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OPTIMALITY AND DUALITY FOR NONSMOOTH SEMI-INFINITE MULTIOBJECTIVE PROGRAMMING WITH SUPPORT FUNCTIONS

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Abstract: In this paper, we consider a nonsmooth semi-infinite multiobjective programming problem involving support functions. We establish sufficient optimality conditions for the primal problem. We formulate Mond-Weir type dual for the primal problem and establish weak, strong and strict converse duality theorems under various generalized convexity assumptions. Moreover, some special cases of our problem and results are presented.

Keywords: Nonsmooth Semi-infinite Multiobjective Optimization, Generalized Convexity, Duality.

MSC: 90C34,90C46,26A5.

1. INTRODUCTION

Semi-infinite multiobjective programming problems arise when more than one objective function is to be optimized over feasible set described by infinite number of inequality constraints. If there is only one objective function, then the problems are reduced to scalar semi-infinite programming problems. Semi-infinite programming problems have been an active research topic due to their applications in several areas of modern research such as in engineering design, mathematical physics, robotics, optimal control, transportation problems, see [8, 11, 16, 23]. Optimality conditions and duality results for semi-infinite programming problems have been studied, see [9, 12, 17, 18, 20, 24, 34, 36]. Caristi *et al.* [3] obtained optimality and duality results for semi-infinite multiobjective programming problems that involved differentiable functions. Kanzi and Nobakhtian [19] obtained several kinds of constraints qualifications, necessary and sufficient optimality conditions for nonsmooth semi-infinite multiobjective programming problems. Recently, Chuong and Kim [7] and Son and Kim [37] obtained optimality and duality for nonsmooth semi-infinite multiobjective programming problems. Many authors have discussed optimality conditions and duality results for nonlinear programming problems containing the square root of a positive semidefinite quadratic function, for example those discussed by Mond [31] and Zang and Mond [38]. Mishra *et al.* [25] proved necessary and sufficient optimality conditions for nondifferential semi-infinite programming problems involving square root of quadratic functions, see, for more details [6, 32, 33, 35]. Furthermore, the term with the square root of a positive semidefinite quadratic function has been replaced by a more general function, namely, the support function of a compact convex set, whose the subdifferential can be simply expressed. Mond and Schechter [30] have constructed symmetric duality of both Wolfe and Mond-Weir types for nonlinear programming problems where the objective contains the support function. Husain *et al.* [13] have obtained optimality and duality for a nondifferentiable nonlinear programming problem involving support function, see for more details [1, 14, 21, 22] and references therein. Convexity and their generalizations play an important role in optimization theory. The class of invex functions was introduced by Hanson [10] and named by Craven [4] as a generalization of convexity. Jayekumar and Mond [15] generalized Hanson's definition to vectorial case. Later, several other generalizations of invex functions have been introduced, for details see Mishra *at el.* [26, 27] and references therein.

This article is organized as follows: In Section 2, definitions and preliminaries are given. In Section 3, we establish the sufficient optimality conditions for multiobjective semi-infinite programming problems involving support functions. In Section 4, we formulate Mond-Weir type dual for multiobjective semi-infinite programming problems involving support functions and establish weak, strong and strict-converse duality theorems under generalized convexity assumptions. In Section 5, we discuss some special cases of the primal and dual problems.

2. DEFINITIONS AND PRELIMINARIES

In this section, we present some definitions and results which will be needed in the sequel. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be the non-negative orthant of $\mathbb{R}^n$. Let $\langle ., . \rangle$ denotes the Euclidean inner product and $\|.\|$ be

Euclidean norm in $\mathbb{R}^n$. Given a nonempty set $D \subseteq \mathbb{R}^n$, we denote the closure of D by $\bar{D}$ and convex cone (containing origin) by cone(D). The native polar cone and the strictly negative polar cone are defined respective by

$$D^{\leq} := \{d \in \mathbb{R}^n | \langle x, d \rangle \leq 0, \ \forall x \in D\},$$

$$D^{<} := \{d \in \mathbb{R}^n | \langle x, d \rangle < 0, \ \forall x \in D\}.$$

Definition 2.1. *[5] Let $D \subseteq \mathbb{R}^n$. The contingent cone $T(D, x)$ at $\bar{x} \in \bar{D}$ is defined by*

$$T(D, \bar{x}) := \{d \in \mathbb{R}^n | \exists \ t_k \downarrow 0, \ \exists \ d_k \to d : \bar{x} + t_k d_k \in D \ \ \forall k \in \mathbb{N}\}.$$

Definition 2.2. *[5] A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz near $x \in \mathbb{R}^n$, if there exist a positive constant K and a neighborhood N of x such that for any $y, z \in N$, one has*

$$|f(y) - f(z)| \leq K \parallel y - z \parallel$$

The function f is said to be locally Lipschitz on $\mathbb{R}^n$ if it is Lipschitz near x for every $x \in \mathbb{R}^n$.

