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SET OPTIMIZATION USING IMPROVEMENT SETS

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Abstract: In this paper we introduce a notion of minimal solutions for set-valued optimization problem in terms of improvement sets by unifying a solution notion, introduced by Kuroiwa [15] for set-valued problems, and a notion of optimal solutions in terms of improvement sets, introduced by Chicco et al. [4] for vector optimization problems. We provide existence theorems for these solutions, and establish lower convergence of the minimal solution sets in the sense of Painlevé-Kuratowski.

Keywords: Set-valued Optimization, Improvement Set, Painlevé-Kuratowski Convergence.

MSC: 49J53, 90C30.

1. INTRODUCTION

In the recent years much attention has been paid to various aspects of set-valued optimization problems due to their occurrence in the areas of decision making such as economics, differential inclusions, and optimal control [2, 11]. In the literature there are two types of well-known criteria of solutions for a set-valued optimization problem, the vector criterion and the set criterion. We consider a set-valued optimization problem

$$\text{(P)} \qquad \begin{array}{rl} \text{Min} & F(x) \\ \text{subject to} & x \in M, \end{array}$$

where $F : X \rightrightarrows Y$ is a set-valued map, X and Y are normed linear spaces, and M is a nonempty subset of X. Let K be a closed convex pointed cone in Y with nonempty interior. The vector criterion involves finding a point $\bar{x} \in M$ such that there exists $\bar{y} \in F(\bar{x})$ with

$$(F(M) - \bar{y}) \cap (-K \backslash \{0\}) = \emptyset,$$

where $F(M) := \bigcup_{x \in M} F(x)$. Such an element $\bar{x}$ is said to be a minimal solution of (P), and the set of all minimal solutions is denoted by Min. This criterion requires the existence of a single element $\bar{y}$ in the image set $F(\bar{x})$ of the solution $\bar{x}$. For more details on the vector criterion, we refer the reader to [3, 8, 17, 21] and the references therein. In contrast to this criterion, the set criterion, introduced by Kuroiwa [15], involves the entire set $F(\bar{x})$ rather than just a single element of this set. Even though the order relations considered in the papers by Kuroiwa [13, 14] and Maeda [18] are quasiorders (or preorders), the solution concept using set criterion is more appropriate for set-valued optimization problems.

However, in practice there are situations, especially in economics ([7, 19]), where one has to deal with order relations which are not necessarily quasiorders. This motivated many authors to consider order relations which are not necessarily quasiorders for optimal solutions of vector optimization problems ([4, 7, 12]). Chicco et al. [4] considered a relation for vector optimization problems which is based on the notion of improvement sets, introduced by Debreu [6]. In [4], authors introduced the concept of E-optimal point of a set in finite dimensional spaces, where E refers to an improvement set, and they gave sufficient conditions for the existence of optimal solutions. This notion unifies the notions of efficient, weak efficient, and approximate efficient points of a vector optimization problem. In the framework of Banach spaces, Oppezzi and Rossi [20] presented several sufficient conditions for the existence of optimal points using improvement sets. Certain stability results have also been investigated in [16, 20, 23] for E-optimal solutions in vector optimization. The solution concept with respect to improvement sets has also been considered in literature, for set-valued problems. Scalarization, Lagrange multiplier theorems and duality results have been established in a real locally convex Hausdorff topological vector space setting, using weak E-optimality [26] and Benson E-optimality [24, 25] for set-valued optimization problems. In all the above papers, dealing with set-valued optimization problems, we note that the vector criterion of solutions, and not the set criterion approach, has been considered.

In the present work a new solution concept is introduced for set-valued problems using improvement set, in the set criterion sense. We refer to these solutions as E-l-minimal solutions and show that this solution concept unifies the notions of E-optimal solutions and optimal solutions in the sense of Kuroiwa [15]. We give existence theorems for E-l-minimal solutions, and establish convergence of E-l-minimal solution sets of perturbed problems to the solution set of the given problem.

The rest of the paper is organized as follows. Section 2 deals with preliminaries required in the sequel. In Section 3, we introduce the notion of E-l-minimal solutions for the set-valued optimization problem (P). A diagrammatic representation has been given to illustrate that E-l-minimal solutions concept unifies various other well-known solution concepts in set-valued optimization. We also present sufficient conditions for the existence of E-l-minimal solutions, where one of the existence theorems is given in the setting of reflexive Banach spaces. In Section 4, we investigate a stability aspect of the set-valued optimization problem, by perturbing the feasible set. In fact, we establish the lower part of Painlevé-Kuratowski convergence for E-l-minimal solution sets.

2. PRELIMINARIES

The cone K induces a partial order relation in Y, in the sense that for $y_1, y_2 \in Y$

$$y_1 \leq y_2 \text{ if and only if } y_2 - y_1 \in K.$$

Let A be a nonempty subset of Y. An element $\hat{a} \in A$ is a minimum of A if $(A - \hat{a}) \cap (-K \backslash \{0\}) = \emptyset$. Also, $\hat{a}$ is a strong minimum of A if $\hat{a} \leq a$, for every $a \in A$. The dual cone of K, denoted by K^*, is defined as

$$K^* := \{y^* \in Y^* : \langle y^*, k\rangle \geq 0, \forall k \in K\},$$

where Y^* denotes the dual space of Y.

