_1^L, \\ C_0^{(43)}, & j_3 = i_3 - 1, & i_3 \in V_2^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_0^{(41)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4 - 1, & i_4 \in V_2^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C_0^{(42)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C_0^{(43)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = 1, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Hence,

$$[A_{0,Q}]_{i_2 j_2} = \begin{cases} C_0^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ C_0^{(2)}, & j_2 = S_{01}, & i_2 = S_{00}, \\ C_0^{(3)}, & j_2 = S_{10}, & i_2 = S_{00}, \\ C_0^{(4)}, & j_2 = S_{11}, & i_2 = S_{00}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

We denote $A_{0,Q}$ as C_0 .

Case (2) We now consider that the inventory level is one, that is $A_{1,1+Q}$. We note that for this case only, the inventory level changes from 1 to $1 + Q$.

Case (2a) Let $i_2 = S_{00}$, $i_3 = 0$ and $i_4 = 0$.

At the time of replenishment, the system state change from $(1, S_{00}, 0, 0)$ to

$(1+Q, S_{00}, 0, 0)$, with intensity of transition β . The sub matrix of the transition rates from $\ll 1, S_{00} \gg$ to $\ll 1+Q, S_{00} \gg$ is given by

$$[C_1^{(1)}]_{i_3 j_3} = \begin{cases} C_1^{(11)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_1^{(11)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (2b) Let $i_2 = S_{01}$, $i_3 = 0$ and $i_4 = 1$.

Replenishment of inventory takes the system state from $(1, S_{01}, 0, 1)$ to $(1+Q, S_{01}, 0, 1)$, with intensity of transition β . The sub matrix of the transition rates from $\ll 1, S_{01} \gg$ to $\ll 1+Q, S_{01} \gg$ is given by

$$[C_1^{(2)}]_{i_3 j_3} = \begin{cases} C_1^{(21)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_1^{(21)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, \quad i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

Case (2c) Let $i_2 = S_{10}$, $i_3 = 1$ and $i_4 = 0$.

Replenishment changes the state of the system from $(1, S_{10}, 1, 0)$ to $(1+Q, S_{10}, 1, 0)$, with intensity of transition β . The sub matrix of the transition rates from $\ll 1, S_{10} \gg$ to $\ll 1+Q, S_{10} \gg$ is given by

$$[C_1^{(3)}]_{i_3 j_3} = \begin{cases} C_1^{(31)}, & j_3 = i_3, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_1^{(31)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (2d) • Let $i_2 = S_{01}$, $i_3 = 0$ and $2 \leq i_4 \leq L$.

The state of the system moves from $(1, S_{01}, 0, i_4)$ to $(1+Q, S_{11}, 1, i_4 - 1)$, $i_4 \in V_2^L$, with the intensity of transition β due to replenishment.

• Let $i_2 = S_{01}$, $1 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

When a replenishment takes place at $(1, S_{01}, i_3, i_4)$, the inventory level reaches to $(1+Q, S_{11}, i_3, i_4)$, $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity of transition β .

The sub matrix of these transition rates from $\ll 1, S_{01} \gg$ to $\ll 1 + Q, S_{11} \gg$ is given by

$$[C_1^{(4)}]_{i_3 j_3} = \begin{cases} C_1^{(41)}, & j_3 = 1, & i_3 = 0, \\ C_1^{(42)}, & j_3 = i_3, & i_3 \in V_1^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_1^{(41)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4 - 1, & i_4 \in V_2^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C_1^{(42)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

Case (2e) • Let $i_2 = S_{10}$, $1 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

A transition from $(1, S_{10}, i_3, i_4)$ to $(1 + Q, S_{11}, i_3, i_4)$, $Q = S - s$, for $i_3 \in V_1^L$, $i_4 \in V_1^L$, takes place with intensity β when a replenishment for Q items occur.

• Let $i_2 = S_{10}$, $2 \leq i_3 \leq L$ and $i_4 = 0$.

At the time of replenishment the system takes from $(1, S_{10}, i_3, 0)$ to $(1 + Q, S_{11}, i_3 - 1, 1)$, $i_3 \in V_2^L$, with the intensity of transition β .

The sub matrix of these transition rates from $\ll 1, S_{10} \gg$ to $\ll 1 + Q, S_{11} \gg$ is given by

$$[C_1^{(5)}]_{i_3 j_3} = \begin{cases} C_1^{(51)}, & j_3 = i_3, & i_3 \in V_1^L, \\ C_1^{(52)}, & j_3 = i_3 - 1, & i_3 \in V_2^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[C_1^{(51)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C_1^{(52)}]_{i_4 j_4} = \begin{cases} \beta, & j_4 = 1, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Hence,

$$[A_{1,1+Q}]_{i_2 j_2} = \begin{cases} C_1^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ C_1^{(2)}, & j_2 = i_2, & i_2 = S_{01}, \\ C_1^{(3)}, & j_2 = i_2, & i_2 = S_{10}, \\ C_1^{(4)}, & j_2 = S_{11}, & i_2 = S_{01}, \\ C_1^{(5)}, & j_2 = S_{11}, & i_2 = S_{10}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

We denote $A_{1,1+Q}$ as C_1 .

Case (3) We now consider the case when the inventory level lies between two to s . We note that for this case, only the inventory level changes from i_1 to $i_1 + Q, i_1 \in V_2^s$. The other system state does not change. Hence, $[A_{i_1, i_1+Q}]_{i_2 j_2} = \beta I_{(3+L^2)(3+L^2)}$.
 More explicitly, for $i_1 \in V_2^s$

$$[C]_{i_2 j_2} = \begin{cases} C^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ C^{(2)}, & j_2 = i_2, & i_2 = S_{10}, \\ C^{(3)}, & j_2 = i_2, & i_2 = S_{01}, \\ C^{(4)}, & j_2 = i_2, & i_2 = S_{11}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where,

$$[C^{(1)}]_{i_3 j_3} = \begin{cases} J_0, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[J_0]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C^{(2)}]_{i_3 j_3} = \begin{cases} J_1, & j_3 = i_3, & i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[J_1]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C^{(3)}]_{i_3 j_3} = \begin{cases} J_2, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[J_2]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C^{(4)}]_{i_3 j_3} = \begin{cases} J_3, & j_3 = i_3, & i_3 \in V_1^L \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[J_3]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

Hence, we denote $A_{i_1, i_1+Q}, i_1 \in V_2^s$ as C.

Next we consider the case $A_{i_1, i_1-1}, i_1 \in V_1^s$. This will occur only when the service completion of the customer or any one of $i_1 (i_1 \in V_1^s)$ item fails.

Case (4) Now we assume that inventory level is one, that is $A_{1,0}$. For this, the following cases occur:

Case (4a) Let $i_2 = S_{00}, i_3 = 0$ and $i_4 = 0$.

Due to perishability of the inventory takes the inventory level from $(1, S_{00}, 0, 0)$ to $(0, S_{00}, 0, 0)$, with intensity of transition γ . The sub matrix of the transition rates from $\ll 1, S_{00} \gg$ to $\ll 0, S_{00} \gg$ is given by

$$[B_1^{(1)}]_{i_3 j_3} = \begin{cases} B_1^{(11)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_1^{(11)}]_{i_4 j_4} = \begin{cases} \gamma, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (4b) Let $i_2 = S_{10}, 1 \leq i_3 \leq L$ and $0 \leq i_4 \leq L$.

Due to the service completion of a customer in queue-1, both queue-1 size and inventory level decrease by one and the state of the process moves from $(1, S_{10}, i_3, i_4)$ to $(0, S_{00}, i_3 - 1, i_4), i_3 \in V_1^L, i_4 \in V_0^L$, with intensity of transition μ_1 . The sub matrix of the transition rates from $\ll 1, S_{10} \gg$ to $\ll 0, S_{00} \gg$ is given by

$$[B_1^{(2)}]_{i_3 j_3} = \begin{cases} B_1^{(21)}, & j_3 = i_3 - 1, \quad i_3 \in V_1^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_1^{(21)}]_{i_4 j_4} = \begin{cases} \mu_1, & j_4 = i_4, \quad i_4 \in V_0^L, \\ 0, & \text{otherwise,} \end{cases}$$

Case (4c) Let $i_2 = S_{01}, 0 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

The service of a customer in queue-2 is completed, both queue-2 size and inventory level decrease by one and the state of the process moves from $(1, S_{01}, i_3, i_4)$ to $(0, S_{00}, i_3, i_4 - 1), i_3 \in V_0^L, i_4 \in V_1^L$, with intensity of transition μ_2 . The sub matrix of the transition rates from $\ll 1, S_{01} \gg$ to $\ll 0, S_{00} \gg$ is given by

$$[B_1^{(3)}]_{i_3 j_3} = \begin{cases} B_1^{(31)}, & j_3 = i_3, \quad i_3 \in V_0^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_1^{(31)}]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = i_4 - 1, \quad i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

Hence, $A_{1,0}$ is given by

$$[A_{1,0}]_{i_2 j_2} = \begin{cases} B_1^{(1)}, & j_2 = i_2, \quad i_2 = S_{00}, \\ B_1^{(2)}, & j_2 = S_{00}, \quad i_2 = S_{10}, \\ B_1^{(3)}, & j_2 = S_{00}, \quad i_2 = S_{01}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

We denote $A_{1,0}$ as B_1 .

Case (5) Now we consider the case that the inventory level is 2, that is $A_{2,1}$. For this, the following cases occur:

Case (5a) Let $i_2 = S_{00}$, $i_3 = 0$ and $i_4 = 0$.

