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OPTIMAL REPLENISHMENT POLICY FOR FUZZY INVENTORY MODEL WITH DETERIORATING ITEMS AND ALLOWABLE SHORTAGES UNDER INFLATIONARY CONDITIONS

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Abstract: This study develops an inventory model to determine ordering policy for deteriorating items with constant demand rate under inflationary condition over a fixed planning horizon. Shortages are allowed and are partially backlogged. In today's wobbling economy, especially for long term investment, the effects of inflation cannot be disregarded as uncertainty about future inflation may influence the ordering policy. Therefore, in this paper a fuzzy model is developed that fuzzify the inflation rate, discount rate, deterioration rate, and backlogging parameter by using triangular fuzzy numbers to represent the uncertainty. For Defuzzification, the well known signed distance method is employed to find the total profit over the planning horizon. The objective of the study is to derive the optimal number of cycles and their optimal length so to maximize the net present value of the total profit over a fixed planning horizon. The necessary and sufficient conditions for an optimal solution are characterized. An algorithm is proposed to find the optimal solution. Finally, the proposed model has been validated with numerical example. Sensitivity analysis has been performed to study the impact of various parameters on the optimal solution, and some important managerial implications are presented.

Keywords: Inventory, Deterioration, Partial Backlogging, Inflation, Triangular Fuzzy Number and Signed Distance Method.

MSC: 90B05.

1. INTRODUCTION

In today's unstable global economy, inflation is an indispensable factor which cannot be ignored. Inflation is a rate at which the general level of prices for goods and services is raising over a period of time. Consequently inflation is also a decline in the real value of money – a loss of purchasing power in the medium of exchange. In the past, many authors have developed different inventory models under inflationary conditions with different assumptions. The pioneer in this field was Buzacott [6], who developed the first EOQ model taking inflation into account. Several researchers have extended their approach to various interesting situations by considering the time value of money, different inflation rates for the internal and external costs, finite replenishment rate and shortages, etc. The models in the literature with such research are developed by Misra [18], Bierman and Thomas [5], Misra [19], Vrat and Padmanabhan [28], Wee and Law [29], [30], Jaggi et al. [13], Dey et al. [8] and Yang et al. [33].

Sometimes in an economy a situation may crop up, where the demand is not fulfilled due to limited stock. This unsatisfied demand is called "shortage", and the period is termed as stock-out period for the firm. Generally, during the stock-out period, the following cases may arise: (i) all the demand is backordered, in which all customers wait until their demand is satisfied, (ii) all the demand is lost due to impatient customers, (iii) a fraction of demand is backordered for the customers who have patience to wait, the demand is satisfied by back-logged items; while others who cannot wait have their demands unsatisfied, and that is termed as lost sale case. The cost for a lost sale ranges from profit loss on the sale to some unspecified loss of good will. The first work in which customer's impatience functions is proposed seems to be that by Abad [1]. Abad derived a pricing and ordering policy for a variable rate of deterioration and partial backlogging. The partial backlogging was assumed to be exponential function of waiting time till the next replenishment. Dye et al. [10] modified this model taking into consideration the backorder cost and lost sale. After this, a lot of work has been done in this direction. Thus in this paper, an optimal replenishment schedule is derived under the assumption of waiting time backordering when units in an inventory are subject to constant deterioration. Widyadana et al. [31] developed a deteriorating inventory problem with and without backorders. Manna et al. [17] developed an order level inventory system for deteriorating items with demand rate as a ramp type function of time. Shah [25] analyzed Economic Production Quantity (EPQ) model for imperfect processes with rework when the units are subject to constant rate of deterioration. Recently, a lot of research work has been carried out for various inventory systems with deteriorating items. Chung and Cardenas-Barron [7] presented a simplified solution procedure to an EOQ model for deteriorating items with stock-dependent demand and two-level trade credit. Wu et al. [32] developed an EOQ model for deteriorating items with expiration dates under two-level trade credit financing. Sarkar et al. [23] presented an inventory model for variable deterioration rate under trade credit policy. Shah and Cardenas-Barron [24] developed retailer's decision for ordering and credit policies for deteriorating items when a supplier offers order-linked credit period.

Furthermore, the existing literature on inventory management reveals that most of the models have been developed in a static environment, where different inventory parameters are assumed to be known precisely. However, in reality, the parameters are inherited to have little deviations from the exact value, which may not follow any probability distribution. In these situations, if these parameters are treated as fuzzy parameters, then it will be more realistic. These types of problems are de-fuzzify first using a suitable fuzzy technique and then the solution procedure can be obtained in the usual manner. Therefore, the researchers have continuously used fuzzy set theory in inventory models so as to make them more practical and realistic. The analysis of Fuzzy Set theory in inventory models was initiated by Zadeh [35]. Subsequently, a lot of research work has been carried out on the defuzzification techniques of fuzzy numbers. Bellman and Zadeh [4] distinguished the difference between randomness and fuzziness by showing that the former deals with uncertainty regarding membership or non membership of an element in a set, while the later is concerned with the degree of uncertainty by which an element belongs to a set. Vujosevic et al. [27] reconsidered the economic order quantity (EOQ) formula by taking imprecise inventory costs. Gen et al. [11] expressed the input data by fuzzy numbers where they used interval mean value concept to solve an inventory problem. Yao and Chiang [34] considered the inventory model with total demand and storing cost as triangular fuzzy numbers. They performed the defuzzification by centroid and signed distance methods. De and Goswami [9] developed an EOQ model with a fuzzy inflation rate and fuzzy deterioration rate, and a delay in payment is also permissible. They derived the fuzzy cost function using Zimmerman [36] and the solution is obtained by Kaufmann and Gupta [14]. Halim et al. [12] developed a fuzzy inventory model for perishable items with stochastic demand, partial backlogging and fuzzy deterioration rate. This model is further extended to consider fuzzy partial backlogging factor. Maity and Maiti [16] proposed a numerical approach to a multi-objective optimal inventory control problem for deteriorating multi-items under fuzzy inflation. Roy et al. [22] established an inventory model for a deteriorating item (seasonal product) with linearly displayed stock dependent demand in imprecise environment (involving both fuzzy and random parameters) under inflation and time value of money. Mahata and Goswami [15] explored an EOQ model for deteriorating items under inflation assuming the demand rate to be a ramp type function of time. They introduced fuzzy multi-objective mathematical programming with the triangular fuzzy number. Valliathal and Uthayakumar [26] discussed a deterministic EOQ model for deteriorating items with partial backlogging under inflation and time discounting. Further, they fuzzify the cost components such as set up cost, purchasing cost, holding cost, shortage cost and opportunity cost due to lost sales by triangular fuzzy numbers. Signed distance method was used to defuzzify the total relevant profit. Ameli et al. [2] developed an inventory model to determine ordering policy for imperfect items with fuzzy defective percentage under fuzzy discounting and inflationary conditions. The signed distance method was used to defuzzify the total profit. Ameli et al. [3] proposed a model under fuzzy inflationary conditions, fuzzy time discounting for defective items with constant defective and deterioration rate and time dependent demand. Recently, Pattnaik [20] developed a single item EOQ model where unit price varies inversely with demand and setup cost increases, with the increase of production under fuzzy technique.