We next recall the notion of Painlevé-Kuratowski convergence as in [22]. For a sequence of sets $\{M_n\}$ in X, we have the notations

$$\text{Li } M_n := \{x \in X : x_n \to x, x_n \in M_n, \text{for sufficiently large } n\},$$

$$\text{Ls } M_n := \{x \in X : x_{n_k} \to x, x_{n_k} \in M_{n_k}, \{n_k\} \text{ is an increasing sequence of integers}\}.$$

We say that the sequence of sets $\{M_n\}$ converges to a set M in the sense of Painlevé-Kuratowski convergence, denoted by $M_n \overset{PK}{\to} M$, if and only if

$$\text{Ls } M_n \subseteq M \subseteq \text{Li } M_n.$$

The relation Ls $M_n \subseteq M$ is referred to as the upper part of the convergence and the relation $M \subseteq$ Li M_n is referred as the lower part of the convergence.

Let $F : X \rightrightarrows Y$ be a set-valued map. The domain and graph of F are defined as

$$\text{dom}F := \{x \in X : F(x) \neq \emptyset\},$$

$$\text{gr}F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

The map F is closed at $x \in \text{dom}F$, if for any sequence $x_n \in \text{dom}F,\ x_n \to x,\ y_n \in F(x_n),\ y_n \to y$, we have $y \in F(x)$. F is said to be closed if F is closed at every $x \in \text{dom}F$. Clearly, F is closed if and only if gr$F$ is closed. Also, F is said to be lower semicontinuous at $x \in \text{dom}F$ if for any $y \in F(x)$ and for any sequence $x_n \in \text{dom}F$ such that $x_n \to x$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y$. F is said to be lower semicontinuous on $M \subseteq \text{dom}F$, if F is lower semicontinuous at every $x \in M$.

Chicco et. al. [4] defined the upper comprehensive set of a nonempty set A in Y, denoted by u-compr(A) as

$$\text{u-compr}(A) := A + K,$$

where K is a closed convex pointed cone in Y with nonempty interior. A nonempty set E in Y is said to be upper comprehensive if u-compr$(E) = E$. A nonempty upper comprehensive set E in Y is said to be an improvement set if $0 \notin E$.

Clearly, K is not an improvement set in Y as $0 \in K$. However, $K\backslash\{0\}$ and interior of K, denoted by int$K$, are improvement sets in Y.

Improvement sets are not necessarily convex sets. For example, the set $E = \mathbb{R}^2_+\backslash([0,1]\times[0,1])$ is an improvement set in $\mathbb{R}^2$ where $K = \mathbb{R}^2_+$. For more examples one may refer to [4].

Clearly, if E is an improvement set of Y, then $E + K = E$. Moreover, from Proposition 3.1 of [4] it follows that if E_1 and E_2 are two improvement sets of Y, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are also improvement sets of Y provided $E_1 \cap E_2 \neq \emptyset$.

Throughout the paper we assume that $M \subseteq$ dom$F$, where M is the feasible set.

The notion of E-minimal solution for a vector optimization problem has been considered in [4] and has been further extended for set-valued problems by Zhao and Yang [24, 25, 26]. We now recall this vector criterion notion for the set-valued problem (P). Given an improvement set E of Y, an element $\bar{x} \in M$ is said to be an E-minimal solution of (P), if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(M) - \bar{y}) \cap (-E) = \emptyset.$$

We denote the set of E-minimal solutions of (P) by E-Min. Clearly, if $E = K\backslash\{0\}$, then E-Min reduces to the set Min.

3. *E-l*-MINIMAL SOLUTIONS

For two nonempty subsets A and B of Y, the following quasiorder relation $\leq^l_K$ is from [9, 13],

$$A \leq^l_K B \text{ if and only if } B \subseteq A + K.$$

In analogy to $\leq^l_K$ we define a non-quasiorder relation using improvement sets. If E is an improvement set in Y then

$$A \leq^l_E B \text{ if and only if } B \subseteq A + E.$$

Remark 3.1. In general, $\leq^l_E$ is neither reflexive nor transitive. However, if $E \subseteq K\backslash\{0\}$, then $\leq^l_E$ is a transitive relation. If $A \leq^l_E B$ and $B \leq^l_E C$, then $B \subseteq A+E$ and $C \subseteq B + E$ which implies that $C \subseteq A + E + E \subseteq A + E + K = A + E$.

For two nonempty sets A and B in Y, it can be seen that $A \leq^l_K B$ and $B \leq^l_K A$ if and only if $A + K = B + K$. A similar result follows for the relation $\leq^l_E$.