Perishability of the inventory takes the inventory level from $(2, S_{00}, 0, 0)$ to $(1, S_{00}, 0, 0)$, with intensity of transition 2γ . The sub matrix of the transition rates from $\ll 2, S_{00} \gg$ to $\ll 1, S_{00} \gg$ is given by

$$[B_2^{(1)}]_{i_3 j_3} = \begin{cases} B_2^{(11)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(11)}]_{i_4 j_4} = \begin{cases} 2\gamma, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (5b) Let $i_2 = S_{10}$, $i_3 = 1$ and $i_4 = 0$.

- A transition from $(2, S_{10}, 1, 0)$ to $(1, S_{10}, 1, 0)$, takes place when any of the item perishes, with intensity of transition γ . The sub matrix of the transition rates from $\ll 2, S_{10} \gg$ to $\ll 1, S_{10} \gg$ is given by

$$[B_2^{(2)}]_{i_3 j_3} = \begin{cases} B_2^{(21)}, & j_3 = i_3, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(21)}]_{i_4 j_4} = \begin{cases} \gamma, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

- At the time of service completion of a customer in queue-1, both queue-1 size and inventory level decrease by one then, the state of the system changes from $(2, S_{10}, 1, 0)$ to $(1, S_{00}, 0, 0)$, with intensity of transition μ_1 . The sub matrix of the transition rates from $\ll 2, S_{10} \gg$ to $\ll 1, S_{00} \gg$ is given by

$$[B_2^{(3)}]_{i_3 j_3} = \begin{cases} B_2^{(31)}, & j_3 = 0, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(31)}]_{i_4 j_4} = \begin{cases} \mu_1, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (5c) Let $i_2 = S_{01}$, $i_3 = 0$ and $i_4 = 1$.

- Due to the perishability, the inventory takes the inventory level from $(2, S_{01}, 0, 1)$ to $(1, S_{01}, 0, 1)$, with intensity of transition γ . The sub matrix of the transition rates from $\ll 2, S_{01} \gg$ to $\ll 1, S_{01} \gg$ is given by

$$[B_2^{(4)}]_{i_3 j_3} = \begin{cases} B_2^{(41)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(41)}]_{i_4 j_4} = \begin{cases} \gamma, & j_4 = i_4, \quad i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

- The service of a customer in queue-2 is completed, both queue-2 size and inventory level decrease by one then, the state of the system changes from $(2, S_{01}, 0, 1)$ to $(1, S_{00}, 0, 0)$, with intensity of transition μ_2 . The sub matrix of the transition rates from $\ll 2, S_{01} \gg$ to $\ll 1, S_{00} \gg$ is given by

$$[B_2^{(5)}]_{i_3 j_3} = \begin{cases} B_2^{(51)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(51)}]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = 0, \quad i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

case (5d) Let $i_2 = S_{11}$, $1 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

- At the time of service completion of a customer in queue-1, both queue-1 size and the inventory level decrease by one and takes the system state from $(2, S_{11}, i_3, i_4)$ to $(1, S_{01}, i_3 - 1, i_4)$, $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity of transition μ_1 . The sub matrix of the transition rates from $\ll 2, S_{11} \gg$ to $\ll 1, S_{01} \gg$ is given by

$$[B_2^{(6)}]_{i_3 j_3} = \begin{cases} B_2^{(61)}, & j_3 = i_3 - 1, \quad i_3 \in V_1^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(61)}]_{i_4 j_4} = \begin{cases} \mu_1, & j_4 = i_4, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

- At the time of service completion of a customer in the queue-2, both queue-2 size and the inventory level decrease by one and takes the system state from $(2, S_{11}, i_3, i_4)$ to $(1, S_{10}, i_3, i_4 - 1)$, $i_3 \in V_1^L, i_4 \in V_1^L$, with intensity of transition μ_2 . The sub matrix of the transition rates from $\ll 2, S_{11} \gg$ to $\ll 1, S_{10} \gg$ is given by

$$[B_2^{(7)}]_{i_3 j_3} = \begin{cases} B_2^{(71)}, & j_3 = i_3, & i_3 \in V_1^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_2^{(71)}]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = i_4 - 1, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

Hence, $A_{2,1}$ is given by

$$[A_{2,1}]_{i_2 j_2} = \begin{cases} B_2^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ B_2^{(2)}, & j_2 = i_2, & i_2 = S_{10}, \\ B_2^{(3)}, & j_2 = S_{00}, & i_2 = S_{10}, \\ B_2^{(4)}, & j_2 = i_2, & i_2 = S_{01}, \\ B_2^{(5)}, & j_2 = S_{00}, & i_2 = S_{01}, \\ B_2^{(6)}, & j_2 = S_{01}, & i_2 = S_{11}, \\ B_2^{(7)}, & j_2 = S_{10}, & i_2 = S_{11}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

We denote $A_{2,1}$ as B_2 .

Case (6) Now, we assume that the inventory level lies between three to S , that is

$$A_{i_1, i_1 - 1}, \quad i_1 \in V_3^S. \text{ For this, we have the following cases:}$$

case (6a) Let $i_2 = S_{00}, i_3 = 0$ and $i_4 = 0$.

A transition from $(i_1, S_{00}, 0, 0)$ to $(i_1 - 1, S_{00}, 0, 0)$, will take place when any one of i_1 items perishes at a rate of γ , and the intensity for this transition is $i_1 \gamma, i_1 \in V_3^S$. The sub matrix of the transition rates from $\ll i_1, S_{00} \gg$ to $\ll i_1 - 1, S_{00} \gg$ is given by

$$[B_{i_1}^{(1)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(11)}, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(11)}]_{i_4 j_4} = \begin{cases} i_1 \gamma, & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

case (6b) Let $i_2 = S_{10}$, $i_3 = 1$ and $i_4 = 0$.

- Due to perishability of the inventory, the inventory level changes from $(i_1, S_{10}, 1, 0)$ to $(i_1 - 1, S_{10}, 1, 0)$, with the intensity of transition $(i_1 - 1)\gamma$, $i_1 \in V_3^S$. The sub matrix of the transition rates from $\ll i_1, S_{10} \gg$ to $\ll i_1 - 1, S_{10} \gg$ is given by

$$[B_{i_1}^{(2)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(21)}, & j_3 = i_3, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(21)}]_{i_4 j_4} = \begin{cases} (i_1 - 1)\gamma, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

- When server-1 completes the service, the state of the system changes from $(i_1, S_{10}, 1, 0)$ to $(i_1 - 1, S_{00}, 0, 0)$, with the intensity of transition μ_1 . The sub matrix of the transition rates from $\ll i_1, S_{10} \gg$ to $\ll i_1 - 1, S_{00} \gg$ is given by

$$[B_{i_1}^{(3)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(31)}, & j_3 = 0, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(31)}]_{i_4 j_4} = \begin{cases} \mu_1, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

Case (6c) Let $i_2 = S_{01}$, $i_3 = 0$ and $i_4 = 1$.

- A transition from $(i_1, S_{01}, 0, 1)$ to $(i_1 - 1, S_{01}, 0, 1)$, takes place when any of i_1 items perishes at a rate of γ , thus the intensity of transition $(i_1 - 1)\gamma$, $i_1 \in V_3^S$. The sub matrix of the transition rates from $\ll i_1, S_{01} \gg$ to $\ll i_1 - 1, S_{01} \gg$ is given by

$$[B_{i_1}^{(4)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(41)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(41)}]_{i_4 j_4} = \begin{cases} (i_1 - 1)\gamma, & j_4 = i_4, \quad i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

- Due to service completion of a customer in queue-2, the state of the system changes from $(i_1, S_{01}, 0, 1)$ to $(i_1 - 1, S_{00}, 0, 0)$, with the intensity of transition μ_2 . The sub matrix of the transition rates from $\ll i_1, S_{01} \gg$ to $\ll i_1 - 1, S_{00} \gg$ is given by

$$[B_{i_1}^{(5)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(51)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(51)}]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = 0, & i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

case (6d) Let $i_2 = S_{11}$, $i_3 = 1$ and $i_4 = 1$.

- At the time of service completion of a customer in queue-1, the state of the system changes from $(i_1, S_{11}, 1, 1)$ to $(i_1 - 1, S_{01}, 0, 1)$, with the intensity of transition μ_1 . The sub matrix of the transition rates from $\ll i_1, S_{11} \gg$ to $\ll i_1 - 1, S_{01} \gg$ is given by

$$[B_{i_1}^{(6)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(61)}, & j_3 = 0, & i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(61)}]_{i_4 j_4} = \begin{cases} \mu_1, & j_4 = i_4, & i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

- The service of a customer in queue-2 is completed, then the state of the system moves from $(i_1, S_{11}, 1, 1)$ to $(i_1 - 1, S_{10}, 1, 0)$, with the intensity of transition μ_2 . The sub matrix of the transition rates from $\ll i_1, S_{11} \gg$ to $\ll i_1 - 1, S_{10} \gg$, is given by

$$[B_{i_1}^{(7)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(71)}, & j_3 = i_3, & i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(71)}]_{i_4 j_4} = \begin{cases} \mu_2, & j_4 = 0, & i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

case (6e) • Let $i_2 = S_{11}$, $1 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

Due to perishability of the inventory, takes the inventory level from (i_1, S_{11}, i_3, i_4) to $(i_1 - 1, S_{11}, i_3, i_4)$, $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity of transition $(i_1 - 2)\gamma$, $i_1 \in V_3^S$.