However, all the above mentioned papers on inflation and shortages has considered the impact of inflation in the modelling, and the prices they charge to their customers during the stock out period are the current prices of the next period, i.e., the time at which the retailer receives his ordered lot, which is quite unreasonable. In a way we are penalizing our loyal customers by charging them increased prices of the next period. Practically, the customer coming during the stock out period should be paid special attention, and must be charged the same price prevailing at that point of time. Keeping up with the above scenario, a model has been developed by incorporating this realistic phenomenon over a fixed planning horizon in a fuzzy environment to_make the model more pertinent. Triangular fuzzy numbers are used to fuzzify the inflation rate, discount rate, deterioration rate, and backlogging parameter. Signed distance method has been used to defuzzify the present value of total profit. Particular attention is placed on investigating the effect of inflation and deterioration on the optimal ordering policy with partially backlogged shortages such that the present value of total profit is maximized over the planning horizon. Finally, the paper concludes with a numerical example and sensitivity analysis performed on important parameters in order to explore important managerial insights.

2. PRELIMINARIES

In order to treat fuzzy inventory model by using signed distance method to defuzzify, we need the following definitions.

Definition 2.1 *(By Pu and Liu ([21], Definition 2.1)) A fuzzy set* $\tilde{a}$ *on* $R = (-\infty, \infty)$ *is called a fuzzy point if its membership function is*

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, x = a \\ 0, x \neq a \end{cases}$$

where the point a is called the support of fuzzy set $\tilde{a}$.

Definition 2.2 *A fuzzy set* $[a_\alpha, b_\alpha]$ *where* $0 \leq \alpha \leq 1$ *and a < b defined on* R *, is called α level of a fuzzy interval if its membership function is*

$$\mu_{[a_\alpha, b_\alpha]}(x) = \begin{cases} \alpha, a \leq x \leq b \\ 0, otherwise \end{cases}$$

Definition 2.3 *A fuzzy number* $\tilde{A} = (a, b, c)$ *where a < b < c, defined on* R*, is called a triangular fuzzy number if its membership function is*

$$\mu_A = \begin{cases} \dfrac{x-a}{b-a}, a \leq x \leq b \\ \dfrac{c-x}{c-b}, b \leq x \leq c \\ 0, \text{Otherwise} \end{cases}$$

When $a = b = c$, *we have a fuzzy point* $(c, c, c) = \tilde{c}$.

The family of all triangular fuzzy numbers on R is denoted as

$$F_N = \{(a, b, c) \mid a < b < c \; \forall \; a, b, c \in R\}.$$

The α -cut of a triangular fuzzy number is shown in Figure 1.

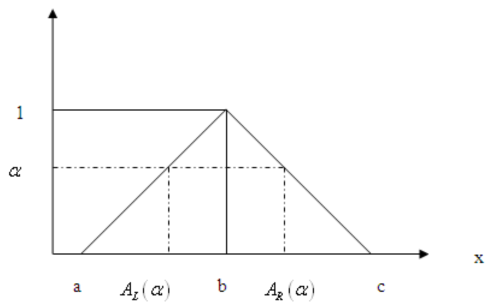

Figure 1: α-cut of a triangular fuzzy number

The α -cut of $\tilde{A} = (a,b,c) \in F_N$, $0 \le \alpha \le 1$, is $A(\alpha) = \left[A_L(\alpha), A_R(\alpha)\right]$.

Where $A_L(\alpha) = a + (b-a)\alpha$ and $A_R(\alpha) = c - (c-b)\alpha$ are the left and right endpoints of $A(\alpha)$.

Definition 2.4 *If* $\tilde{A} = (a,b,c)$ *is a triangular fuzzy number then, the signed distance of* $\tilde{A}$ *is defined as* $d(\tilde{A},\tilde{0}) = \int_0^1 d\left(\left[A_L(\alpha)_\alpha, A_R(\alpha)_\alpha\right],\tilde{0}\right) = \frac{1}{4}(a+2b+c)$.

3. ASSUMPTIONS AND NOTATIONS

The mathematical model has been developed on the basis of the following assumptions and notations.

3.1. Assumptions

1. Demand is deterministic and occurs uniformly over the period.
2. Replenishment is instantaneous, but its size is finite.
3. The time horizon of the inventory system is finite.
4. Lead time is constant.
5. Shortages are partially backlogged and are fulfilled at the beginning of the next cycle.
6. During the stock-out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment. So that the backlogging rate for the negative inventory is $e^{-\delta(T-t)}$; where $\delta(>0)$ denotes the backlogging parameter and (T – t) is waiting time for the next replenishment.
7. A constant fraction $\theta(0 \le \theta \le 1)$ of the on-hand inventory deteriorates per unit time.
8. There is no repair or replenishment of the deteriorated items during the inventory cycle.
9. A Discount Cash Flow (DCF) approach is used to consider the various costs at various times.
10. Triangular fuzzy numbers are used to fuzzify the model.
11. Signed distance method is applied to defuzzify the model.

3.2. Notations

H	the length of the whole planning horizon
n	the number of replenishment over [$0,H$]
T	the replenishment cycle and $H = nT$
A_0	the ordering cost per order ($/order) at time zero
h_0	the holding cost per unit per unit time ($/unit/unit time) at time zero
C_0	the purchase cost per unit ($/unit) at time zero
S_0	the shortage cost per unit per unit time ($/unit/unit time) at time zero
L_0	The lost sale cost per unit per unit time ($/unit/unit time) at time zero
$p_0\left(p_0 > C_0\right)$	selling price per unit ($/unit) of item at time zero
θ	the deterioration rate
$I_i\left(t\right)$	the inventory level at time t , in i[th] interval, where $t \in \left[0, H\right]$
t_i	the time at which the inventory level reaches zero in i[th] replenishment cycle (i=1, 2, 3……), with $t_i = iT = i\left(\frac{H}{n}\right)$
q_i	the maximum inventory level in i[th] cycle
IB_i	the maximum backorder units during stock-out period in i[th] cycle
Q_i	the economic order quantity in i[th] cycle
r	the discount rate, representing the time value of money
α	the inflation rate
$\tilde{\theta}$	the deterioration rate in fuzzy sense
$\tilde{\delta}$	the backlogging Parameter in fuzzy sense
$\tilde{\alpha}$	the inflation rate in fuzzy sense
$\tilde{r}$	the discount rate in fuzzy sense
$TP\left(k,n\right)$	the present worth of crisp total profit
$\tilde{TP}(k,n)$	the present worth of fuzzy total profit
$TP_{dS}(k,n)$	the present worth of defuzzify total profit

4. MATHEMATICAL FORMULATION

The planning horizon H has been divided into n equal cycles of length T, i.e., n replenishment cycles. Let us consider the i[th] cycle, i.e. $t_{i-1} \le t \le t_i$, where $t_0 = 0, t_n = H, t_i - t_{i-1} = T$ and $t_i = iT(i = 1, 2, ..., n)$. At the beginning of the i[th] cycle, the inventory system starts with zero inventories at time t_{i-1} and shortages are allowed to accumulate up to t_{i1}. However, some shortages are lost and some are backordered. The order quantity received at t_{i1} is used partly to make up for the backorders, which are accumulated in the previous cycle from time t_{i-1} to t_{i1}; and it is partly used to satisfy the demand of the current cycle. Thus, the on hand inventory at t_{i1} gradually reduces to zero at t_i, due to demand and deterioration of items.