Lemma 3.1. *Let A and B be two nonempty subsets of Y and E be an improvement set of Y such that $E \subseteq K\backslash\{0\}$. Then, $A \leq^l_E B$ and $B \leq^l_E A$ if and only if*

$$A + E = B + E = A + K = B + K.$$

Proof. If $A \leq^l_E B$, then $B \subseteq A+E$, which implies $B+K \subseteq A+E+K = A+E \subseteq A+K$. Similarly, if $B \leq^l_E A$, then $A+K \subseteq B+E \subseteq B+K$. The conclusion follows from the above two relations.

Converse implication holds as $A \subseteq A+K = B+E$ and $B \subseteq B+K = A+E$. □

However the above result may not hold if $E \subseteq K\backslash\{0\}$ fails to hold, as it is shown in the next example.

Example 3.1. Let $A = \{(1,1),(2,0)\}$ and $B = \{(0,2)\}$. Let $K = \mathbb{R}^2_+$ and $E = \{(y_1, y_2) : y_1 + y_2 \geq 0\}\backslash\{(0,0)\}$. Then, it can be seen that all four sets $A+E, B+E, A+K$ and $B+K$ are distinct.

We now define the notion of E-l-minimal solution for problem (P).

Definition 3.1. Given an improvement set E of Y, an element $\bar{x} \in M$ is said to be an E-l-minimal solution of (P) if

$$F(x) \leq^l_E F(\bar{x}), x \in M \Rightarrow F(\bar{x}) \leq^l_E F(x).$$

We denote the set of all E-l-minimal solutions of (P) by E-l-Min.

Kuroiwa [13, 14] gave the notion of l-minimal solution where the cone K was considered instead of the set E in the above definition. Similarly, l-weak minimal solution is defined if int$K$ is considered instead of E in the above definition. We denote the set of l-minimal solutions and l-weak minimal solutions by l-Min and l-WMin, respectively. We also observe that the notion of E-l-minimal solution unifies the notion of l-WMin but not of l-Min.

We next establish that an E-minimal solution is an E-l-minimal solution.

Proposition 3.1. *Let E be an improvement set of Y then,*

$$E\text{-Min} \subseteq E\text{-}l\text{-Min}.$$

Proof. Let $\bar{x} \in M$ be an E-minimal solution of (P), then there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(M) - \bar{y}) \cap (-E) = \emptyset.$$

Thus, there does not exist any $x \in M$ such that $F(x) \leq^l_E F(\bar{x})$, and thus, $\bar{x}$ is an E-l-minimal solution. □

Remark 3.2. It is known that Min $\not\subseteq$ l-Min, for instance see Example 3.1 of [10]. However, from Proposition 3.1, we observe that when non-quasiorder relation $\leq^l_E$ is considered, such an inclusion holds.

We now give an example to show that an E-l-minimal solution may not be E-minimal.

Example 3.2. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$ and $M = [0, 1]$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(1,1),(2,0)\}, & \text{if } x \neq 0, \\ \{(0,2)\}, & \text{if } x = 0. \end{cases}$$

Let $E = \{(y_1, y_2) : y_1 + y_2 \geq 0\} \backslash \{(0, 0)\}$. Then, E-l-Min $= [0, 1]$ whereas the set of E-minimal solutions is empty.

Remark 3.3. The following observations justify that E-l-minimal solutions extend and unify various types of approximate solutions.

(a) If $E = \varepsilon_+ e + K$, for $\varepsilon_+ > 0$ and $e \in \text{int}K$ with $\|e\| < 1$ then, E-l-Min reduces to the set of ε_+-l-minimal solutions which we denote by ε_+-l-Min (see [1]).

(b) If $E = \varepsilon e + \text{int}K$, for $\varepsilon \geq 0$ and $e \in \text{int}K$ with $\|e\| < 1$ then, E-l-Min reduces to the set of ε-l-weak minimal solutions which we denote by ε-l-WMin (see [1]).

(c) As in the proof of Proposition 2.7 in [9], it can be deduced that ε-l-Min $\subseteq \varepsilon$-l-WMin.

(d) $E = \varepsilon e + K \backslash \{0\}$ for $\varepsilon \geq 0$ and $e \in \text{int}K$, then E-Min reduces to the set of ε-efficient solutions which we denote by ε-Min (see [1]).

(e) $E = \varepsilon e + \text{int}K$ for $\varepsilon \geq 0$ and $e \in \text{int}K$, then E-Min reduces to the set of ε-weak efficient solutions, which we denote by ε-WMin (see [21]).

When $\varepsilon = 0$ in (e), the solution set is denoted by WMin.

We now summarize the relations between different solution concepts in a diagrammatic form.