- Let $i_2 = S_{11}$, $2 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.
When server-1 completes the service of a customer in queue-1, the system state moves from (i_1, S_{11}, i_3, i_4) to $(i_1 - 1, S_{11}, i_3 - 1, i_4)$, $i_3 \in V_2^L$, $i_4 \in V_1^L$, with intensity of transition μ_1 .
- Let $i_2 = S_{11}$, $1 \leq i_3 \leq L$ and $2 \leq i_4 \leq L$.
When server-2 completes the service of a customer in queue-2, the system state moves from (i_1, S_{11}, i_3, i_4) to $(i_1 - 1, S_{11}, i_3, i_4 - 1)$, $i_3 \in V_1^L$, $i_4 \in V_2^L$, with intensity of transition μ_2 .

Note that in the following cases jockeying of a customer will occur during the service completion of a customer in any of the queues.

- Let $i_2 = S_{11}$, $i_3 = 1$ and $2 \leq i_4 \leq L$.
At the time of service completion of a customer in queue-1, the state moves from $(i_1, S_{11}, 1, i_4)$ to $(i_1 - 1, S_{11}, 1, i_4 - 1)$, $i_4 \in V_2^L$, with intensity of transition μ_1 .
- Let $i_2 = S_{11}$, $2 \leq i_3 \leq L$ and $i_4 = 1$.
Due to service completion in queue-2, the state moves from $(i_1, S_{11}, i_3, 1)$ to $(i_1 - 1, S_{11}, i_3 - 1, 1)$, $i_3 \in V_2^L$, with intensity of transition μ_2 . The sub matrix of the transition rates from $\ll i_1, S_{11} \gg$ to $\ll i_1 - 1, S_{11} \gg$ is given by

$$[B_{i_1}^{(8)}]_{i_3 j_3} = \begin{cases} B_{i_1}^{(81)}, & j_3 = i_3, & i_3 = 1, \\ B_{i_1}^{(82)}, & j_3 = i_3 - 1, & i_3 \in V_2^L, \\ B_{i_1}^{(83)}, & j_3 = i_3, & i_3 \in V_2^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where

$$[B_{i_1}^{(81)}]_{i_4 j_4} = \begin{cases} (i_1 - 2)\gamma, & j_4 = i_4, & i_4 \in V_1^L, \\ \mu_1 + \mu_2, & j_4 = i_4 - 1, & i_4 \in V_2^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[B_{i_1}^{(82)}]_{i_4 j_4} = \begin{cases} \mu_1 + \mu_2, & j_4 = 1, & i_4 = 1, \\ \mu_1, & j_4 = i_4, & i_4 \in V_2^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[B_{i_1}^{(83)}]_{i_4 j_4} = \begin{cases} (i_1 - 2)\gamma, & j_4 = i_4, & i_4 \in V_1^L, \\ \mu_2, & j_4 = i_4 - 1, & i_4 \in V_2^L, \\ 0, & \text{otherwise,} \end{cases}$$

Hence $A_{i_1, i_1 - 1}$ is given by

$$[A_{i_1, i_1 - 1}]_{i_2 j_2} = \begin{cases} B_{i_1}^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ B_{i_1}^{(2)}, & j_2 = i_2, & i_2 = S_{10}, \\ B_{i_1}^{(3)}, & j_2 = S_{00}, & i_2 = S_{10}, \\ B_{i_1}^{(4)}, & j_2 = i_2, & i_2 = S_{01}, \\ B_{i_1}^{(5)}, & j_2 = S_{00}, & i_2 = S_{01}, \\ B_{i_1}^{(6)}, & j_2 = S_{01}, & i_2 = S_{11}, \\ B_{i_1}^{(7)}, & j_2 = S_{10}, & i_2 = S_{11}, \\ B_{i_1}^{(8)}, & j_2 = i_2, & i_2 = S_{11}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

We denote A_{i_1, i_1-1} as B_{i_1} , $i_1 = 3, 4, \dots, S$.

Finally, we consider the case A_{i_1, i_1} , $i_1 = 0, 1, \dots, S$. This will occur only when the inventory level remains unchanged. Here, due to each of the following mutually exclusive cases, a transition results in:

1. An arrival of customer may occur,
2. Impatience of a customer may occur.

Case (7) When the inventory level is zero, that is $A_{0,0}$, we have the following cases.

Case (7a) Let $i_2 = S_{00}$, $i_3 = i_4$.

- An arrival of a customer may choose queue-1, then the state of the arrival process moves from $(0, S_{00}, i_3, i_4)$ to $(0, S_{00}, i_3 + 1, i_4)$, $i_3 \in V_0^{L-1}$, $i_4 \in V_0^{L-1}$, with intensity of transition $p\lambda$.
- An arrival of a customer may choose queue-2, then the state of the arrival process moves from $(0, S_{00}, i_3, i_4)$ to $(0, S_{00}, i_3, i_4 + 1)$, $i_3 \in V_0^{L-1}$, $i_4 \in V_0^{L-1}$, with intensity of transition $q\lambda$.

Case (7b) Let $i_2 = S_{00}$, $0 \leq i_3 \leq L - 1$ and $1 \leq i_4 \leq L$.

If $i_3 < i_4$, an arrival of a customer increases the number of customer waiting in queue-1 by one, then the state of system moves from $(0, S_{00}, i_3, i_4)$ to $(0, S_{00}, i_3 + 1, i_4)$, $i_3 \in V_0^{L-1}$, $i_4 \in V_1^L$, with the intensity of transition λ .

Case (7c) Let $i_2 = S_{00}$, $1 \leq i_3 \leq L$ and $0 \leq i_4 \leq L - 1$.

If $i_3 > i_4$, an arrival of a customer increases the number of customer waiting in queue-2 by one, then the state of system moves from $(0, S_{00}, i_3, i_4)$ to $(0, S_{00}, i_3, i_4 + 1)$, $i_3 \in V_1^L$, $i_4 \in V_0^{L-1}$, with the intensity of transition λ .

Case (7d) Let $i_2 = S_{00}$, $1 \leq i_3 \leq L$ and $0 \leq i_4 \leq L$.

A customer leaves from queue-1 without getting service and the state of the process moves from $(0, S_{00}, i_3, i_4)$ to $(0, S_{00}, i_3 - 1, i_4)$, $i_3 \in V_1^L$, $i_4 \in V_0^L$, with intensity $i_3\alpha_1$.

Case (7e) Let $i_2 = S_{00}$, $0 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

A customer leaves from queue-2 without getting service and the state of the process moves from $(0, S_{00}, i_3, i_4)$ to $(0, S_{00}, i_3, i_4 - 1)$, $i_3 \in V_0^L$, $i_4 \in V_1^L$, with intensity $i_4\alpha_1$.

The transition rate for any of the transitions not considered in above cases from 7a to 7e, when inventory level is zero, is zero. The intensity of passage in the state $(0, i_2, i_3, i_4)$ is given by

$$-\sum_{(0, i_2, i_3, i_4) \neq (0, j_2, j_3, j_4)} a((0, i_2, i_3, i_4); (0, j_2, j_3, j_4))$$

Using the above arguments from cases(7a-7e), we have constructed the following matrices

For $i_3 = 0, 1, 2, \dots, L - 1$

$$[A^{(0i_3)}]_{i_4 j_4} = \begin{cases} p\lambda, & j_4 = i_4, & i_4 = i_3, \\ \lambda, & j_4 = i_4, & i_3 < i_4 \leq L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 1, 2, \dots, L$

$$[A_{(i_3 0)}]_{i_4 j_4} = \begin{cases} i_3 \alpha_1, & j_4 = i_4, & i_4 \in V_0^L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 0, 1, 2, \dots, L - 1$

$$[A_{(i_3 i_3)}]_{i_4 j_4} = \begin{cases} q\lambda, & j_4 = i_4 + 1, & i_4 = i_3, \\ \lambda, & j_4 = i_4 + 1, & 0 \leq i_4 < i_3, \\ i_4 \alpha_2, & j_4 = i_4 - 1, & i_4 \in V_1^L, \\ -(\lambda + i_3 \alpha_1 + i_4 \alpha_2 + \beta), & j_4 = i_4, & i_4 \in V_0^L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = L$

$$[A_{(i_3 i_3)}]_{i_4 j_4} = \begin{cases} \lambda, & j_4 = i_4 + 1, & i_4 \in V_0^{L-1}, \\ i_4 \alpha_2, & j_4 = i_4 - 1, & i_4 \in V_1^L, \\ -(i_3 \alpha_1 + i_4 \alpha_2 + \beta + \lambda \delta_{i_4 L}), & j_4 = i_4, & i_4 \in V_0^L, \\ 0, & \text{otherwise,} \end{cases}$$

Combining these matrices in a suitable form, we get

$$[A_0^{(1)}]_{i_3 j_3} = \begin{cases} A^{(0i_3)}, & j_3 = i_3 + 1, & i_3 \in V_0^{L-1}, \\ A_{(i_3 0)}, & j_3 = i_3 - 1, & i_3 \in V_1^L, \\ A_{(i_3 i_3)}, & j_3 = i_3, & i_3 \in V_0^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

Hence, the matrix A_{00} is given by

$$[A_0]_{i_2 j_2} = \begin{cases} A_0^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

and is denoted by A_0 .

Case (8) When the inventory level is one, that is $A_{1,1}$, we have the following cases.