Now, $t_{i-1} = t_{i1} - kT \Rightarrow \quad t_{i1} = \left\{(i-1) + k\right\} \dfrac{H}{n}, (i = 1, 2, ..., n),\ (0 \le k \le 1),$

where kT is the fraction of the cycle having shortages.

The model begins with shortages and ends without shortages, which is depicted graphically in Figure 2.

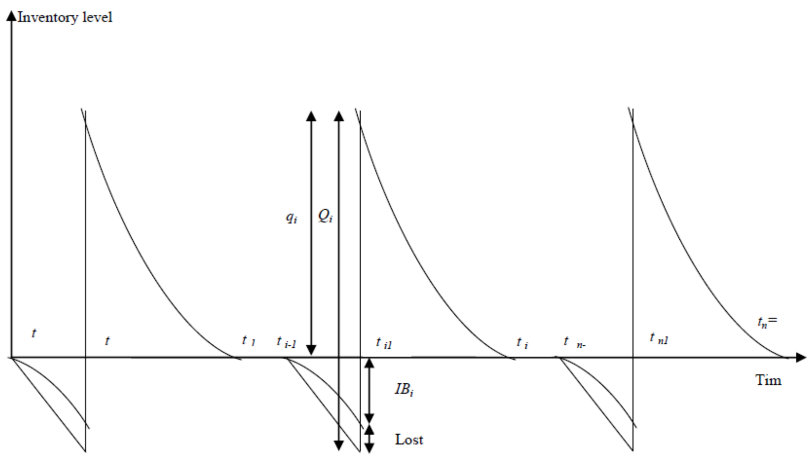

Figure 2: Pictorial Representation of Inventory system

Moreover, in the present paper fuzziness is allowed to some parameters so as to make the model more practical. However, in such a scenario, the model is divided into two sections:

(i) Crisp Model
(ii) Fuzzy Model

4.1. Crisp model

The inventory level at any time t during the i[th] replenishment cycle is governed by the following differential equations:

During the time interval $\left[t_{i1}, t_i\right]$ the inventory is depleted by the combined effect of demand and deterioration. Hence, the inventory level at any time *t* is

$$\frac{dI_i(t)}{dt} + \theta I_i(t) = -D \qquad , \; t_{i1} \le t \le t_i \quad i = 1,2,...,n \tag{1}$$

The solution of the above differential equation with the boundary condition $I(t_i) = 0$ is

$$I_i(t) = \frac{D}{\theta}\left\{e^{\theta(t_i - t)} - 1\right\} \tag{2}$$

Further, the time interval $\left[t_{i-1}, t_{i1}\right]$ is the stock out period, with partially backlogged shortages. In this case only some fraction i.e. $e^{-\delta(T-t)}$ of the total shortages is backlogged while the rest is lost, where $t \in \left[t_{i-1}, t_{i1}\right]$. Hence, the inventory level at any time t during the time interval $\left[t_{i-1}, t_{i1}\right]$ in the i[th] replenishment cycle is governed by the following differential equation:

$$\frac{dI_i(t)}{dt} = -De^{-\delta(t_{i1} - t)}, \quad t_{i-1} \le t \le t_{i1} \quad i = 1,2,...,n \tag{3}$$

After using the boundary condition $I(t_{i-1}) = 0$, the solution of the differential equation (3) is

$$I_i(t) = \frac{D}{\delta}\left\{e^{-\delta(t_{i1} - t_{i-1})} - e^{-\delta(t_{i1} - t)}\right\} \; i = 1,2,...,n \tag{4}$$

The maximum amount of positive inventory is

$$q_i = I(t_{i1}) = \frac{D}{\theta}\left\{e^{\theta(t_i - t_{i1})} - 1\right\} \text{ (using Equation (2))} \tag{5}$$

The maximum number of backordered units is

$$IB_i = -I(t_{i1}) = \frac{D}{\delta}\left\{1 - e^{-\delta(t_{i1} - t_{i-1})}\right\} \text{(using Equation (4))} \tag{6}$$

Hence, the economic order quantity Q_i for i[th] cycle is

$$Q_i = q_i + IB_i = \frac{D}{\theta}\left\{e^{\theta(t_i - t_{i1})} - 1\right\} + \frac{D}{\delta}\left\{1 - e^{-\delta(t_{i1} - t_{i-1})}\right\} \tag{7}$$

Moreover, the present model has been developed under inflationary conditions. Hence, one simple way of modeling is to assume α, the constant rate of inflation. Therefore, the various costs as the ordering cost, unit cost of the item, inventory carrying cost, shortage cost, cost due to lost sales, and selling price at any time t are given by

$A(t) = A_0\, e^{\alpha t}, C(t) = C_0\, e^{\alpha t}, h(t) = h_0\, e^{\alpha t}, S(t) = S_0\, e^{\alpha t}, L(t) = L_0\, e^{\alpha t}, p(t) = p_0\, e^{\alpha t}$

The model begins at time t_0, when shortages start to accumulate at the price of $p(t_0) = p_0$ till t_{11}. At t_{11}, an ordered lot is received in the system from which the backorders accumulated till t_{11} are satisfied at the same prevalent price $p(t_0)$, when the customers have placed their order. However, due to inflation the selling price of the items would be $p(t_1) = p_0\, e^{\alpha t_1}$ for the customers coming during the period (t_{11}, t_1). The same process would be followed for the ith cycle.

Now, the present value of the profit for the ith replenishment cycle is given by sales revenue- ordering cost- purchasing cost- holding cost- shortage cost- cost due to lost sales. Thus, by assuming continuous compounding of inflation and time value of money, the present value of the various costs for ith cycle are evaluated as follows:

Present worth of the revenue for ith cycle is

[1] Present worth of the revenue from the sales is $p(t_{i1})e^{-rt_{i1}} \int_{t_{i1}}^{t_i} D e^{-rt} dt$

[2] Now to calculate the present worth of the revenue from the shortages, the selling price of the items should be $p\left(t_{(i-1)1}\right)$, i.e., the price prevalent when the shortages start to occur.

[3] Present worth of the revenue from the shortages is $p\left(t_{(i-1)1}\right)e^{-rt_{i1}} \int_{t_{i-1}}^{t_{i1}} D e^{-\delta(t_{i1}-t)} dt$

Hence, the total present worth of the revenue is

$$SR_i = p(t_{i1})e^{-rt_{i1}} \int_{t_{i1}}^{t_i} D e^{-rt} dt + p\left(t_{(i-1)1}\right)e^{-rt_{i1}} \int_{t_{i-1}}^{t_{i1}} D e^{-\delta(t_{i1}-t)} dt$$

$$= p_0 D\left[e^{(\alpha-r)t_{i1}}\left\{\frac{\left(e^{-rt_{i1}} - e^{-rt_i}\right)}{r}\right\} + e^{\alpha t_{(i-1)1}}\, e^{-rt_{i1}}\left\{\frac{\left(1-e^{-\delta(t_{i1}-t_{i-1})}\right)}{\delta}\right\}\right], i = 1,2,...,n. \quad (8)$$

Present worth of the ordering cost for ith cycle is

$$A_i = A(t_{i1})e^{-rt_{i1}} = A_0\, e^{(\alpha-r)t_{i1}},\ i = 1,2,...,n. \quad (9)$$