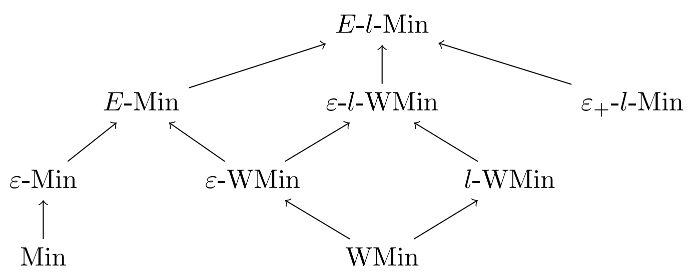

Figure 3.1: Unification of different solution concepts ($\varepsilon \geq 0, \varepsilon_+ > 0$)

Proposition 3.2. *If E is an improvement set of Y, then*

$$E\text{-Min} \subseteq \{\bar{x} \in M : \nexists\, x \in M \text{ such that } F(x) \leq^l_E F(\bar{x})\} \subseteq E\text{-}l\text{-Min}. \quad (1)$$

Proof. Let $\bar{x} \in E$-Min. If there exists $x \in M$ such that $F(x) \leq_E^l F(\bar{x})$ then, $F(\bar{x}) \subseteq F(x) + E \subseteq F(M) + E$, which contradicts the fact that $\bar{x} \in E$-Min. The second inclusion is obvious. □

Remark 3.4. From Example 3.2 it is clear that the second containment in Proposition 3.2 is strict.

In the next theorem, some sufficient conditions are given, to establish that, the last two sets in (1) coincide.

Proposition 3.3. *Let E be an improvement set of Y and $\bar{x} \in M$ be such that either of the following conditions hold*

(i) *$E \subseteq K \backslash \{0\}$ and $F(\bar{x})$ has a minimum;*
(ii) *E is a convex set and $F(\bar{x})$ has a strong minimum.*

Then $\bar{x}$ is an E-l-minimal solution of (P) *if and only if there is no $x \in M$ such that $F(x) \leq_E^l F(\bar{x})$.*

Proof. It is enough to show that there is no $x \in M$ such that $F(x) \leq_E^l F(\bar{x})$, if $\bar{x} \in E$-l-Min. On the contrary, suppose that there exists $x \in M$ such that $F(x) \leq_E^l F(\bar{x})$. By definition we have $F(\bar{x}) \leq_E^l F(x)$. Using the above two relations we get

$$F(\bar{x}) \subseteq F(\bar{x}) + E + E. \tag{2}$$

If (i) holds, then $F(\bar{x}) \subseteq F(\bar{x}) + K \backslash \{0\}$, as $E \subseteq K \backslash \{0\}$. Let $\bar{y}$ be a minimum of $F(\bar{x})$ then,

$$(F(\bar{x}) - \bar{y}) \cap (-K \backslash \{0\}) = \emptyset. \tag{3}$$

As $F(\bar{x}) \subseteq F(\bar{x}) + K \backslash \{0\}$, there exist $y \in F(\bar{x})$ and $k \in K \backslash \{0\}$, such that $\bar{y} = y + k$, which contradicts (3).

If (ii) holds, then there exists $\bar{y} \in F(\bar{x})$ such that $y - \bar{y} \in K$ for every $y \in F(\bar{x})$. As E is a convex set we have $E + E = 2E$. Therefore, from (2) we have $\bar{y} = y_1 + 2e$ for some $y_1 \in F(\bar{x})$ and $e \in E$. Hence, $-e \in K$, which together with the fact $e \in E$, implies that $0 \in E + K = E$ which is a contradiction. □

Remark 3.5. The above theorem may not hold in the absence of any of the conditions in (i) or (ii).

Example 3.3. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2, K = \mathbb{R}^2_+$ and $M = [0, 1]$.

(i) Let $F : X \rightrightarrows Y$ be defined as $F(x) = \{(y, y) : y > 0\}$ for every $x \in X$ and $E = \text{int}\mathbb{R}^2_+$ then, E-l-Min $= [0, 1]$. For any $\bar{x} \in E$-l-Min we observe that $F(\bar{x})$ does not have a minimum and $F(x) \leq_E^l F(\bar{x})$ for every $x \in M$.

(ii) Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(0,1)\}, & \text{if } x \neq 0, \\ \{(1,0)\}, & \text{if } x = 0, \end{cases}$$

and $E = \{(y_1, y_2) : y_1 + y_2 \geq 0\} \backslash \{(0,0)\} \not\subset K \backslash \{0\}$ then E-l-Min $= [0,1]$. Also, E is not convex and for any $\bar{x} \in E$-l-Min, we observe that $F(\bar{x})$ has a strong minimum and there exists $x \in M$ such that $F(x) \leq^l_E F(\bar{x})$.

(iii) Let $F : X \rightrightarrows Y$ be defined as $F(x) = \{(y, 1-y) : 0 < y \leq 1\}$ for every $x \in X$ and $E = \{(y_1, y_2) : y_1 + y_2 \geq 0, y_1 \geq 0\} \backslash \{(0,0)\} \not\subset K \backslash \{0\}$ then, E-l-Min $= [0,1]$. Also, E is a convex set and for any $\bar{x} \in E$-l-Min, we observe that $F(\bar{x})$ has a minimum but no strong minimum and $F(x) \leq^l_E F(\bar{x})$ for every $x \in M$.