Case (8a) Let $i_2 = S_{00}, i_3 = i_4$.

- If at arrival a customer choose queue-1, then the state of the system moves from $(1, S_{00}, 0, 0)$ to $(1, S_{10}, 1, 0)$, with intensity of transition $p\lambda$.
- If at arrival a customer choose queue-2, then the state of the system moves from $(1, S_{00}, 0, 0)$ to $(1, S_{01}, 0, 1)$, with intensity of transition $q\lambda$.

Case (8b) Let $i_2 = S_{10}$, $i_3 = i_4$.

- At arrival a customer may choose queue-1, then the state of the system moves from $(1, S_{10}, i_3, i_4)$ to $(1, S_{10}, i_3 + 1, i_4)$, $i_3 \in V_1^{L-1}$, $i_4 \in V_1^{L-1}$, with intensity of transition $p\lambda$.
- At arrival a customer may choose queue-2, then the state of the system moves from $(1, S_{10}, i_3, i_4)$ to $(1, S_{10}, i_3, i_4 + 1)$, $i_3 \in V_1^{L-1}$, $i_4 \in V_1^{L-1}$, with intensity of transition $q\lambda$.

Case (8c) Let $i_2 = S_{01}$, $i_3 = i_4$.

- If at arrival a customer choose queue-1, then the system changes from $(1, S_{01}, i_3, i_4)$ to $(1, S_{01}, i_3 + 1, i_4)$, $i_3 \in V_1^{L-1}$, $i_4 \in V_1^{L-1}$, with intensity of transition $p\lambda$.
- If at arrival a customer choose queue-2, then the system changes from $(1, S_{01}, i_3, i_4)$ to $(1, S_{01}, i_3, i_4 + 1)$, $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity of transition $q\lambda$.

case (8d) If $i_3 < i_4$, arrival of a customer increases the number of customer waiting in queue-1 by one.

- Let $i_2 = S_{10}$, $1 \leq i_3 \leq L - 1$ and $2 \leq i_4 \leq L$, then the state of arrival process moves from $(1, S_{10}, i_3, i_4)$ to $(1, S_{10}, i_3 + 1, i_4)$, $i_3 \in V_1^{L-1}$, $i_4 \in V_2^L$, with the intensity of transition λ .
- Let $i_2 = S_{01}$, $0 \leq i_3 \leq L - 1$ and $1 \leq i_4 \leq L$, then the state of arrival process moves from $(1, S_{01}, i_3, i_4)$ to $(1, S_{01}, i_3 + 1, i_4)$, $i_3 \in V_0^{L-1}$, $i_4 \in V_1^L$, with the intensity of transition λ .

Case (8e) If $i_3 > i_4$, arrival of a customer increases the number of customer waiting in queue-2 by one.

- Let $i_2 = S_{10}$, $1 \leq i_3 \leq L$ and $0 \leq i_4 \leq L - 1$, then the state of arrival process moves from $(1, S_{10}, i_3, i_4)$ to $(1, S_{10}, i_3, i_4 + 1)$, $i_3 \in V_1^L$, $i_4 \in V_0^{L-1}$, with the intensity of transition λ .
- Let $i_2 = S_{01}$, $2 \leq i_3 \leq L$ and $1 \leq i_4 \leq L - 1$, then the state of arrival process moves from $(1, S_{01}, i_3, i_4)$ to $(1, S_{01}, i_3, i_4 + 1)$, $i_3 \in V_2^L$, $i_4 \in V_1^{L-1}$, with the intensity of transition λ .

Case (8f) Let $i_2 = S_{10}$, $2 \leq i_3 \leq L$ and $0 \leq i_4 \leq L$.

The waiting customer leaves from queue-1 without getting service and the state of the process moves from $(1, S_{10}, i_3, i_4)$ to $(1, S_{10}, i_3 - 1, i_4)$, $i_3 \in V_2^L$, $i_4 \in V_0^L$, with intensity of transition $(i_3 - 1)\alpha_1$.

Case (8g) Let $i_2 = S_{10}$, $1 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

The waiting customer leaves from queue-2 without getting service and the state of the process moves from $(1, S_{10}, i_3, i_4)$ to $(1, S_{10}, i_3, i_4 - 1)$, $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity $i_4\alpha_2$.

Case (8h) Let $i_2 = S_{01}$, $1 \leq i_3 \leq L$ and $1 \leq i_4 \leq L$.

A customer leaves from queue-1 without getting service and the state of the process moves from $(1, S_{01}, i_3, i_4)$ to $(1, S_{01}, i_3 - 1, i_4)$, $i_3 \in V_1^L$, $i_4 \in V_1^L$, with intensity of transition $i_3\alpha_1$.

Case (8i) Let $i_2 = S_{01}$, $0 \leq i_3 \leq L$ and $2 \leq i_4 \leq L$.

A customer leaves from queue-2 without getting service and the state of the process moves from $(1, S_{01}, i_3, i_4)$ to $(1, S_{01}, i_3, i_4 - 1)$, $i_3 \in V_0^L$, $i_4 \in V_2^L$, with intensity $(i_4 - 1)\alpha_2$.

The transition rate for any of the transitions not considered in above cases from 8a to 8i, when inventory level is one, is zero. The intensity of passage in the state $(1, i_2, i_3, i_4)$ is given by

$$-\sum_{(1, i_2, i_3, i_4) \neq (1, j_2, j_3, j_4)} a((1, i_2, i_3, i_4); (1, j_2, j_3, j_4))$$

Using the above arguments from case(8a to 8i), we have constructed the following matrices

$$[A_1]_{i_2 j_2} = \begin{cases} A_1^{(1)}, & j_2 = i_2, & i_2 = S_{00}, \\ A_1^{(2)}, & j_2 = S_{10}, & i_2 = S_{00}, \\ A_1^{(3)}, & j_2 = S_{01}, & i_2 = S_{00}, \\ A_1^{(4)}, & j_2 = i_2, & i_2 = S_{10}, \\ A_1^{(5)}, & j_2 = i_2, & i_2 = S_{01}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_1^{(1)}]_{i_3 j_3} = \begin{cases} A_1^{(11)}, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_1^{(11)}]_{i_4 j_4} = \begin{cases} -(\lambda + \gamma + \beta), & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_1^{(2)}]_{i_3 j_3} = \begin{cases} A_1^{(21)}, & j_3 = 1, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_1^{(21)}]_{i_4 j_4} = \begin{cases} p\lambda, & j_4 = i_4, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_1^{(3)}]_{i_3 j_3} = \begin{cases} A_1^{(31)}, & j_3 = i_3, & i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_1^{(31)}]_{i_4 j_4} = \begin{cases} q\lambda, & j_4 = 1, & i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_1^{(4)}]_{i_3 j_3} = \begin{cases} M^{(0i_3)}, & j_3 = i_3 + 1, & i_3 \in V_1^{L-1}, \\ M_{(i_3 0)}, & j_3 = i_3 - 1, & i_3 \in V_2^L, \\ M_{(i_3 i_3)}, & j_3 = i_3, & i_3 \in V_1^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

For $i_3 = 1, 2, \dots, L - 1$

$$[M^{(0i_3)}]_{i_4 j_4} = \begin{cases} p\lambda, & j_4 = i_4, & i_4 = i_3, \\ \lambda, & j_4 = i_4, & i_3 < i_4 \leq L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 2, 3, \dots, L$

$$[M_{(i_3 0)}]_{i_4 j_4} = \begin{cases} (i_3 - 1)\alpha_1, & j_4 = i_4, & i_4 \in V_0^L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 1, 2, \dots, L - 1$

$$[M_{(i_3 i_3)}]_{i_4 j_4} = \begin{cases} -(\lambda + (i_3 - 1)\alpha_1 + i_4\alpha_2 + \mu_1 + \beta), & j_4 = i_4, & i_4 \in V_0^L, \\ q\lambda, & j_4 = i_4 + 1, & i_4 = i_3, \\ \lambda, & j_4 = i_4 + 1, & 0 \leq i_4 < i_3, \\ i_4\alpha_2, & j_4 = i_4 - 1, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = L$

$$[M_{(i_3 i_3)}]_{i_4 j_4} = \begin{cases} \lambda, & j_4 = i_4 + 1, & i_4 \in V_0^{L-1}, \\ i_4\alpha_2, & j_4 = i_4 - 1, & i_4 \in V_1^L, \\ -(\lambda\delta_{i_4 L} + (i_3 - 1)\alpha_1 + i_4\alpha_2 + \mu_1 + \beta), & j_4 = i_4, & i_4 \in V_0^L, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_1^{(5)}]_{i_3 j_3} = \begin{cases} N^{(0i_3)}, & j_3 = i_3 + 1, & i_3 \in V_0^{L-1}, \\ N_{(i_3 0)}, & j_3 = i_3 - 1, & i_3 \in V_1^L, \\ N_{(i_3 i_3)}, & j_3 = i_3, & i_3 \in V_0^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

For $i_3 = 0, 1, 2, \dots, L - 1$

$$[N^{(0i_3)}]_{i_4 j_4} = \begin{cases} p\lambda\delta_{i_3 0}, & j_4 = i_4, & i_4 = i_3, \\ \lambda, & j_4 = i_4, & i_3 < i_4 \leq L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 1, 2, \dots, L$

$$[N_{(i_3 0)}]_{i_4 j_4} = \begin{cases} i_3\alpha_1, & j_4 = i_4, & i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 0, 1, 2, \dots, L - 1$

$$[N_{(i_3 i_3)}]_{i_4 j_4} = \begin{cases} -(\lambda + i_3\alpha_1 + (i_4 - 1)\alpha_2 + \mu_2 + \beta), & j_4 = i_4, & i_4 \in V_1^L, \\ q\lambda\delta_{i_3 0}, & j_4 = i_4 + 1, & i_4 = i_3, \\ \lambda, & j_4 = i_4 + 1, & 1 \leq i_4 < i_3, \\ (i_4 - 1)\alpha_2, & j_4 = i_4 - 1, & i_4 \in V_2^L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = L$

$$[N_{(i_3 i_3)}]_{i_4 j_4} = \begin{cases} \lambda, & j_4 = i_4 + 1, \quad i_4 \in V_1^{L-1}, \\ (i_4 - 1)\alpha_2, & j_4 = i_4 - 1, \quad i_4 \in V_2^L, \\ -(\lambda\delta_{i_4 L} + i_3\alpha_1 + (i_4 - 1)\alpha_2 + \mu_2 + \beta), & j_4 = i_4, \quad i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

Here we denote $A_{1,1}$, as A_1 .