Definition 2.3. *[5] The Clarke generalized directional derivative [5] of a locally Lipschitz function f at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$, denoted by $f^o(x; d)$, is defined as*

$$f^o(x; d) = \limsup_{t \downarrow 0, \ y \to x} \frac{f(y + td) - f(y)}{t}$$

where y is a vector in $\mathbb{R}^n$.

Definition 2.4. *[5] The Clarke generalized subdifferential of f at $x \in \mathbb{R}^n$ is denoted by $\partial_c f(x)$, defined as*

$$\partial_c f(x) = \{\xi \in \mathbb{R}^n : f^o(x; d) \geq \langle \xi, d \rangle, \forall d \in \mathbb{R}^n\}.$$

Definition 2.5. *[26] A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be invex at $x^* \in \mathbb{R}^n$ if there exists an n-dimensional vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(x) - f(x^*) \geq \ \langle \xi, \eta(x, x^*) \rangle,$$

for each $x \in \mathbb{R}^n$ and every $\xi \in \partial_c f(x^*)$.
The function f is said to be invex near $x^* \in \mathbb{R}^n$ if it is invex at each point of neighborhood of $x^* \in \mathbb{R}^n$.

Definition 2.6. *[26] A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strictly invex at $x^* \in \mathbb{R}^n$ if there exists an n-dimensional vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(x) - f(x^*) > \ \langle \xi, \eta(x, x^*) \rangle,$$

for each $x \in \mathbb{R}^n$, $x \neq x^*$ and every $\xi \in \partial_c f(x^*)$.
The function f is said to be strictly invex near $x^* \in \mathbb{R}^n$ if it is strictly invex at each point of neighborhood of $x^* \in \mathbb{R}^n$.

Definition 2.7. *[26] A locally Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *is said to be pseudo invex at* $x^* \in \mathbb{R}^n$ *if for all* $x \in \mathbb{R}^n$*, there exists an n-dimensional vector valued function* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\langle \xi, \eta(x, x^*) \rangle \geq 0, \text{ for some } \xi \in \partial_c f(x^*) \Rightarrow f(x) \geq f(x^*).$$

Definition 2.8. *[26] A locally Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *is said to be strictly pseudo invex at* $x^* \in \mathbb{R}^n$ *if for all* $x \in \mathbb{R}^n$*,* $x \neq x^*$*, there exists an n-dimensional vector valued function* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\langle \xi, \eta(x, x^*) \rangle \geq 0, \text{ for some } \xi \in \partial_c f(x^*) \Rightarrow f(x) > f(x^*).$$

Definition 2.9. *[26] A locally Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *is said to be quasi-invex at* x^* *if there exists an n-dimensional vector valued function* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$f(x) \leq f(x^*) \Rightarrow \langle \xi, \eta(x, x^*) \rangle \leq 0,$$

for each $x \in \mathbb{R}^n$ *and every* $\xi \in \partial_c f(x^*)$*.*
The function f is said to be quasi-invex near $x^* \in \mathbb{R}^n$ *if it is quasi-invex at each point of neighborhood of* $x^* \in \mathbb{R}^n$*.*

Remark 2.1. *[26]*

1. *Every invex function is also quasi-invex for the same* η*, but not conversely.*
2. *Every invex function is also pseudo-invex for the same* η*, but not conversely.*
3. *Every strictly invex function is also strictly pseudo-invex for the same* η*, but not conversely.*

Let C be a nonempty compact convex set in $\mathbb{R}^n$. The support function $S(.|C) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$S(x|C) = \max\{\langle z, x \rangle : z \in C\}.$$