In the following theorem we discuss certain properties of the set of E-l-minimal solutions.

Theorem 3.1. *Let E, E_1 and E_2 be improvement sets of Y. Then the following hold.*

(a) *If $E_1 \subseteq E_2 \subseteq K \backslash \{0\}$, then E_2-l-Min $\subseteq E_1$-l-Min.*

(b) *If $E_1 \cup E_2 \subseteq K \backslash \{0\}$, then $(E_1 \cup E_2)$-l-Min $\subseteq E_1$-l-Min $\cap$ E_2-l-Min.*

(c) *If $E_1 \cup E_2 \subseteq K \backslash \{0\}$, then E_1-l-Min $\cup$ E_2-l-Min $\subseteq (E_1 \cap E_2)$-l-Min.*

(d) *If $E \subseteq K \backslash \{0\}$, then E-l-Min $\subseteq \bigcap_{E'} E'$-$l$-Min, where $E' \subset E, E' \neq E$ is an improvement set.*

Proof. (a) If $\bar{x} \in E_2$-l-Min and $F(x) \leq^l_{E_1} F(\bar{x})$, for some $x \in M$, then $F(x) \leq^l_{E_2} F(\bar{x})$, as $E_1 \subseteq E_2$. Since $\bar{x} \in E_2$-l-Min, it follows that $F(\bar{x}) \leq^l_{E_2} F(x)$ and hence, by Lemma 3.1, we have $F(x) + E_2 = F(\bar{x}) + E_2$. Also, as $F(\bar{x}) \leq^l_{E_2} F(x)$ and $F(x) \leq^l_{E_1} F(\bar{x})$, we have

$$F(x) \subseteq F(\bar{x}) + E_2 \subseteq F(x) + E_1 + E_2 = F(\bar{x}) + E_1 + E_2.$$

As $E_2 \subseteq K \backslash \{0\}$ and $E_1 + K = E_1$, we have $F(x) \subseteq F(\bar{x}) + E_1$, which is equivalent to $F(\bar{x}) \leq^l_{E_1} F(x)$.

(b) Since $E_1 \subseteq E_1 \cup E_2 \subseteq K \backslash \{0\}$ therefore, by part (a), we have

$(E_1 \cup E_2)$-l-Min $\subseteq E_1$-l-Min.

Similarly, we have

$(E_1 \cup E_2)$-l-Min $\subseteq E_2$-l-Min.

(c) We observe that $E_1 \cap E_2 \neq \emptyset$ as

$$E_1 \cap E_2 = (E_1 + K) \cap (E_2 + K)$$

and int$K$ is nonempty. Rest of the proof follows on the lines of (b).

(d) Follows from (a).

□

Remark 3.6. (i) The condition $E_1 \subseteq E_2 \subseteq K\backslash\{0\}$ cannot be relaxed in Theorem 3.1(a). Consider problem (P) with $X = M = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \text{int}\mathbb{R}^2_+, & \text{if } x \neq 0, \\ \{(y_1, y_2) : y_1 + y_2 > 0\}, & \text{if } x = 0. \end{cases}$$

Let $E_1 = \text{int}\mathbb{R}^2_+$ and $E_2 = \{(y_1, y_2) : y_1 + y_2 > 0\}$. It can be seen that E_1-l-Min $= \{0\}$ and E_2-l-Min $= \mathbb{R}$.

(ii) The following example shows that equality in Theorem 3.1(b) is not true in general. Consider problem (P) with $X = M = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x \neq 0, \\ A, & \text{if } x = 0, \end{cases}$$

where $A = ([0, 1] \times [0, 1])\backslash([0, 1/2[\times[0, 1/2[)$. Let $E_1 = (0, 1/2) + K$ and $E_2 = (1/2, 0) + K$. Then $E_1 \cup E_2 \subseteq K\backslash\{0\}$. Clearly, $0 \in E_1$-l-Min $\cap$ E_2-l-Min but $0 \notin (E_1 \cup E_2)$-l-Min.

(iii) The equality in Theorem 3.1(c) is not true in general. Consider problem (P) with $X = M = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } x = 0, \\ \{(0, y) : -1 \leq y \leq 0\}, & \text{if } x > 0, \\ \{(y, 0) : -1 \leq y \leq 0\}, & \text{if } x < 0. \end{cases}$$

Let $E_1 = (0, 1) + K$ and $E_2 = (1, 0) + K$. Then, $E_1 \cup E_2 \subseteq K\backslash\{0\}$. Clearly, $0 \in (E_1 \cap E_2)$-l-Min but $0 \notin E_1$-l-Min $\cup$ E_2-l-Min.