Arguments similar to above yield, for $i_1 = 2, 3, \dots, S$.

$$[A_{i_1}]_{i_2 j_2} = \begin{cases} A_{i_1}^{(1)}, & j_2 = i_2, \quad i_2 = S_{00}, \\ A_{i_1}^{(2)}, & j_2 = S_{10}, \quad i_2 = S_{00}, \\ A_{i_1}^{(3)}, & j_2 = S_{01}, \quad i_2 = S_{00}, \\ A_{i_1}^{(4)}, & j_2 = i_2, \quad i_2 = S_{10}, \\ A_{i_1}^{(5)}, & j_2 = S_{11}, \quad i_2 = S_{10}, \\ A_{i_1}^{(6)}, & j_2 = i_2, \quad i_2 = S_{01}, \\ A_{i_1}^{(7)}, & j_2 = S_{11}, \quad i_2 = S_{01}, \\ A_{i_1}^{(8)}, & j_2 = i_2, \quad i_2 = S_{11}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(1)}]_{i_3 j_3} = \begin{cases} A_{i_1}^{(11)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{(i_1)}^{(1)}]_{i_4 j_4} = \begin{cases} -(\lambda + i_1\gamma + \beta H(s - i_1)), & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(2)}]_{i_3 j_3} = \begin{cases} K_0, & j_3 = 1, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[K_0]_{i_4 j_4} = \begin{cases} p\lambda, & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(3)}]_{i_3 j_3} = \begin{cases} K_1, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[K_1]_{i_4 j_4} = \begin{cases} q\lambda, & j_4 = 1, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(4)}]_{i_3 j_3} = \begin{cases} A_{i_1}^{(41)}, & j_3 = i_3, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{(i_1)}^{(41)}]_{i_4 j_4} = \begin{cases} -(\lambda + \mu_1 + (i_1 - 1)\gamma + \beta H(s - i_1)), & j_4 = i_4, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(5)}]_{i_3 j_3} = \begin{cases} K_2, & j_3 = i_3, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[K_2]_{i_4 j_4} = \begin{cases} \lambda, & j_4 = 1, \quad i_4 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(6)}]_{i_3 j_3} = \begin{cases} A_{i_1}^{(61)}, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{(i_1)}^{(61)}]_{i_4 j_4} = \begin{cases} -(\lambda + \mu_2 + (i_1 - 1)\gamma + \beta H(s - i_1)), & j_4 = i_4, \quad i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(7)}]_{i_3 j_3} = \begin{cases} K_3, & j_3 = 1, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[K_3]_{i_4 j_4} = \begin{cases} \lambda, & j_4 = i_4, \quad i_4 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[A_{i_1}^{(8)}]_{i_3 j_3} = \begin{cases} G^{(0i_3)}, & j_3 = i_3 + 1, \quad i_3 \in V_1^{L-1}, \\ G_{(i_3 0)}, & j_3 = i_3 - 1, \quad i_3 \in V_2^L, \\ G_{(i_1 i_3)}, & j_3 = i_3, \quad i_3 \in V_1^L, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

For $i_3 = 1, 2, \dots, L - 1$

$$[G^{(0i_3)}]_{i_4 j_4} = \begin{cases} p\lambda, & j_4 = i_4, \quad i_4 = i_3, \\ \lambda, & j_4 = i_4, \quad i_3 < i_4 \leq L, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_3 = 2, 3, \dots, L$

$$[G_{(i_3 0)}]_{i_4 j_4} = \begin{cases} (i_3 - 1)\alpha_1, & j_4 = i_4, \quad i_4 \in V_1^L, \\ 0, & \text{otherwise,} \end{cases}$$

$(L + 1) \times 1$. $C_0^{(41)}, C_0^{(42)}, C_0^{(43)}, C_1^{(51)}$ and $C_1^{(52)}$ are matrices of size $(L + 1) \times L$. $C_1^{(2)}$ and $C_1^{(3)}$ are matrices of size $L(L + 1) \times 1$. $C_1^{(4)}$ and $C_1^{(5)}$ are matrices of size $L(L + 1) \times L^2$. $C_1^{(1)}, C_1^{(11)}, C^{(1)}, C^{(2)}, C^{(3)}, J_0, J_1, J_2, B_2^{(1)}, B_2^{(11)}, B_2^{(3)}, B_2^{(31)}, B_2^{(5)}, B_2^{(51)}, B_{i_1}^{(1)}, B_{i_1}^{(2)}, B_{i_1}^{(3)}, B_{i_1}^{(4)}, B_{i_1}^{(5)}, B_{i_1}^{(11)}, B_{i_1}^{(21)}, B_{i_1}^{(31)}, B_{i_1}^{(41)}, B_{i_1}^{(51)}, A_1^{(1)}, A_1^{(11)}, A_{i_1}^{(1)}, A_{i_1}^{(11)}, A_{i_1}^{(2)}, A_{i_1}^{(3)}, A_{i_1}^{(4)}, A_{i_1}^{(41)}, A_{i_1}^{(6)}, A_{i_1}^{(61)}, K_0$, and K_1 are square matrices of size 1. $C_1^{(41)}, C_1^{(42)}, B_2^{(61)}, B_{i_1}^{(81)}, B_{i_1}^{(82)}, B_{i_1}^{(83)}, J_3, G^{(0i_3)}, G^{(i_30)}, G^{(i_3i_3)}, N^{(0i_3)}, N^{(i_30)}$ and $N^{(i_3i_3)}$ are square matrices of size L . $C_1^{(21)}, B_{i_1}^{(61)}$ and $B_{i_1}^{(71)}$ are matrices of size $L \times 1$. $C^{(4)}, B_{(i_1)}^{(8)}$ and $A_{i_1}^{(8)}$ are square matrices of size L^2 . $B_1^{(1)}$ is a matrix of size $1 \times (L + 1)^2$. $B_1^{(2)}$ and $B_1^{(3)}$ are matrices of size $L(L + 1) \times (L + 1)^2$. $B_1^{(31)}$ and $B_2^{(71)}$ are matrices of size $L \times (L + 1)$. $B_1^{(21)}, A^{(0i_3)}, A_{(i_30)}, A_{(i_3i_3)}, M^{(0i_3)}, M_{(i_30)}$ and $M_{(i_3i_3)}$ are square matrices of size $L + 1$. $B_2^{(2)}, B_2^{(4)}, A_1^{(2)}$ and $A_1^{(3)}$ are matrices of size $1 \times L(L + 1)$. $B_1^{(11)}, B_2^{(21)}$ and $A_1^{(21)}$ are matrices of size $1 \times (L + 1)$. $B_2^{(41)}, A_1^{(31)}, K_2$ and K_3 are matrices of size $1 \times L$. $B_2^{(6)}$ and $B_2^{(7)}$ are matrices of size $L^2 \times L(L + 1)$. $B_{i_1}^{(6)}$ and $B_{i_1}^{(7)}$ are matrices of size $L^2 \times 1$. $A_1^{(4)}$ and $A_1^{(5)}$ are square matrices of size $L(L + 1)$. $A_{i_1}^{(5)}$ and $A_{i_1}^{(7)}$ are matrices of order $1 \times L^2$.