Present worth of the purchase cost for ith cycle is

$$PC_i = Q_i C(t_{i1})e^{-rt_{i1}} = Q_i C_0\, e^{(\alpha-r)t_{i1}}$$

$$PC_i = C_0\, e^{(\alpha-r)t_{i1}} \left[\frac{D}{\theta}\left\{ e^{\theta(t_i - t_{i1})} - 1 \right\} + \frac{D}{\delta}\left\{ 1 - e^{-\delta(t_{i1} - t_{i-1})} \right\} \right], i = 1,2,...,n. \quad (10)$$

Present worth of the inventory holding cost for i^th cycle is

$$HC_i = h(t_{i1}) e^{-r t_{i1}} \int_{t_{i1}}^{t_i} I_i(t) e^{-rt}\, dt$$

$$= h_0\, e^{(\alpha-r)t_{i1}} \int_{t_{i1}}^{t_i} \frac{D}{\theta}\left\{ e^{\theta(t_i - t)} - 1 \right\} e^{-rt}\, dt$$

$$= \frac{h_0 D}{\theta} e^{(\alpha-r)t_{i1}} \left[\frac{\left(e^{-r t_i} - e^{-r t_{i1}} \right)}{r} + \frac{\left(e^{\theta(t_i - t_{i1})} e^{-r t_{i1}} - e^{-r t_i} \right)}{(\theta + r)} \right],\ i = 1,2,...,n. \quad (11)$$

Present worth of the shortage cost for i^th cycle is

$$SC_i = S(t_{i1}) e^{-r t_{i1}} \int_{t_{i-1}}^{t_{i1}} -I_i(t) e^{-rt}\, dt$$

$$= S_0\, e^{(\alpha-r)t_{i1}} \int_{t_{i-1}}^{t_{i1}} \frac{D}{\delta}\left\{ e^{-\delta(t_{i1} - t_{i-1})} - e^{-\delta(t_{i1} - t)} \right\} e^{-rt}\, dt$$

$$= \frac{S_0 D}{\delta}\left[\frac{\left(e^{-r t_{i1}} - e^{-\delta(t_{i1} - t_{i-1})} e^{-r t_{i-1}} \right)}{(\delta - r)} + \frac{e^{-\delta(t_{i1} - t_{i-1})}\left(e^{-r t_{i1}} - e^{-r t_{i-1}} \right)}{r} \right], i = 1,2,...,n. \quad (12)$$

Present worth of the cost due to lost sales for i^th cycle is

$$LC_i = L(t_{i1}) e^{-r t_{i1}} \int_{t_{i-1}}^{t_{i1}} D\left\{ 1 - e^{-\delta(t_{i1} - t)} \right\} dt$$

$$= DL_0 e^{(\alpha-r)t_{i1}} \left[(t_{i1} - t_{i-1}) + \frac{\left(e^{-\delta(t_{i1} - t_{i-1})} - 1 \right)}{\delta} \right], i = 1,2,...,n. \quad (13)$$

Now, the present value of the total profit TP_i for the i^th cycle is given by the following expression:

$$TP_i = SR_i - \left(OC_i + PC_i + HC_i + LC_i + SC_i \right), i = 1,2,...,n. \quad (14)$$

The present worth of the total profit of the system during the entire time horizon H is

$$TP(k,n)=\sum_{i=1}^{n}TP_i=\sum_{i=1}^{n}\left\{SR_i-\left(OC_i+PC_i+HC_i+LC_i+SC_i\right)\right\} \tag{15}$$

Substituting the values of $SR_i, OC_i, PC_i, HC_i, SC_i$ and LC_i from equations (8), (9), (10), (11), (12) and (13), respectively, in equation (15) and after simplification, we get

$$\begin{aligned}TP(k,n)={}&p_0De^{(\alpha-r)kH/n}\left\{\left(\frac{1-e^{(\alpha-2r)H}}{1-e^{(\alpha-2r)H/n}}\right)\left(\frac{e^{-rkH/n}-e^{-rH/n}}{r}\right)+\left(\frac{1-e^{(\alpha-r)H(n-1)/n}}{1-e^{(\alpha-r)H/n}}\right)\left(\frac{e^{-rH/n}\left(1-e^{-\delta kH/n}\right)}{\delta}\right)\right\}+p_0D\\&\left(\frac{e^{-rH/n}\left(1-e^{-\delta kH/n}\right)}{\delta}\right)-e^{(\alpha-r)kH/n}\left\{A_0\left(\frac{1-e^{(\alpha-r)H}}{1-e^{(\alpha-r)H/n}}\right)+C_0D\left(\frac{1-e^{(\alpha-r)H}}{1-e^{(\alpha-r)H/n}}\right)\left(\frac{e^{\theta(1-k)H/n}-1}{\theta}+\frac{1-e^{-\delta kH/n}}{\delta}\right)\right.\\&+h_0D\left(\frac{1-e^{(\alpha-2r)H}}{1-e^{(\alpha-2r)H/n}}\right)\left(\frac{e^{-rH/n}-e^{-rkH/n}}{r\theta}+\frac{e^{-rkH/n}e^{\theta(1-k)H/n}-e^{-rH/n}}{\theta(\theta+r)}\right)+S_0D\left(\frac{1-e^{(\alpha-2r)H}}{1-e^{(\alpha-2r)H/n}}\right)\\&\left.\left(\frac{e^{-rkH/n}-e^{-\delta kH/n}}{\delta(\delta-r)}+\frac{e^{-\delta kH/n}\left(e^{-rkH/n}-1\right)}{\delta r}\right)+L_0D\left(\frac{1-e^{(\alpha-r)H}}{1-e^{(\alpha-r)H/n}}\right)\left(\frac{kH}{n}+\frac{e^{-\delta kH/n}-1}{\delta}\right)\right\}\end{aligned} \tag{16}$$

4.2. Fuzzy model

In the real world transactions, owing to the unstable environments, it would not be easy to determine the exact value of the parameters. Thus, the decision maker determines that the approximate value of these parameters.

Thus in the present study, to develop the fuzzy model, some parameters viz. inflation rate, discount rate, deterioration rate, and backlogging parameter are taken as triangular fuzzy numbers and denoted by $\tilde{\alpha},\tilde{r},\tilde{\theta},\tilde{\delta}$ respectively. These fuzzy parameters are represented by $\tilde{\alpha}=(\alpha_1,\alpha_2,\alpha_3),\tilde{r}=(r_1,r_2,r_3),\tilde{\theta}=(\theta_1,\theta_2,\theta_3),\ \tilde{\delta}=(\delta_1,\delta_2,\delta_3)$ where $\alpha_1<\alpha_2<\alpha_3, r_1<r_2<r_3, \theta_1<\theta_2<\theta_3$ and $\delta_1<\delta_2<\delta_3$.Since, the inflation rate, discount rate, deterioration rate, and backlogging parameter are fuzzy numbers, therefore using equation (16), the fuzzy profit is given by