(iv) Equality holds in Theorem 3.1(d) if we take $E' \subseteq E$ such that E' is an improvement set. In general, equality does not hold in Theorem 3.1(d). For example, consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(0, 0)\}, & \text{if } x \neq 0, \\ A, & \text{if } x = 0, \end{cases}$$

where $A = \{(y_1, y_2) : y_1 + y_2 \geq 1, y_1 \geq 0, y_2 \geq 0\}$, If $E = A$, then it can be seen that $0 \in E'$-l-Min for every $E' \subset E, E' \neq E$ but $0 \notin E$-l-Min.

Given $x_0 \in M$, we denote the level set of x_0 by $L(x_0)$ where

$$L(x_0) := \{x \in M : F(x) \leq^l_E F(x_0)\}.$$

Even though $\leq^l_E$ is not a reflexive relation, still, it is possible that $x_0 \in L(x_0)$ for some $x_0 \in M$. For instance, consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2, K =$

$\mathbb{R}^2_+, E = \mathrm{int}\mathbb{R}^2_+$ and $M =]0,1]$. Let $F : X \rightrightarrows Y$ be defined as $F(x) =]0,x[\times]0,x[$, for every $x \in]0,1]$. Then $x_0 \in L(x_0)$ for every $x_0 \in]0,1]$.

We now give an existence result for E-l-minimal solutions.

Theorem 3.2. *Let $x_0 \in M, E \subseteq K\backslash\{0\}$ be an improvement set. Let $y^* \in K^*\backslash\{0\}$ be such that the following conditions hold*

(i) $\inf \langle y^*, F(x)\rangle$ *is finite for each* $x \in M$;

(ii) *if* $F(x_1) \leq^l_E F(x_2), x_1 \neq x_2, x_1, x_2 \in M$, *then* $\inf \langle y^*, F(x_1)\rangle < \inf \langle y^*, F(x_2)\rangle$;

(iii) *if* $L(x_0)\backslash\{x_0\} \neq \emptyset$, *then* $\inf_{x\in L(x_0)\backslash\{x_0\}}\inf \langle y^*, F(x)\rangle$ *exists.*

Then E-l-Min $\neq \emptyset$.

Proof. If $L(x_0)\backslash\{x_0\} = \emptyset$, then $x_0 \in E$-l-Min. If $L(x_0)\backslash\{x_0\} \neq \emptyset$, then by (iii) we have $\inf_{x\in L(x_0)\backslash\{x_0\}}\inf \langle y^*, F(x)\rangle$ exists. Let

$$\inf_{x\in L(x_0)\backslash\{x_0\}}\inf \langle y^*, F(x)\rangle = \inf \langle y^*, F(x_1)\rangle \tag{4}$$

for some $x_1 \in L(x_0)\backslash\{x_0\}$. As $E \subseteq K\backslash\{0\}$, it follows that $L(x_1) \subseteq L(x_0)$. If $x_0 \in L(x_1)$, then by (ii) we have $\inf \langle y^*, F(x_0)\rangle < \inf \langle y^*, F(x_1)\rangle$. Also, $x_1 \in L(x_0)$ thus again using (ii), we obtain $\inf \langle y^*, F(x_1)\rangle < \inf \langle y^*, F(x_0)\rangle$, which is a contradiction. Hence, $x_0 \notin L(x_1)$. If $L(x_1)\backslash\{x_1\} \neq \emptyset$, there exists $x_2 \in L(x_1), x_2 \neq x_1$. Clearly, $x_2 \neq x_0$. Since $F(x_2) \leq^l_E F(x_1)$, therefore by (ii) we have

$$\inf \langle y^*, F(x_2)\rangle < \inf \langle y^*, F(x_1)\rangle . \tag{5}$$

As $x_2 \in L(x_1) \subseteq L(x_0)$, it follows from (4) that

$$\inf \langle y^*, F(x_1)\rangle \leq \inf \langle y^*, F(x_2)\rangle$$

which contradicts (5). Hence, $L(x_1)\backslash\{x_1\} = \emptyset$ and $x_1 \in E$-l-Min. □

Conditions (i) and (ii) are similar to the conditions taken by Kuroiwa in [13] to establish the existence of an l-minimal solution.

We now give an example to illustrate that the above theorem fails in the absence of condition (iii).

Example 3.4. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+, M = [0,1]$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} \{(x,t) : x \leq t \leq 1\}, & \text{if } 0 < x \leq 1, \\ \{(1,1)\}, & \text{otherwise.} \end{cases}$$

Let $E = K\backslash\{0\}$. Then $y^* \in K^*\backslash\{0\}$ is of the form (y^*_1, y^*_2) such that $y^*_1 \geq 0, y^*_2 \geq 0, y^*_1 + y^*_2 \neq 0$. Clearly, for any $y^* \in K^*\backslash\{0\}$, we observe that conditions (i) and (ii) hold but (iii) does not hold. It can be easily seen that E-l-Min $= \emptyset$.

We now give sufficient conditions for the existence of E-minimal solutions, and hence of E-l-minimal solutions (Proposition 3.1) assuming Y to be a reflexive Banach space.