3.1. Steady state analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), Y(t), X_1(t), X_2(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1, i_2, i_3, i_4)} = \lim_{t \rightarrow \infty} Pr[L(t) = i_1, Y(t) = i_2, X_1(t) = i_3, X_2(t) = i_4 | L(0), Y(0), X_1(0), X_2(0)] \text{ exists.}$$

Let $\Phi = (\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(S)})$, each vector $\Phi^{(i_1)}$ being partitioned as follows

$$\begin{aligned} \Phi^{(0)} &= (\Phi^{(0, S_{00})}), \\ \Phi^{(1)} &= (\Phi^{(1, S_{00})}, \Phi^{(1, S_{01})}, \Phi^{(1, S_{10})}), \\ \Phi^{(i_1)} &= (\Phi^{(i_1, S_{00})}, \Phi^{(i_1, S_{01})}, \Phi^{(i_1, S_{10})}, \Phi^{(i_1, S_{11})}), i_1 \in V_2^S; \end{aligned}$$

where

$$\begin{aligned} \Phi^{(0, S_{00})} &= (\Phi^{(0, S_{00}, 0)}, \Phi^{(0, S_{00}, 1)}, \Phi^{(0, S_{00}, 2)}, \dots, \Phi^{(0, S_{00}, L)}), \\ \Phi^{(1, S_{00})} &= (\Phi^{(1, S_{00}, 0)}), \\ \Phi^{(1, S_{01})} &= (\Phi^{(1, S_{01}, 0)}, \Phi^{(1, S_{01}, 1)}, \Phi^{(1, S_{01}, 2)}, \dots, \Phi^{(1, S_{01}, L)}), \\ \Phi^{(1, S_{10})} &= (\Phi^{(1, S_{10}, 1)}, \Phi^{(1, S_{10}, 2)}, \Phi^{(1, S_{10}, 3)}, \dots, \Phi^{(1, S_{10}, L)}), \\ \Phi^{(i_1, S_{00})} &= (\Phi^{(i_1, S_{00}, 0)}), i_1 \in V_2^S; \\ \Phi^{(i_1, S_{10})} &= (\Phi^{(i_1, S_{10}, 1)}), i_1 \in V_2^S; \\ \Phi^{(i_1, S_{01})} &= (\Phi^{(i_1, S_{01}, 0)}), i_1 \in V_2^S; \\ \Phi^{(i_1, S_{11})} &= (\Phi^{(i_1, S_{11}, 1)}, \Phi^{(i_1, S_{11}, 2)}, \Phi^{(i_1, S_{11}, 3)}, \dots, \Phi^{(i_1, S_{11}, L)}), i_1 \in V_2^S; \end{aligned}$$

Further, the above vectors also partitioned as follows:

$$\begin{aligned} \Phi^{(0,S_{00},i_3)} &= (\phi^{(0,S_{00},i_3,0)}, \phi^{(0,S_{00},i_3,1)}, \dots, \phi^{(0,S_{00},i_3,L)}), i_3 \in V_0^L; \\ \Phi^{(1,S_{00},0)} &= (\phi^{(1,S_{00},0,0)}); \\ \Phi^{(1,S_{01},i_3)} &= (\phi^{(1,S_{01},i_3,1)}, \phi^{(1,S_{01},i_3,2)}, \dots, \phi^{(1,S_{01},i_3,L)}), i_3 \in V_0^L; \\ \Phi^{(1,S_{10},i_3)} &= (\phi^{(1,S_{10},i_3,0)}, \phi^{(1,S_{10},i_3,1)}, \dots, \phi^{(1,S_{10},i_3,L)}), i_3 \in V_1^L; \\ \Phi^{(i_1,S_{00},0)} &= (\phi^{(i_1,S_{00},0,0)}); \\ \Phi^{(i_1,S_{10},1)} &= (\phi^{(i_1,S_{10},1,0)}); \\ \Phi^{(i_1,S_{01},0)} &= (\phi^{(i_1,S_{01},0,1)}); \\ \Phi^{(i_1,S_{11},i_3)} &= (\phi^{(i_1,S_{11},i_3,1)}, \phi^{(i_1,S_{11},i_3,2)}, \dots, \phi^{(i_1,S_{11},i_3,L)}), i_3 \in V_1^L. \end{aligned}$$

Then the steady state probability Φ satisfies

$$\begin{aligned} \Phi\Theta &= \mathbf{0} \quad \text{and} \tag{1} \\ \sum_{(i_1,i_2,i_3,i_4)} \sum_{(i_1,i_2,i_3,i_4)} \phi^{(i_1,i_2,i_3,i_4)} &= 1. \tag{2} \end{aligned}$$

The equation (1) yields the following set of equations:

$$\begin{aligned} \Phi^{i_1} B_{i_1} + \Phi^{i_1-1} A_{i_1-1} &= \mathbf{0}, \quad i_1 = 1, 2, \dots, Q, \\ \Phi^{i_1} B_{i_1} + \Phi^{i_1-1} A_{i_1-1} + \Phi^{(i_1-1-Q)} C_0 &= \mathbf{0}, \quad i_1 = Q + 1, \tag{*} \\ \Phi^{i_1} B_{i_1} + \Phi^{i_1-1} A_{i_1-1} + \Phi^{(i_1-1-Q)} C_1 &= \mathbf{0}, \quad i_1 = Q + 2, \\ \Phi^{i_1} B_{i_1} + \Phi^{i_1-1} A_{i_1-1} + \Phi^{(i_1-1-Q)} C &= \mathbf{0}, \quad i_1 = Q + 3, Q + 4, \dots, S, \\ \Phi^S A_S + \Phi^S C &= \mathbf{0}. \end{aligned}$$

After lengthy simplifications, the above equations, except (*), yields

$$\begin{aligned} \phi^{i_1} &= (-1)^{Q-i_1} \phi^Q \prod_{j=Q}^{i_1+1} \Omega B_j A_{j-1}^{-1}, \quad i_1 = Q - 1, Q - 2, \dots, 0 \\ &= (-1)^Q \phi^Q \sum_{j=0}^{s-1} \left[\left\{ \left(\prod_{k=Q}^{(s+1)-j} \Omega B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \right\} \left\{ \left(\prod_{l=S-j}^{i_1+1} \Omega B_l A_{l-1}^{-1} \right) C_1 A_i^{-1} \right\} \right], \quad i_1 = Q + 1 \\ &= (-1)^{2Q-i_1+1} \phi^Q \sum_{j=0}^{S-i_1} \left[\left(\prod_{k=Q}^{s+1-j} \Omega B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\prod_{l=S-j}^{i_1+1} \Omega B_l A_{l-1}^{-1} \right) \right], \\ & \quad i_1 = S, S - 1, \dots, Q + 2 \end{aligned}$$

where ϕ^Q can be obtained by solving,

$$\phi^{Q+1} B_{Q+1} + \phi^Q A_Q + \phi^0 C_0 = \mathbf{0} \text{ and } \sum_{i_1=0}^S \phi^{i_1} \mathbf{e} = 1,$$

that is

$$\phi^Q \left[(-1)^Q \phi^Q \sum_{j=0}^{s-1} \left[\left\{ \left(\begin{matrix} (s+1)-j \\ k=Q \end{matrix} \right) B_k A_{k-1}^{-1} \right\} C A_{s-j}^{-1} \right] \left\{ \left(\begin{matrix} i_1+1 \\ l=s-j \end{matrix} \right) B_l A_{l-1}^{-1} \right\} C_1 A_i^{-1} \right] B_{Q+1} + A_Q + \left[(-1)^Q \sum_{j=Q}^1 B_j A_{j-1}^{-1} \right] C_0 \right] = \mathbf{0},$$

and

$$\phi^Q \left[\sum_{i_1=0}^{Q-1} \left((-1)^{Q-i_1} \sum_{j=Q}^{i_1+1} B_j A_{j-1}^{-1} \right) + I + (-1)^Q \sum_{j=0}^{s-1} \left[\left\{ \left(\begin{matrix} (s+1)-j \\ k=Q \end{matrix} \right) B_k A_{k-1}^{-1} \right\} C A_{s-j}^{-1} \right] \left\{ \left(\begin{matrix} i_1+1 \\ l=s-j \end{matrix} \right) B_l A_{l-1}^{-1} \right\} C_1 A_i^{-1} \right] + \sum_{i_1=Q+1}^S \left((-1)^{2Q-i_1+1} \sum_{j=0}^{s-i_1} \left[\left(\begin{matrix} (s+1)-j \\ k=Q \end{matrix} \right) B_k A_{k-1}^{-1} \right] C A_{s-j}^{-1} \left(\begin{matrix} i_1+1 \\ l=s-j \end{matrix} \right) B_l A_{l-1}^{-1} \right] \right] \mathbf{e} = \mathbf{1}.$$

4. SYSTEM PERFORMANCE MEASURES

In this section, we derive some measures of system performance in the steady state. Using this, we derive the total expected cost rate.

4.1. Expected inventory level

Let η_I denote the expected inventory level in the steady state. Then

$$\eta_I = \sum_{i_1=1}^S i_1 \Phi^{(i_1, S_{00}, 0, 0)} + \sum_{i_1=2}^S \left(i_1 \Phi^{(i_1, S_{10}, 1, 0)} + i_1 \Phi^{(i_1, S_{01}, 0, 1)} + \sum_{i_3=1}^L \sum_{i_4=1}^L i_1 \Phi^{(i_1, S_{11}, i_3, i_4)} \right) + \sum_{i_3=0}^L \sum_{i_4=1}^L \Phi^{(1, S_{01}, i_3, i_4)} + \sum_{i_3=1}^L \sum_{i_4=0}^L \Phi^{(1, S_{10}, i_3, i_4)}$$

4.2. Expected reorder rate

Let η_R denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s + 1$ to s . This may occur in the following cases:

- server-1 or server-2 may completes the service for a customer,
- an item may perish.