$$\begin{aligned}\tilde{TP}(k,n) = {} & p_0 D e^{\left(\tilde{\alpha}-\tilde{r}\right)kH/n}\left\{\left(\frac{1-e^{\left(\tilde{\alpha}-2\tilde{r}\right)H}}{1-e^{\left(\tilde{\alpha}-2\tilde{r}\right)H/n}}\right)\left(\frac{e^{-\tilde{r}kH/n}-e^{-\tilde{r}H/n}}{\tilde{r}}\right)+\left(\frac{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H(n-1)/n}}{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H/n}}\right)\left(\frac{e^{-\tilde{r}H/n}\left(1-e^{-\tilde{\delta}kH/n}\right)}{\tilde{\delta}}\right)\right\}+p_0 D \\ & \left(\frac{e^{-\tilde{r}H/n}\left(1-e^{-\tilde{\delta}kH/n}\right)}{\tilde{\delta}}\right)-e^{\left(\tilde{\alpha}-\tilde{r}\right)kH/n}\left\{A_0\left(\frac{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H}}{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H/n}}\right)+C_0 D\left(\frac{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H}}{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H/n}}\right)\left(\frac{e^{\tilde{\theta}(1-k)H/n}-1}{\tilde{\theta}}+\frac{1-e^{-\tilde{\delta}kH/n}}{\tilde{\delta}}\right)\right. \\ & +h_0 D\left(\frac{1-e^{\left(\tilde{\alpha}-2\tilde{r}\right)H}}{1-e^{\left(\tilde{\alpha}-2\tilde{r}\right)H/n}}\right)\left(\frac{e^{-\tilde{r}H/n}-e^{-\tilde{r}kH/n}}{\tilde{r}\tilde{\theta}}+\frac{e^{-\tilde{r}kH/n}e^{\tilde{\theta}(1-k)H/n}-e^{-\tilde{r}H/n}}{\tilde{\theta}\left(\tilde{\theta}+\tilde{r}\right)}\right)+S_0 D\left(\frac{1-e^{\left(\tilde{\alpha}-2\tilde{r}\right)H}}{1-e^{\left(\tilde{\alpha}-2\tilde{r}\right)H/n}}\right) \\ & \left.\left(\frac{e^{-\tilde{r}kH/n}-e^{-\tilde{\delta}kH/n}}{\tilde{\delta}\left(\tilde{\delta}-\tilde{r}\right)}+\frac{e^{-\tilde{\delta}kH/n}\left(e^{-\tilde{r}kH/n}-1\right)}{\tilde{\delta}\tilde{r}}\right)+L_0 D\left(\frac{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H}}{1-e^{\left(\tilde{\alpha}-\tilde{r}\right)H/n}}\right)\left(\frac{kH}{n}+\frac{e^{-\tilde{\delta}kH/n}-1}{\tilde{\delta}}\right)\right\}\end{aligned} \tag{17}$$

Now, by using Signed distance Method, the fuzzy profit is converted into the corresponding crisp one. Hence, the present worth of the total fuzzy profit of the system during the entire time horizon H is

$$TP_{dS}(k,n) = \left(X1 + 2 * X2 + X3\right)/4 \tag{18}$$

Where

$$X1 = p_0 D e^{(\alpha_1 - r_3)kH/n}\left\{\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)\left(\frac{e^{-r_3kH/n}-e^{-r_1H/n}}{r_3}\right)+\left(\frac{1-e^{(\alpha_3-r_1)H(n-1)/n}}{1-e^{(\alpha_1-r_3)H/n}}\right)\left(\frac{e^{-r_3H/n}\left(1-e^{-\delta_1kH/n}\right)}{\delta_3}\right)\right\}+p_0D$$
$$\left(\frac{e^{-r_3kH/n}\left(1-e^{-\delta_1kH/n}\right)}{\delta_3}\right)-e^{(\alpha_3-r_1)kH/n}\left\{A_0\left(\frac{1-e^{(\alpha_1-r_3)H}}{1-e^{(\alpha_3-r_1)H/n}}\right)+C_0D\left(\frac{1-e^{(\alpha_1-r_3)H}}{1-e^{(\alpha_3-r_1)H/n}}\right)\left(\frac{e^{\theta_3(1-k)H/n}-1}{\theta_1}+\frac{1-e^{-\delta_3kH/n}}{\delta_1}\right)\right.$$
$$+h_0D\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right)\left(\frac{e^{-r_1H/n}-e^{-r_3kH/n}}{r_1\theta_1}+\frac{e^{-r_1kH/n}e^{\theta_3(1-k)H/n}-e^{-r_3H/n}}{\theta_1\left(\theta_1+r_1\right)}\right)+S_0D\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right)$$
$$\left.\left(\frac{e^{-r_1kH/n}-e^{-\delta_3kH/n}}{\delta_1\left(\delta_1-r_3\right)}+\frac{e^{-\delta_1kH/n}\left(e^{-r_1kH/n}-1\right)}{\delta_1r_1}\right)+L_0D\left(\frac{1-e^{(\alpha_1-r_3)H}}{1-e^{(\alpha_3-r_1)H/n}}\right)\left(\frac{kH}{n}+\frac{e^{-\delta_1kH/n}-1}{\delta_1}\right)\right\}$$

$$X2 = p_0 D e^{(\alpha_2 - r_2)kH/n}\left\{\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right)\left(\frac{e^{-r_2kH/n}-e^{-r_2H/n}}{r_2}\right)+\left(\frac{1-e^{(\alpha_2-r_2)H(n-1)/n}}{1-e^{(\alpha_2-r_2)H/n}}\right)\left(\frac{e^{-r_2H/n}\left(1-e^{-\delta_2kH/n}\right)}{\delta_2}\right)\right\}+p_0D$$
$$\left(\frac{e^{-r_2H/n}\left(1-e^{-\delta_2kH/n}\right)}{\delta_2}\right)-e^{(\alpha_2-r_2)kH/n}\left\{A_0\left(\frac{1-e^{(\alpha_2-r_2)H}}{1-e^{(\alpha_2-r_2)H/n}}\right)+C_0D\left(\frac{1-e^{(\alpha_2-r_2)H}}{1-e^{(\alpha_2-r_2)H/n}}\right)\left(\frac{e^{\theta_2(1-k)H/n}-1}{\theta_2}+\frac{1-e^{-\delta_2kH/n}}{\delta_2}\right)\right.$$
$$+h_0D\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right)\left(\frac{e^{-r_2H/n}-e^{-r_2kH/n}}{r_2\theta_2}+\frac{e^{-r_2kH/n}e^{\theta_2(1-k)H/n}-e^{-r_2H/n}}{\theta_2\left(\theta_2+r_2\right)}\right)+S_0D\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right)$$
$$\left.\left(\frac{e^{-r_2kH/n}-e^{-\delta_2kH/n}}{\delta_2\left(\delta_2-r_2\right)}+\frac{e^{-\delta_2kH/n}\left(e^{-r_2kH/n}-1\right)}{\delta_2r_2}\right)+L_0D\left(\frac{1-e^{(\alpha_2-r_2)H}}{1-e^{(\alpha_2-r_2)H/n}}\right)\left(\frac{kH}{n}+\frac{e^{-\delta_2kH/n}-1}{\delta_2}\right)\right\}$$