Example 2.1. *If* $C = [0, 1]$*, then the support function* $S(\cdot|C) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is given by*

$$S(x|C) = \max\{zx : z \in C\}.$$

$$S(x|C) = \frac{|x| + x}{2}$$

The support function, being convex and everywhere finite, has a Clark subdifferential [5], in the sense of convex analysis. Its subdifferential is given by

$$\partial S(x|C) = \{z \in C : \langle z, x \rangle = S(x|C)\}.$$

For any nonempty set $S \subseteq \mathbb{R}^n$, the normal cone to S at the point $x \in S$ is denoted by $N_S(x)$ and defined as follows:

$$N_S(x) = \{y \in \mathbb{R}^n : \langle y, z - x \rangle \leq 0, \forall z \in S\}.$$

It is easy to verify that for a compact convex set C, $y \in N_c(x)$ if and only if $S(y|C) = \langle x, y\rangle$ or equivalently, x is in the subdifferential of S at y.

In this paper, we consider the following nonsmooth semi-infinite multiobjective programming problem :

$$\text{(MOSIP)} \qquad \min\left(f_1(x) + S(x|C_1), ..., f_p(x) + S(x|C_p)\right)$$
$$\text{subject to} \quad g_i(x) \leq 0, \quad i \in I$$
$$x \in \mathbb{R}^n,$$

where I is an index set which is possibly infinite, $f_j, j = 1, 2, ..., p$ and $g_i, i \in I$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$. Let M denote the feasible solutions of (MOSIP).

$$M := \{x \in \mathbb{R}^n | g_i(x) \leq 0 \ \forall i \in I\}.$$

Let $x^* \in M$. We denote $I(x^*) = \{i \in I : g_i(x^*) = 0\}$, the index set of active constraints and let

$$F(x^*) := \bigcup_{j=1}^{p} \partial_c \left(f_j(x^*) + S(x^*|C_j)\right),$$
$$G(x^*) := \bigcup_{i \in I(x^*)} \partial_c g_i(x^*).$$

The following constraint qualifications are generalization of constraint qualifications from [19] for multiobjective programming problem with support functions (MOSIP).

Definition 2.10. *We say that:*

(a) The Abedie contraint qualification(ACQ) holds at $x^* \in M$ if

$$G^{\leq}(x^*) \subseteq T(M, x^*).$$

(b) The Basic constraint qualification (BCQ) holds at $x^* \in M$ if

$$T^{\leq}(M, x^*) \subseteq cone(G(\bar{x})).$$

(c) The Regular contraint qualification(RCQ) holds at $\bar{x} \in M$ if

$$F^{<}(x^*) \cap G^{\leq}x^*) \subseteq T(M, x^*).$$

Definition 2.11. *A feasible point $x^* \in M$ is said to be weakly efficient solution for (MOSIP) if there is no $x \in M$ such that*

$$f_j(x) + S(x|C_j) < f_j(x^*) + S(x^*|C_j) \ \textit{for all} \ \ j = 1, 2, ..., p.$$

3. OPTIMALITY CONDITIONS

In this section, we prove the sufficient optimality conditions for considered nonsmooth semi-infinite multiobjective programming problem (MOSIP). For this, Theorem 3.4 (ii) from Kanzi and Nobakhtian [19] can be generalized for the multiobjective semi-infinite programming problem with support functions(MOSIP) as follows:

Theorem 3.1. *[Necessary optimality conditions] Let x^* be a weakly efficient solution for (MOSIP) and assume that a suitable constraints qualification from Definition 2.10 holds at x^*. If cone(G (x^*)) is closed, then there exist $\tau_j \geq 0$, $z_j \in C_j$ (for $j = 1, 2, ..., p$)and $\lambda_i \geq 0$ (for $i \in I(x^*)$) with $\lambda_i \neq 0$ for finitely many indices i, such that*

$$0 \in \sum_{j=1}^{p} \tau_j[\partial_c f_j(x^*) + z_j] + \sum_{i \in I(x^*)} \lambda_i \partial_c g_i(x^*), \tag{1}$$

$$\sum_{j=1}^{p} \tau_j = 1, \tag{2}$$

$$\langle z_j, x^* \rangle = S(x^*|C_j), j = 1, ..., p. \tag{3}$$

Theorem 3.2. *[Sufficient optimality conditions] Let x^* be feasible for (MOSIP) and $I(x^*)$ is nonempty. Assume that there exist $\tau_j > 0$, $z_j \in C_j$ (for $j = 1, 2, ..., p$) and scalars $\lambda_i \geq 0$ for $i \in I(x^*)$) with $\lambda_i \neq 0$ for finitely many indices i, such that necessary optimality conditions (1)-(3) hold at x^*. If $\tau_j(f_j(.) + \langle z_j, . \rangle)$, for $j = 1, 2, ..., p$ are pseudo-invex and $\lambda_i g_i(\cdot), i \in I(x^*)$ are quasi-invex at x^* with respect to the same η, then x^* is a weakly efficient solution for (MOSIP).*