Hence, we get

$$\eta_R = (s + 1)\gamma \phi^{(s+1, S_{00}, 0, 0)} + (\mu_1 + s\gamma) \phi^{(s+1, S_{10}, 1, 0)} + (\mu_2 + s\gamma) \phi^{(s+1, S_{01}, 0, 1)} + \sum_{i_3=1}^L \sum_{i_4=1}^L (\mu_1 + \mu_2 + (s - 1)\gamma) \phi^{(s+1, S_{11}, i_3, i_4)}$$

4.3. *Expected perishable rate*

Let η_P denote the expected perishable rate for the $i_1 - th$ inventory level which is given by

$$\eta_P = \sum_{i_1=1}^S i_1 \gamma \phi^{(i_1, S_{00}, 0, 0)} + \sum_{i_1=2}^S \left((i_1 - 1) \gamma \phi^{(i_1, S_{10}, 1, 0)} + (i_1 - 1) \gamma \phi^{(i_1, S_{01}, 0, 1)} + \sum_{i_3=1}^L \sum_{i_4=1}^L (i_1 - 2) \gamma \phi^{(i_1, S_{11}, i_3, i_4)} \right)$$

4.4. *Expected number of customers in queue-1*

Let η_{W1} denote the expected number of customers in queue-1. Hence, η_{W1} is given by

$$\eta_{W1} = \sum_{i_3=1}^L \left(\sum_{i_4=0}^L (i_3 \phi^{(0, S_{00}, i_3, i_4)} + i_3 \phi^{(1, S_{10}, i_3, i_4)}) + \sum_{i_4=1}^L i_3 \phi^{(1, S_{01}, i_3, i_4)} \right) + \sum_{i_1=2}^S \left(\phi^{(i_1, S_{10}, 1, 0)} + \sum_{i_3=1}^L \sum_{i_4=1}^L i_3 \phi^{(i_1, S_{11}, i_3, i_4)} \right)$$

4.5. *Expected number of customers in queue-2*

Let η_{W2} denote the expected number of customer in queue-2. Hence, η_{W2} is given by

$$\eta_{W2} = \sum_{i_3=0}^L \sum_{i_4=1}^L (i_4 \phi^{(0, S_{00}, i_3, i_4)} + i_4 \phi^{(1, S_{01}, i_3, i_4)}) + \sum_{i_1=2}^S \left(\phi^{(i_1, S_{01}, 0, 1)} + \sum_{i_3=1}^L \sum_{i_4=1}^L i_4 \phi^{(i_1, S_{11}, i_3, i_4)} \right) + \sum_{i_3=1}^L \sum_{i_4=1}^L i_4 \phi^{(1, S_{10}, i_3, i_4)}$$

4.6. *Expected balking rate*

Let η_{BR} denote the expected balking rate in the steady state which is given by

$$\eta_{BR} = \lambda \left(\phi^{(0, S_{00}, L, L)} + \phi^{(1, S_{01}, L, L)} + \phi^{(1, S_{10}, L, L)} + \sum_{i_1=2}^S \phi^{(i_1, S_{11}, L, L)} \right)$$

4.7. *Expected renegeing rate in queue-1*

Let η_{R1} denote the expected renegeing rate in the queue-1. Then,

$$\eta_{R1} = \sum_{i_3=1}^L \left(\sum_{i_4=0}^L (i_3 \alpha_1 \phi^{(0, S_{00}, i_3, i_4)} + (i_3 - 1) \alpha_1 \phi^{(1, S_{10}, i_3, i_4)}) + \sum_{i_4=1}^L i_3 \alpha_1 \phi^{(1, S_{01}, i_3, i_4)} \right) + \sum_{i_1=2}^S \sum_{i_3=1}^L \sum_{i_4=1}^L (i_3 - 1) \alpha_1 \phi^{(i_1, S_{11}, i_3, i_4)}$$

4.8. *Expected renegeing rate in queue-2*

Let η_{R2} denote the expected renegeing rate in the queue-2. Then,

$$\eta_{R2} = \sum_{i_3=0}^L \sum_{i_4=1}^L (i_4 \alpha_2 \phi^{(0, S_{00}, i_3, i_4)} + (i_4 - 1) \alpha_2 \phi^{(1, S_{01}, i_3, i_4)}) + \sum_{i_3=1}^L \sum_{i_4=1}^L i_4 \alpha_2 \phi^{(1, S_{10}, i_3, i_4)} + \sum_{i_1=2}^S \sum_{i_3=1}^L \sum_{i_4=1}^L (i_4 - 1) \alpha_2 \phi^{(i_1, S_{11}, i_3, i_4)}$$

4.9. *Probability that both the servers are idle*

Let η_{SI} denote the probability that both the servers are idle is given by

$$\eta_{SI} = \sum_{i_3=0}^L \sum_{i_4=0}^L \phi^{(0, S_{00}, i_3, i_4)} + \sum_{i_1=1}^S \phi^{(i_1, S_{00}, 0, 0)}$$

4.10. *Probability that both the servers are busy*

Let η_{SB} denote the probability that both the servers are busy is given by

$$\eta_{SB} = \sum_{i_1=2}^S \sum_{i_3=1}^L \sum_{i_4=1}^L \phi^{(i_1, S_{11}, i_3, i_4)}$$

4.11. *Probability that both the servers are idle when the inventory level is positive*

Let η_{SP} denote the probability that both the Servers are idle when the inventory level is positive is given by

$$\eta_{SP} = \sum_{i_1=1}^S \phi^{(i_1, S_{00}, 0, 0)}$$

5. TOTAL EXPECTED COST RATE

We assume various cost elements associated with different system performance measures, given as follows:

- c_h – inventory carrying cost per unit per unit time,
- c_s – setup cost per order,
- c_p – perishable rate per unit per unit time,
- c_{w1} – waiting time cost of a customer in the queue-1 per unit time,
- c_{w2} – waiting time cost of a customer in the queue-2 per unit time,
- c_l – cost per customer lost per unit time,
- c_{r1} – renegeing cost of a customer in the queue-1 per unit time,
- c_{r2} – renegeing cost of a customer in the queue-2 per unit time,

We construct the function for the expected total cost per unit time as follows:

$$TC(S, s, L) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_{w1}\eta_{W1} + c_{w2}\eta_{W2} + c_l\eta_{BR} + c_{r1}\eta_{R1} + c_{r2}\eta_{R2}$$

where η 's are as given in the above measures of system performance. Since the computation of ϕ 's are recursive, it is very difficult to show the convexity of the total expected cost rate. However, we present, in the next section some numerical examples to illustrate the results of this work.

6. NUMERICAL ILLUSTRATIONS

In this section, we discuss some numerical examples that reveal the possible convexity of the total expected cost rate. A typical three dimensional plot of the total expected cost function $TC(s, S, 9)$ is given in Figure 1. Some 2-dimensional plot for variation of system parameters on performance measures are presented through Figure 2 to Figure 16, and the results confirm with what one would expect. Table 1, gives the total expected cost rate as a function of s and S by fixing other variables as constant. After obtaining the local optima, S^* and s^* , the sensitivity analysis is carried out to see how the changes in S and s affect the total expected cost rate(Figure 1). We have computed the values of $\frac{TC(s,S,9)}{TC(s^*,S^*,9)}$, by fixing the parameters and costs as: $\lambda = 5, \beta = 0.008, \gamma = 0.02, \mu_1 = 7, \mu_2 = 4, \alpha_1 = 3, \alpha_2 = 0.5, p = 0.5, q = 0.5, c_h = .004, c_s = 50, c_p = 0.12, c_{w1} = 0.01, c_{w2} = 6, c_b = 7, c_{r1} = 3, c_{r1} = 8$. In that Table, underlined value denotes the column minimum and in **bold** faced value denotes the row minimum. Hence, both underlined and bold faced value refer to the optimal value of the function. It appears that the total expected cost rate is more sensitive to the changes in s than that to in S .

In the following numerical examples, we select the cost values as $c_h = .004, c_s = 50, c_p = 0.12, c_{w1} = 0.01, c_{w2} = 6, c_b = 7, c_{r1} = 3, c_{r1} = 8$.

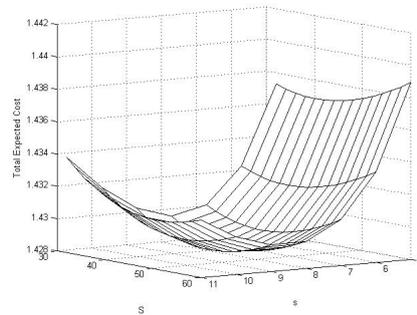


Figure 1: Effect of S and s on Total Expected Cost Rate

Table 1: Total expected cost rate as a function of S and s

S	s	6	7	8	9	10
49		1.002418	1.000745	1.000259	1.000572	1.001445
50		1.002364	1.000658	1.000133	1.000403	1.001230
51		<u>1.002349</u>	1.000611	1.000049	1.000280	1.001063
52		1.002369	<u>1.000600</u>	1.000005	1.000198	1.000940
53		1.002422	1.000626	1.000000	1.000156	1.000859
54		1.002508	1.000686	1.000029	<u>1.000153</u>	1.000819
55		1.002624	1.000779	1.000093	1.000185	<u>1.000816</u>
56		1.002770	1.000902	1.000190	1.000252	1.000851

Example 6.1.

In this example, we look at the impact of the demand rate λ , the perishable rate γ , the lead time rate β , service rates μ_1 and μ_2 for server-1 and server-2, respectively, on the total expected cost rate $TC(s, S, 9)$. Towards this end, we first fix the parameter values as $\alpha_1 = 8$, $\alpha_2 = 0.9$, $p = 0.5$, $q = 0.5$. From Figures 2 to 5, we observe the following:

1. The optimal expected cost rate increases when λ and γ increase.
2. The optimal expected cost rate decreases when β , μ_1 and μ_2 increase.

Example 6.2.