and

$$X3 = p_0 D e^{(\alpha_3 - r_1)kH/n}\left\{\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right)\left(\frac{e^{-r_1kH/n}-e^{-r_3H/n}}{r_1}\right)+\left(\frac{1-e^{(\alpha_1-r_3)H(n-1)/n}}{1-e^{(\alpha_3-r_1)H/n}}\right)\left(\frac{e^{-r_1H/n}\left(1-e^{-\delta_3kH/n}\right)}{\delta_1}\right)\right\}+p_0D$$
$$\left(\frac{e^{-r_1H/n}\left(1-e^{-\delta_3kH/n}\right)}{\delta_1}\right)-e^{(\alpha_1-r_3)kH/n}\left\{A_0\left(\frac{1-e^{(\alpha_3-r_1)H}}{1-e^{(\alpha_1-r_3)H/n}}\right)+C_0D\left(\frac{1-e^{(\alpha_3-r_1)H}}{1-e^{(\alpha_1-r_3)H/n}}\right)\left(\frac{e^{\theta_1(1-k)H/n}-1}{\theta_3}+\frac{1-e^{-\delta_1kH/n}}{\delta_3}\right)\right.$$
$$+h_0D\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)\left(\frac{e^{-r_3H/n}-e^{-r_1kH/n}}{r_3\theta_3}+\frac{e^{-r_3kH/n}e^{\theta_1(1-k)H/n}-e^{-r_1H/n}}{\theta_3\left(\theta_3+r_3\right)}\right)+S_0D\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)$$
$$\left.\left(\frac{e^{-r_3kH/n}-e^{-\delta_1kH/n}}{\delta_3\left(\delta_3-r_1\right)}+\frac{e^{-\delta_3kH/n}\left(e^{-r_3kH/n}-1\right)}{\delta_3r_3}\right)+L_0D\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)\left(\frac{1-e^{-r_1kH/n}}{r_3}+\frac{e^{-\delta_3kH/n}-e^{-r_1kH/n}}{\left(\delta_3-r_1\right)}\right)\right\}$$

Now, the problem is to determine the optimal values of *k* and *n* which maximize $TP_{dS}(k,n)$. Since, the profit function $TP_{dS}(k,n)$, is a function of two variables *k* and *n*, where *k* is a continuous and *n* is a discrete variable, therefore, for any given value of $n = n_0$ (say), the necessary condition for $TP_{dS}(k,n)$ to be maximum is