Proof : We proceed by contradiction. Suppose, contrary to the result, that $x^* \in M$ is not a weakly efficient solution for (MOSIP). Then, there exists a feasible point $x \in M$ for (MOSIP) such that

$$f_j(x) + S(x|C_j) < f_j(x^*) + S(x^*|C_j), \quad \text{for all} \quad j = 1, ..., p,$$

thus, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + S(x|C_j) \right] < \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + S(x^*|C_j) \right]. \tag{4}$$

Since $\langle z, x \rangle \leq S(x|C)$ and the assumption $\langle z_j, x^* \rangle = S(x^*|C_j), j = 1, ..., p$, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \langle z_j, x \rangle \right] < \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + \langle z_j, x^* \rangle \right]. \tag{5}$$

From (1), there exist $\xi_j \in \partial_c f_j(x^*)$ and $\zeta_i \in \partial_c g_i(x^*)$ such that

$$\sum_{j=1}^{p} \tau_j \left(\xi_j + z_j \right) + \sum_{i \in I(x^*)} \lambda_i \zeta_i = 0. \tag{6}$$

Since x is a feasible point for (MOSIP) and $\lambda_i g_i(x^*) = 0, i \in I(x^*)$

$$\sum_{i \in I(x^*)} \lambda_i g_i(x) \leq \sum_{i \in I(x^*)} \lambda_i g_i(x^*), \tag{7}$$

Thus, from pseudo-invexity of $\tau_i\left(f_i(\cdot) + \langle z_i, \cdot\rangle\right)$, for $i = 1, 2, ..., p$ we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \langle z_j, x\rangle\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + \langle z_j, x^*\rangle\right],$$

which contradicts (5). This completes the proof.

The following corollary is a direct consequence of Remark 2.1 and Theorem 3.2.

Corollary 3.1. *Let x^* be feasible for (MOSIP) and $I(x^*)$ is nonempty. Assume that there exist $\tau_j > 0$, $z_j \in C_j$ (for $j = 1, 2, ..., p$) and scalars $\lambda_i \geq 0$ (for $i \in I(x^*)$) with $\lambda_i \neq 0$ for finitely many indices i, such that necessary optimality conditions (1)-(3) hold at x^*. If $\tau_j(f_j(.) + \langle z_j, .\rangle)$, for $j = 1, 2, ..., p$ are invex and $\lambda_i g_i(\cdot), i \in I(x^*)$ are invex at x^* with respect to the same η, then x^* is a weakly efficient solution for (MOSIP).*

We now give an example to illustrate the above theorem for a particular multiobjective semi-infinite programming problem.

Example 3.1. *We consider the following problem:*

$$\text{(MOSIP)} \qquad \min\left(f_1(x) + S(x|C_1),\ f_2(x) + S(x|C_2)\right)$$
$$\text{subject to} \quad g_i(x) \leq 0, \quad \forall i \in I$$
$$x \in \mathbb{R},$$

where, $I := \{2, 3, ...\}$ and $f_1,\ f_2,\ S(x|C_1),\ S(x|C_2)$ are functions defined as: $f_1(x) = -x, f_2(x) = x^2, S(x|C_1) = S(x|C_2) = |x|$ for $C_1 = C_2 = [-1, 1]$ and

$$g_i(x) = \begin{cases} \frac{1}{i}x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

The feasible solution for problem (MOSIP) is $M := (-\infty, 0]$ and for $\bar{x} = 0 \in M, I(\bar{x}) = I$. It is easy to verify that all defined functions are locally Lipschitz at $\bar{x} = 0$. Also, $\partial f_1(\bar{x}) = -1, \partial f_2(\bar{x}) = 0, \partial g_i(\bar{x}) = [\frac{1}{i}, 1], i = 2, 3, \cdots$.
Clearly necessary optimality conditions (1) − (3) of Theorem 3.1 hold at $\bar{x} \in M$, as there exist $\tau_1 = \tau_2 = \frac{1}{2}, z_1 = -1, z_2 = 0, \lambda = (1, 0, 0, ...), \xi_1 = -1, \xi_2 = 0, \zeta_i = 1,$ for $i \in I$, such that

$$\sum_{j=1}^{2} \tau_j \left(\xi_j + z_j\right) + \sum_{i \in I(x^*)} \lambda_i \zeta_i = \frac{1}{2}(-1-1) + 0 + 1 = 0.$$