Theorem 3.3. *Let Y be a reflexive Banach space and E be an improvement set. If there exists $y^* \in Y^* \backslash \{0\}$ such that the following hold*

(i) $\langle y^*, e\rangle > 0$, *for every* $e \in E$;
(ii) $\inf\langle y^*, F(M)\rangle$ *is finite*;
(iii) $F(M)$ *is weakly closed and bounded*;

then E-Min $\neq \emptyset$.

Proof. For each $n \in \mathbb{N}$ there exists $y_n \in F(M)$ such that

$$\inf \langle y^*, F(M)\rangle \leq \langle y^*, y_n\rangle \leq \inf \langle y^*, F(M)\rangle + 1/n. \tag{6}$$

Since $F(M) \subseteq Y$ is bounded and Y is reflexive Banach space, it follows that the sequence $\{y_n\}$ admits a weakly convergent subsequence. Hence, from (6) and the fact that $F(M)$ is weakly closed, it follows that there exists $y \in F(M)$ such that

$$\langle y^*, y\rangle \leq \inf \langle y^*, F(M)\rangle . \tag{7}$$

By (i) we have $\langle y^*, y - e\rangle < \langle y^*, y\rangle$ for every $e \in E$, which together with (ii), yields that $(y - E) \cap F(M) = \emptyset$. □

Remark 3.7. A similar existence theorem for a vector optimization problem has been considered by Oppezzi and Rossi [20] (Theorem 3.1).

Remark 3.8. From Example 3.2, it can be seen that the above theorem fails in the absence of condition (i). Also, in the problem considered in Example 3.4, we observe that E-Min $= \emptyset$ as $F(M)$ is not weakly closed.

4. LOWER CONVERGENCE IN THE SENSE OF PAINLEVÉ-KURATOWSKI

In this section we establish the stability aspects of problem (P) by perturbing the feasible region. The main aim of this section is to establish lower convergence for the E-l-minimal solution sets in the sense of Painlevé-Kuratowski.

Consider the following perturbed set-valued problem obtained by perturbing the feasible region of problem (P)

$$(\mathrm{P}_n) \qquad \mathrm{Min} \quad F(x) \qquad \text{subject to} \quad x \in M_n,$$

where $n \in \mathbb{N}$ and $M_n \subseteq \mathrm{dom}F$ is a perturbation of the set feasible set M.

We denote the set of all E-l-minimal solutions of (P_n) by E-l-Min(P_n). For any set A
cl A denotes the closure of the set A.

We first establish lower and upper Painlevé-Kuratowski convergence of the sequence of sets $\{F(M_n)\}$ to the set $F(M)$.

Theorem 4.1. *Let $M_n \overset{PK}{\to} M$ and* cl $\bigcup_{n\in\mathbb{N}} M_n$ *be sequentially compact. Then the following hold.*

(a) *If F is lower semicontinuous on M then* $F(M) \subseteq \text{Li } F(M_n)$.
(b) *If F is a closed map then* $\text{Ls } F(M_n) \subseteq F(M)$.

Proof. (a) Let $y \in F(M)$ then, there exists $x \in M$ such that $y \in F(x)$. Since $M_n \overset{PK}{\to} M$, there exist $x_n \in M_n$ such that $x_n \to x$. Using the lower semicontinuity of F at x, it follows that there exist $y_n \in F(x_n)$ such that $y_n \to y$. Hence, $F(M) \subseteq \text{Li } F(M_n)$.

(b) Let $y \in \text{Ls } F(M_n)$, hence there exists a subsequence $y_{n_k} \in F(x_{n_k})$ such that $y_{n_k} \to y$ where $x_{n_k} \in M_{n_k}$. Since $M_n \overset{PK}{\to} M$ and $\text{cl} \bigcup_{n\in\mathbb{N}} M_n$ is sequentially compact, it follows that the sequence x_{n_k} has a convergent subsequence. Without loss of generality, we assume that $x_{n_k} \to x \in M$. Now using the fact that the map F is closed, we have $y \in F(x)$. $\square$

Remark 4.1. If F is a vector valued map, then the Theorem 4.1 reduces to a particular case of Theorem 3.1 of [16].

We now establish lower part of convergence for the E-l-minimal solution sets in the sense of Painlevé-Kuratowski. Let $F^E : X \rightrightarrows Y$ be defined as

$$F^E(x) = \begin{cases} F(x) + E, & \text{if } x \in \text{dom}F, \\ \emptyset, & \text{if } x \notin \text{dom}F. \end{cases}$$

Theorem 4.2. *Let E be an improvement set of $Y, M_n \overset{PK}{\to} M$, $\text{cl} \bigcup_{n\in\mathbb{N}} M_n$ be sequentially compact and F^E be closed. Let $\bar{x} \in M$ be such that F is lower semicontinuous at $\bar{x}$ and for every $x \in X$ either of the following conditions hold*

(i) $E \subseteq K\backslash\{0\}$ *and $F(x)$ has a minimum;*
(ii) *E is a convex set and $F(x)$ has a strong minimum.*

If $\bar{x} \in E\text{-}l\text{-Min}$ *then* $\bar{x} \in \text{Li } E\text{-}l\text{-Min}(\text{P}_n)$.