$\frac{\partial TP_{dS}(k,n_0)}{\partial k} = 0$, which gives

$$
\left[\begin{array}{l}
\frac{p_0D(\alpha_1-r_3)e^{(\alpha_1-r_3)kH/n}}{4}\Bigg\{\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)\left(\frac{e^{-r_3kH/n}-e^{-r_1H/n}}{r_3}\right)+\left(\frac{1-e^{(\alpha_3-r_1)H(n-1)/n}}{1-e^{(\alpha_1-r_3)H/n}}\right)\left(\frac{e^{-r_3H/n}\left(1-e^{-\delta_1kH/n}\right)}{\delta_3}\right)\Bigg\}+\frac{p_0De^{(\alpha_1-r_3)kH/n}}{4} \\
\Bigg\{\left(\frac{1-e^{(\alpha_3-r_1)H(n-1)/n}}{1-e^{(\alpha_1-r_3)H/n}}\right)\left(\frac{e^{-r_3H/n}e^{-\delta_1kH/n}\delta_1}{\delta_3}\right)-\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)e^{-r_3kH/n}\Bigg\}+\frac{p_0De^{-r_3kH/n}}{4}\left(\frac{\delta_1e^{-\delta_1kH/n}-r_3\left(1-e^{-\delta_1kH/n}\right)}{\delta_3}\right)-e^{(\alpha_3-r_1)kH/n} \\
\frac{(\alpha_3-r_1)}{4}\Bigg\{\left(\frac{1-e^{(\alpha_1-r_3)H}}{1-e^{(\alpha_3-r_1)H/n}}\right)\Bigg(A_0+C_0D\left(\frac{e^{\theta_3(1-k)H/n}-1}{\theta_1}+\frac{1-e^{-\delta_3kH/n}}{\delta_1}\right)+L_0\left(\frac{kH}{n}+\frac{e^{-\delta_1kH/n}-1}{\delta_1}\right)\Bigg)+D\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right) \\
\Bigg(h_0\left(\frac{e^{-r_1H/n}-e^{-r_3kH/n}}{r_1\theta_1}+\frac{e^{-r_1kH/n}e^{\theta_3(1-k)H/n}-e^{-r_3H/n}}{\theta_1(\theta_1+r_1)}\right)+S_0\left(\frac{e^{-r_1kH/n}-e^{-\delta_3kH/n}}{\delta_1(\delta_1-r_3)}+\frac{e^{-\delta_1kH/n}\left(e^{-r_1kH/n}-1\right)}{\delta_1r_1}\right)\Bigg)\Bigg\}-\frac{e^{(\alpha_3-r_1)kH/n}D}{4} \\
\Bigg\{\Bigg(C_0\left(\frac{-\theta_3e^{\theta_3(1-k)H/n}}{\theta_1}+\frac{\delta_3e^{-\delta_3kH/n}}{\delta_1}\right)+L_0\left(1-e^{-\delta_1kH/n}\right)\Bigg)\left(\frac{1-e^{(\alpha_1-r_3)H}}{1-e^{(\alpha_3-r_1)H/n}}\right)+\Bigg(\Bigg(\frac{r_3e^{-r_3kH/n}}{r_1\theta_1}-\frac{r_1e^{-r_1kH/n}e^{\theta_3(1-k)H/n}+\theta_3e^{-r_1kH/n}e^{\theta_3(1-k)H/n}}{\theta_1(\theta_1+r_1)}\Bigg) \\
h_0+S_0\left(\frac{\delta_3e^{-\delta_3kH/n}-r_1e^{-r_1kH/n}}{\delta_1(\delta_1-r_3)}-\frac{e^{-\delta_1kH/n}\left(e^{-r_1kH/n}-1\right)}{r_1}-\frac{e^{-\delta_1kH/n}e^{-r_1kH/n}}{\delta_1}\right)\Bigg)\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right)\Bigg\}+\frac{p_0D(\alpha_2-r_2)e^{(\alpha_2-r_2)kH/n}}{2} \\
\Bigg\{\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right)\left(\frac{e^{-r_2kH/n}-e^{-r_2H/n}}{r_2}\right)+\left(\frac{1-e^{(\alpha_2-r_2)H(n-1)/n}}{1-e^{(\alpha_2-r_2)H/n}}\right)\left(\frac{e^{-r_2H/n}\left(1-e^{-\delta_2kH/n}\right)}{\delta_2}\right)\Bigg\}+\frac{p_0De^{(\alpha_2-r_2)kH/n}}{4} \\
\Bigg\{\left(\frac{1-e^{(\alpha_2-r_2)H(n-1)/n}}{1-e^{(\alpha_2-r_2)H/n}}\right)\left(\frac{e^{-r_2H/n}e^{-\delta_2kH/n}\delta_2}{\delta_2}\right)-\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right)e^{-r_2kH/n}\Bigg\}+\frac{p_0De^{-r_2kH/n}}{2}\left(\frac{\delta_2e^{-\delta_2kH/n}-r_2\left(1-e^{-\delta_2kH/n}\right)}{\delta_2}\right)-e^{(\alpha_2-r_2)kH/n} \\
\frac{(\alpha_2-r_2)}{2}\Bigg\{\left(\frac{1-e^{(\alpha_2-r_2)H}}{1-e^{(\alpha_2-r_2)H/n}}\right)\Bigg(A_0+C_0D\left(\frac{e^{\theta_2(1-k)H/n}-1}{\theta_2}+\frac{1-e^{-\delta_2kH/n}}{\delta_2}\right)+L_0\left(\frac{kH}{n}+\frac{e^{-\delta_2kH/n}-1}{\delta_2}\right)\Bigg)+D\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right) \\
\Bigg(h_0\left(\frac{e^{-r_2H/n}-e^{-r_2kH/n}}{r_2\theta_2}+\frac{e^{-r_2kH/n}e^{\theta_2(1-k)H/n}-e^{-r_2H/n}}{\theta_2(\theta_2+r_2)}\right)+S_0\left(\frac{e^{-r_2kH/n}-e^{-\delta_2kH/n}}{\delta_2(\delta_2-r_2)}+\frac{e^{-\delta_2kH/n}\left(e^{-r_2kH/n}-1\right)}{\delta_2r_2}\right)\Bigg)\Bigg\}-\frac{e^{(\alpha_2-r_2)kH/n}D}{4} \\
\Bigg\{\Bigg(C_0\left(\frac{-\theta_2e^{\theta_2(1-k)H/n}}{\theta_2}+\frac{\delta_2e^{-\delta_2kH/n}}{\delta_2}\right)+L_0\left(1-e^{-\delta_2kH/n}\right)\Bigg)\left(\frac{1-e^{(\alpha_2-r_2)H}}{1-e^{(\alpha_2-r_2)H/n}}\right)+\Bigg(\Bigg(\frac{r_2e^{-r_2kH/n}}{r_2\theta_2}-\frac{r_2e^{-r_2kH/n}e^{\theta_2(1-k)H/n}+\theta_2e^{-r_2kH/n}e^{\theta_2(1-k)H/n}}{\theta_2(\theta_2+r_2)}\Bigg) \\
h_0+S_0\left(\frac{\delta_2e^{-\delta_2kH/n}-r_2e^{-r_2kH/n}}{\delta_2(\delta_2-r_2)}-\frac{e^{-\delta_2kH/n}\left(e^{-r_2kH/n}-1\right)}{r_2}-\frac{e^{-\delta_2kH/n}e^{-r_2kH/n}}{\delta_2}\right)\Bigg)\left(\frac{1-e^{(\alpha_2-2r_2)H}}{1-e^{(\alpha_2-2r_2)H/n}}\right)\Bigg\} \\
+\frac{p_0D(\alpha_3-r_1)e^{(\alpha_3-r_1)kH/n}}{4}\Bigg\{\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right)\left(\frac{e^{-r_1kH/n}-e^{-r_3H/n}}{r_1}\right)+\left(\frac{1-e^{(\alpha_1-r_3)H(n-1)/n}}{1-e^{(\alpha_3-r_1)H/n}}\right)\left(\frac{e^{-r_1H/n}\left(1-e^{-\delta_3kH/n}\right)}{\delta_1}\right)\Bigg\}+\frac{p_0De^{(\alpha_3-r_1)kH/n}}{4} \\
\Bigg\{\left(\frac{1-e^{(\alpha_1-r_3)H(n-1)/n}}{1-e^{(\alpha_3-r_1)H/n}}\right)\left(\frac{e^{-r_1H/n}e^{-\delta_3kH/n}\delta_3}{\delta_1}\right)-\left(\frac{1-e^{(\alpha_1-2r_3)H}}{1-e^{(\alpha_3-2r_1)H/n}}\right)e^{-r_1kH/n}\Bigg\}+\frac{p_0De^{-r_1kH/n}}{4}\left(\frac{\delta_3e^{-\delta_3kH/n}-r_1\left(1-e^{-\delta_3kH/n}\right)}{\delta_1}\right)-e^{(\alpha_1-r_3)kH/n} \\
\frac{(\alpha_1-r_3)}{4}\Bigg\{\left(\frac{1-e^{(\alpha_3-r_1)H}}{1-e^{(\alpha_1-r_3)H/n}}\right)\Bigg(A_0+C_0D\left(\frac{e^{\theta_1(1-k)H/n}-1}{\theta_3}+\frac{1-e^{-\delta_1kH/n}}{\delta_3}\right)+L_0\left(\frac{kH}{n}+\frac{e^{-\delta_3kH/n}-1}{\delta_3}\right)\Bigg)+D\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right) \\
\Bigg(h_0\left(\frac{e^{-r_3H/n}-e^{-r_1kH/n}}{r_3\theta_3}+\frac{e^{-r_3kH/n}e^{\theta_1(1-k)H/n}-e^{-r_1H/n}}{\theta_3(\theta_3+r_3)}\right)+S_0\left(\frac{e^{-r_3kH/n}-e^{-\delta_1kH/n}}{\delta_3(\delta_3-r_1)}+\frac{e^{-\delta_3kH/n}\left(e^{-r_3kH/n}-1\right)}{\delta_3r_3}\right)\Bigg)\Bigg\}-\frac{e^{(\alpha_1-r_3)kH/n}D}{4} \\
\Bigg\{\Bigg(C_0\left(\frac{-\theta_1e^{\theta_1(1-k)H/n}}{\theta_3}+\frac{\delta_1e^{-\delta_1kH/n}}{\delta_3}\right)+L_0\left(1-e^{-\delta_3kH/n}\right)\Bigg)\left(\frac{1-e^{(\alpha_3-r_1)H}}{1-e^{(\alpha_1-r_3)H/n}}\right)+\Bigg(\Bigg(\frac{r_1e^{-r_1kH/n}}{r_3\theta_3}-\frac{r_3e^{-r_3kH/n}e^{\theta_1(1-k)H/n}+\theta_1e^{-r_3kH/n}e^{\theta_1(1-k)H/n}}{\theta_3(\theta_3+r_3)}\Bigg) \\
h_0+S_0\left(\frac{\delta_1e^{-\delta_1kH/n}-r_3e^{-r_3kH/n}}{\delta_3(\delta_3-r_1)}-\frac{e^{-\delta_3kH/n}\left(e^{-r_3kH/n}-1\right)}{r_3}-\frac{e^{-\delta_3kH/n}e^{-r_3kH/n}}{\delta_3}\right)\Bigg)\left(\frac{1-e^{(\alpha_3-2r_1)H}}{1-e^{(\alpha_1-2r_3)H/n}}\right)\Bigg\}
\end{array}\right]=0
$$

Further, for the present worth of total profit $TP_{dS}(k,n)$, to be concave, the following sufficient condition must be satisfied $\frac{\partial^2 TP_{ds}(k,n)}{\partial k^2} \le 0$.

Since, the second derivative of the present worth of total profit $TP_{dS}(k,n)$ is a very complicated function, thus it is very difficult to prove concavity mathematically. Therefore, the concavity of the present worth of total profit has been established graphically (on several data sets) (Figure 3).

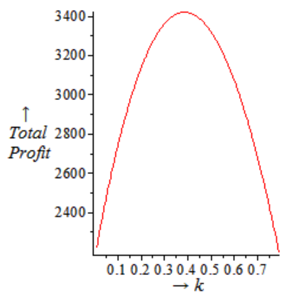

Figure 3: Total profit per unit time with respect to k

5. SOLUTION PROCEDURE

In order to obtain the values of k and n which maximize the total profit $TP_{dS}(k,n)$, the following procedure (Jaggi et al. [13]) is adopted.