In this example, we study the impact of the demand rate λ , the lead time rate β , service rates μ_1 and μ_2 for server-1 and server-2 respectively, impatience rates α_1 and α_2 of queue-1 and queue-2 respectively and system level on the expected number of customer in each queue. Towards this end, we first fix the parameter values as $\gamma = 0.02$, $p = 0.5$, $q = 0.5$. From Figures 6 to 11, we observe the following:

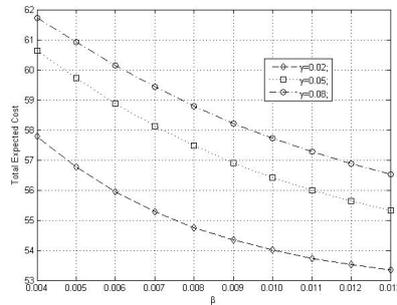


Figure 2: TC vs β for different values of γ

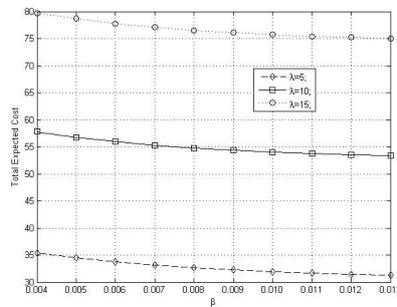


Figure 3: TC vs β for different values of λ

1. The expected number of customers in the waiting hall(queue-1 and queue-2) increases when λ and L increase.
2. The expected number of customers in the queue-1 and queue-2 decreases when reorder rate and service rates μ_1 and μ_2 of server-1 and server-2 respectively decrease.
3. The expected number of customers in the queue-1 and queue-2 decreases when the impatience rates α_1 and α_2 of queue-1 and queue-2 respectively increase.

Example 6.3.

In this example, we look at the impact of the demand rate λ , the perishable rate γ , the lead time rate β , service rates μ_1 and μ_2 for server-1 and server-2, respectively, on the expected loss rate. Towards this end, we first fix the parameter values as $\alpha_1 = 8$, $\alpha_2 = 0.9$, $p = 0.5$, $q = 0.5$. From Figures 12 to 14, we observe the following:

1. The expected loss rate increases when λ and γ increase.
2. The expected loss rate decreases when β , μ_1 and μ_2 increase.

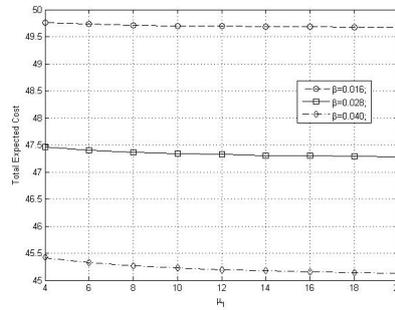


Figure 4: TC vs μ_1 for different values of β

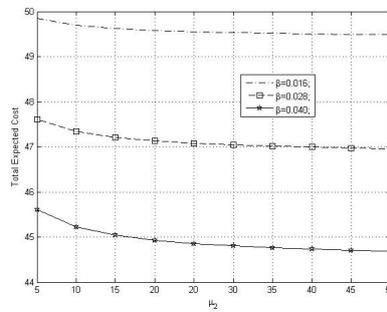


Figure 5: TC vs μ_2 for different values of β

Example 6.4.

In this example, we look at the impact of the lead time rate β , service rates μ_1 and μ_2 for server-1 and server-2 respectively on the expected reneing rate of each queue. Towards this end, we first fix the parameter values as $\lambda = 5, \gamma = 0.04; \alpha_1 = 8, \alpha_2 = 0.9, p = 0.5, q = 0.5$. From Figures 14 to 16, we observe the following:

1. The expected reneing rate of each queue decreases when β, μ_1 and μ_2 increase.

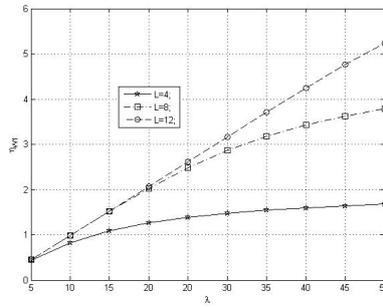


Figure 6: η_{W1} vs λ for different values of L

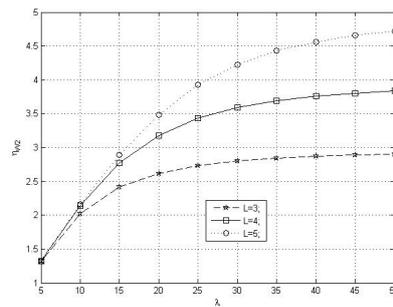


Figure 7: η_{W2} vs λ for different values of L

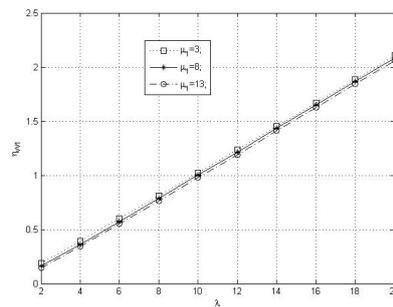


Figure 8: η_{W1} vs λ for different values of μ_1

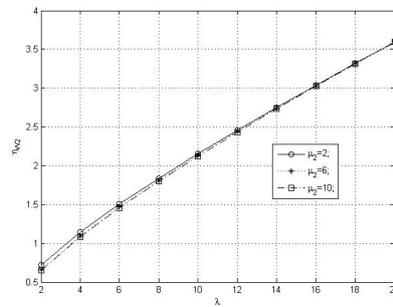


Figure 9: η_{W2} vs λ for different values of μ_2

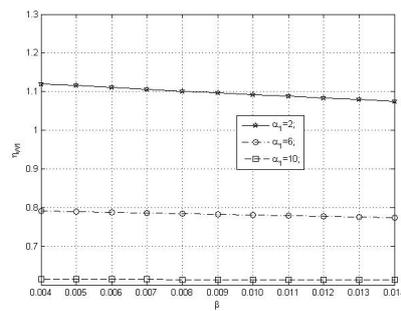


Figure 10: η_{W1} vs β for different values of α_1

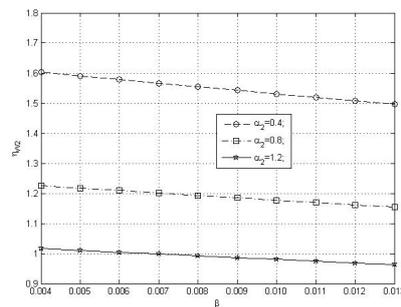


Figure 11: η_{W2} vs β for different values of α_2

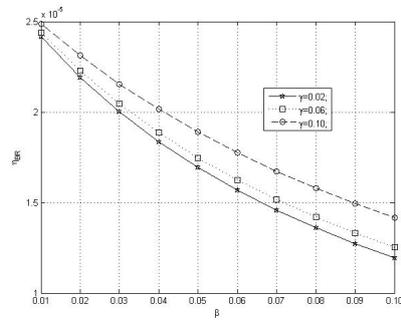


Figure 12: η_{BR} vs β for different values of γ

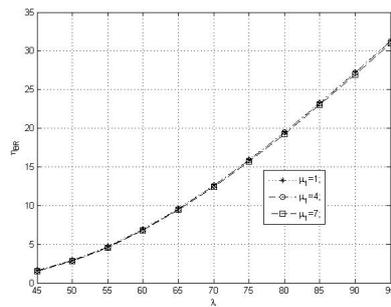


Figure 13: η_{BR} vs λ for different values of μ_1

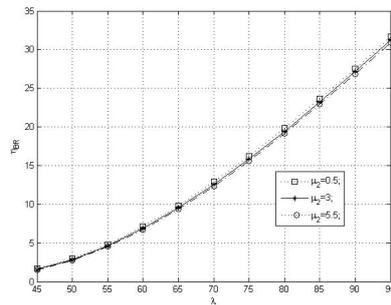


Figure 14: η_{BR} vs λ for different values of μ_2

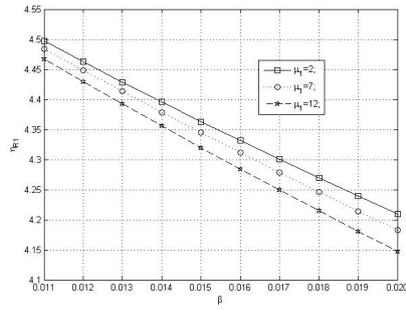


Figure 15: η_{w2} vs λ for different values of μ_2

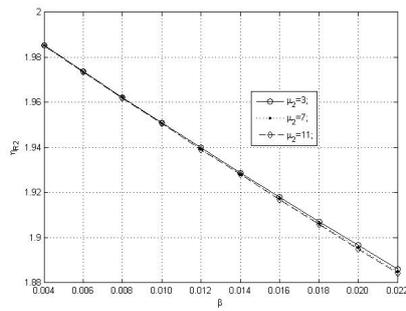


Figure 16: η_{w1} vs β for different values of α_1

7. CONCLUDING REMARKS

We have studied a continuous review stochastic queueing-inventory system with two parallel queues and jockeying. The model is analyzed within the framework of Markov processes. Joint probability distribution of the number of customers in the system (queue-1 and queue-2), status of the server and the inventory level is obtained in the steady state. Various system performance measures are derived and the long-run total expected cost rate is calculated. By assuming a suitable cost structure on the queueing-inventory system, we have presented extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times that follow PH-distributions.

Acknowledgement: The authors would like to thank the anonymous referees for their perceptive comments and valuable suggestions on a previous draft of this paper to improve its quality. The first author's research was supported by University Grants Commission (UGC), Government of India through UGC-BSR Research Start-Up-Grant project F.30-82/2014 (BSR).

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