Proof. Let $\bar{x} \in E\text{-}l\text{-Min}$ then by Proposition 3.3, for every $x \in M$ we have

$$F(\bar{x}) \not\subseteq F(x) + E. \tag{8}$$

Since $\bar{x} \in M$ and $M_n \overset{PK}{\to} M$, therefore there exist $x_n \in M_n$ such that $x_n \to \bar{x}$. It suffices to show that $x_n \in E\text{-}l\text{-Min}(\text{P}_n)$ for sufficiently large n. Suppose that there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \notin E\text{-}l\text{-Min}(\text{P}_{n_k})$. Again, using Proposition 3.3, it follows that there exist $u_{n_k} \in M_{n_k}$ such that

$$F(x_{n_k}) \subseteq F(u_{n_k}) + E. \tag{9}$$

Since $M_n \overset{PK}{\to} M$ and $\text{cl} \bigcup_{n\in\mathbb{N}} M_n$ is sequentially compact, therefore without loss of generality, $u_{n_k} \to \bar{u} \in M$. Let $\bar{z} \in F(\bar{x})$. Since F is lower semicontinuous at $\bar{x}$, therefore there exist $z_{n_k} \in F(x_{n_k})$ such that $z_{n_k} \to \bar{z}$. By (9) we have that $z_{n_k} \in F(u_{n_k}) + E$. Since F^E is a closed map, we obtain $\bar{z} \in F(\bar{u}) + E$, and hence $F(\bar{x}) \subseteq F(\bar{u}) + E$, which contradicts (8). $\square$

We now give an example to show that the conclusion of the Theorem 4.2 may not hold if F^E is not a closed map.

Example 4.1. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}, K = \mathbb{R}_+$ and $M = [-1, 1]$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} [-1,1], & \text{if } x \in [-1,1], \\ [-3,1], & \text{otherwise.} \end{cases}$$

Let $E = 2 + K, M_n = [-1 - \frac{1}{n}, 1 + \frac{1}{n}]$ then, $M_n \overset{PK}{\to} M$ and cl $\bigcup_{n \in \mathbb{N}} M_n$ is sequentially compact. The map F is lower semicontinuous at 0 and for every $x \in X$ the conditions in (i), (ii) of Theorem 4.2 hold but F^E is not closed. It can be seen that $0 \in E$-l-Min but $0 \notin$ Li E-l-Min(P$_n$).

The following example illustrates that Theorem 4.2 fails to hold in the absence of the lower semicontinuity assumption of the map F.

Example 4.2. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}, K = \mathbb{R}_+$ and $M = [-1, 1]$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} [-1,1], & \text{if } x \in \{-1,0,1\}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $E = 1 + K, M_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$ then, $M_n \overset{PK}{\to} M$ and cl $\bigcup_{n \in \mathbb{N}} M_n$ is sequentially compact. Here, F^E is closed and for every $x \in X$, the conditions in (i), (ii) of Theorem 4.2 hold but the map F is not lower semicontinuous at the points $-1, 0$ and 1. It can be seen that $-1 \in E$-l-Min but $-1 \notin$ Li E-l-Min(P$_n$). Also, $1 \in E$-l-Min but $1 \notin$ Li E-l-Min(P$_n$).

We now give an example to illustrate that upper convergence of the E-l-minimal solution sets in the sense of Painlevé-Kuratowski may not hold under the conditions of Theorem 4.2.

Example 4.3. Consider problem (P) with $X = \mathbb{R}, Y = \mathbb{R}, K = \mathbb{R}_+$ and $M = [-1, 1]$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \begin{cases} [0,1], & \text{if } x \in (-1,1), \\ \{-1\}, & \text{otherwise.} \end{cases}$$

Let $E = 1 + K, M_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$ then, $M_n \overset{PK}{\to} M$ and cl $\bigcup_{n \in \mathbb{N}} M_n$ is sequentially compact. Here, the map F is lower semicontinuous on $(-1, 1)$, F^E is closed and for every $x \in X$, the conditions in (i), (ii) of Theorem 4.2 hold. It can be seen that $0 \in$ Ls E-l-Min(P$_n$) but $0 \notin E$-l-Min.

5. CONCLUSION

In recent years, solutions of set-valued optimization problems using set criterion have been extensively studied. In this paper, we introduced a unifying notion of E-l-minimal solutions using improvement sets, which is evident from the diagrammatic representation in Figure 3.1. Apart from investigating some of the properties of E-l-minimal solution sets, we derived sufficient conditions for the existence of these solutions. We also established lower part of Painlevé-Kuratowski convergence for E-l-minimal solution sets. It would be worthwhile to investigate if complete convergence could be established for these solution sets under appropriate conditions.

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