Step 1: Solve Equation (19) for k by substituting $n = n_p$ and $n = n_p + 1$, the corresponding values of k are k_{n_p} and $k_{n_{p+1}}$, respectively, $(n_p = 1, 2, ...)$.

Step 2: Compute $TP_{dS}(k_{n_p}, n_p)$ and $TP_{dS}(k_{n_p+1}, n_p + 1)$.

Step 3: If $TP_{dS}(k_{n_p}, n_p) \ge TP_{dS}(k_{n_p+1}, n_p + 1)$, then the optimal values of k and n are $k = k_{n_p}$ and $n = n_p$. The optimal value of T can be obtained using the relation $T = H / n$ while the optimal value of $TP_{dS}(k,n)$ can be obtained by substituting k and n in Equation (18) and optimal lot size (Q_i) for $i = 1, 2, ..., n$ can be obtained from Equation (7). Else, go to Step 4.

Step 4: Replace n_p by $n_p + 1$ and go to Step 1.

6. NUMERICAL EXAMPLE

Let us consider an inventory system with following data:

$$A = 400, D = 300, C = 50, h = 10, S = 35, L = 42,$$
$$p = 75, H = 2, \theta_1 = .075, \theta_2 = .08, \theta_3 = .085, \delta_1 = .82,$$
$$\delta_2 = .9, \delta_3 = .98, \alpha_1 = .04, \alpha_2 = .05, \alpha_3 = .06, r_1 = .11, r_2 = .12, r_3 = .13$$

Fuzzy Model: Using the solution procedure and with the help of software MS- Excel and Maple-15, we get the results as follows

$n=6, k=.38, T=121.6, TP_{dS}(k,n)=3421.22, Q=98$.

Crisp Model: However, if the parameters are assumed to be deterministic in nature i.e.,

if $\theta_1=\theta_2=\theta_3=.08, \delta_1=\delta_2=\delta_3=.9, \alpha_1=\alpha_2=.\alpha_3=.05, r_1=r_2=r_3=.12$, then fuzzy model is converted into crisp model, and the following solution is obtained:

$n=5, k=.28, T=146, TP(k,n)=6994.86, Q=119$.

It is observed that the total profit decreases as the uncertainties in the model increases. However, the decision maker moves to a more realistic and practical situation when the parameters are treated as fuzzy parameters.

7. SENSITIVITY ANALYSIS

In this section, sensitivity analysis has been performed to study the effects of inflation $(\tilde{\alpha})$, deterioration $(\tilde{\theta})$ and backlogging parameter $(\tilde{\delta})$ on the optimal number of cycles (n), optimal cycle length (T), present worth of total average profit $(TP_{dS}(k,n))$ and economic order quantity Q the results are summarized in Table 1 and Table 2.

Table 1: Impact of $\tilde{\alpha}$ and $\tilde{\theta}$ on the optimal replenishment policy

$\tilde{\alpha}\downarrow$	$\tilde{\theta}\rightarrow$	.080=(.075,.080, .085)	.090=(.081,.090,.095)	.100=(.090,.100, .110)
	n	6	7	8
	T (days)	121.6	104.2	91.2
.05=(.040,.050,.060)	Q (units)	98	86	65
	$TP_{dS}(k,n)$	3421.22	3222.62	3149.75
	n	5	6	7
	T (days)	146	121.6	104.2
.06= (.054,.060,.066)	Q (units)	104	91	76
	$TP_{dS}(k,n)$	3658.61	3435.10	3295.93
	n	4	5	6
	T (days)	182.5	146	121.6
.07=(.067,.070,.072)	Q (units)	107	96	80
	$TP_{dS}(k,n)$	3898.67	3656.29	3467.66

From Table 1 following inferences can be made:

For any fixed deterioration rate $\tilde{\theta}$, if the rate of inflation $\tilde{\alpha}$ increases (with constant discount rate) then, the optimal number of cycles n decreases (i.e. cycle length T

increases) and the present value of total profit $TP_{dS}(k,n)$ and economic order quantity also increases significantly. Due to mounting inflation and uncertainty, organization begin hoarding out of concern that prices will increase in future. Therefore, mounting inflation compels the companies to keep a high level of inventories so as to manage their profits efficiently.

Further, for any fixed inflation rate $\tilde{\alpha}$ if the deterioration rate $\tilde{\theta}$ increases then, the optimal number of cycles n increases (i.e. cycle length T decreases), whereas the present value of total profit $TP_{dS}(k,n)$ and economic order quantity decreases considerably. Since due to deterioration the value of goods decreases, hence it is optimal for the retailer to order less quantity for a shorter period of time so as to manage the loss due to deterioration.

Table 2: Impact of $\tilde{\delta}$ on the optimal replenishment policy

$\tilde{\delta}$	n	T (days)	Q(units)	$TP_{dS}(k,n)$
$.72=(.61,.72,.82)$	7	137.6	105	3637.24
$.81=(.65,.81,.95)$	6	143.1	102	3524.18
$.99=(.75,.99,1.18)$	5	143.2	96	3401.58
$1.08=(.79,1.08,1.32)$	4	144.2	94	3369.74

Table 2 reveals that if the value of backlogging parameter $\tilde{\delta}$ increases then, the optimal number of cycle's n decreases (i.e. cycle length T increases) however the present value of total profit $TP_{dS}(k,n)$ and economic order quantity decreases substantially. Since a major portion from the order size is utilized for satisfying the backlogged demand, and an increasing backlogging rate implies less of backlogged demand; therefore the order quantity decreases, which eventually results in lower profits.

8. CONCLUSION

In the present study, a fuzzy inventory model has been developed for deteriorating items with partially backlogged shortages over a finite planning horizon. The present value of various costs is computed using continuous compounding of inflation and time value of money. Moreover, in the present model, special attention has been paid to the loyal customers coming during the stock out period, who wait for their order till the next replenishment period. These customers are charged the same price prevailing at that point of time and not the increased prices of the next period, which has not been considered in the previous research. Further, the inflation rate, discount rate, deterioration rate, and backlogging parameter are assumed as triangular fuzzy number to make the model more practical. Finally, signed distance method has been employed to defuzzify the present value of total profit during planning horizon.

The findings have been validated by a numerical example. From the numerical example, it is observed that the optimal profit of fuzzy model is lower than that of crisp one since the average profit decreases when the uncertainties in the model are accounted in a large manner. Moreover, sensitivity analysis reveals that if backlogged demand is lower, then the order quantity decreases. Further ahead, it is observed that inflation rate and deterioration rate play opposite characteristics in the optimal replenishment policies. While high inflation rate recommends a large order quantity, at the same time, it is advisable to order less quantity if the deterioration rate is increasing. Thus, it is important to determine a tradeoff between these two parameters so as to arrive at the optimal replenishment policy.

For future investigation, the model can be extended by incorporating some more practical situations with different types of variable demand such as price dependent demand, stock dependent demand, linear time dependent demand, etc. The model can also be studied under the effect of permissible delay in payments, imperfect production, and inspection process.

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