It is verified that $\tau_i(f_i(x) + \langle z_i, x\rangle)$, for $i = 1, 2$ are pseudo-invex at $\bar{x}$ and $\lambda_i g_i(x)$ are quasi-invex at $\bar{x}$ with respect to $\eta(x, \bar{x}) = x - \bar{x}$.
We observe that there is no $x \in M$, such that

$$f_j(x) + S(x|C_j) < f_j(\bar{x}) + S(\bar{x}|C_j) \ \text{for all} \ \ j = 1, 2.$$

Hence, $\bar{x} = 0$ is a weakly efficient solution for (MOSIP).

4. DUALITY

Many authors have formulated Mond-Weir type dual and established duality results in various optimization problems with support functions; see [1, 2, 13, 21, 22, 30] and the references therein. Following the above mentioned works, we formulate Mond-Weir type dual for nonsmooth semi-infinite programming problem with support function (MOSIP) and establish duality theorems.

$$\text{(MOSID)} \;\; \text{Max}\left(f_1(y) + \langle z_1, y\rangle, ..., f_p(y) + \langle z_p, y\rangle\right)$$

$$0 \in \sum_{j=1}^{p} \tau_j \left(\partial_c f_j(y) + z_j\right) + \sum_{i\in I} \lambda_i \partial_c g_i(y), \tag{8}$$

$$\sum_{i\in I} \lambda_i g_i(y) \geq 0, \tag{9}$$

We now discuss the weak, strong and strict converse duality for the pair (MOSIP) and (MOSID).

Theorem 4.1. *[Weak Duality] Let x be feasible for (MOSIP) and $(y, \tau, \lambda, z_1, ..., z_p)$ be feasible for (MOSID). If $\tau_j(f_j(\cdot)+\langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ are pseudo-invex and $\lambda_i g_i(\cdot), i \in I$ are quasi-invex at y with respect to the same η. Then the following cannot hold:*

$$f_j(x) + S(x|C_j) < f_j(y) + \langle z_j, y\rangle \quad \text{for all} \quad j = 1, ..., p.$$

Proof:
Let x be feasible for (MOSIP) and $(y, \tau, \lambda, z_1, ..., z_p)$ be feasible for (MOSID), then

$$0 \in \sum_{j=1}^{p} \tau_j \left(\partial_c f_j(y) + z_j\right) + \sum_{i\in I} \lambda_i \partial_c g_i(y), \tag{10}$$

$$\sum_{i\in I} \lambda_i g_i(y) \geq 0, \tag{11}$$

According to (10), there exist $\xi_j \in \partial_c f_j(y)$ and $\zeta_i \in \partial_c g_i(y)$ such that

$$\sum_{j=1}^{p} \tau_j \left(\xi_j + z_j\right) + \sum_{i\in I} \lambda_i \zeta_i = 0 \tag{12}$$

We proceed to the result of the theorem by contradiction. Assume that

$$f_j(x) + S(x|C_j) < f_j(y) + \langle z_j, y\rangle \quad \text{for all} \quad j = 1, ..., p.$$

Thus, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + (S(x|C_j)\right] < \sum_{j=1}^{p} \tau_j \left[f_j(y) + \langle z_j, y\rangle\right]. \tag{13}$$

Using the inequality $\langle z, x\rangle \leq S(x|C)$, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \langle z_j, x\rangle\right] < \sum_{j=1}^{p} \tau_j \left[f_j(y) + \langle z_j, y\rangle\right]. \tag{14}$$

As x is feasible for (MOSIP) and $(y, \tau, \lambda, z_1, ..., z_p)$ is feasible for (MOSID), we have

$$\sum_{i\in I} \lambda_i g_i(x) \leq \sum_{i\in I} \lambda_i g_i(y).$$

From definition of quasi-invexity, we have

$$\left\langle \sum_{i\in I} \lambda_i \zeta_i, \eta(x, y)\right\rangle \leq 0, \tag{15}$$

for each $x \in X$ and every $\zeta_i \in \partial_c g_i(x)$.
Multiplying (12) by $\eta(x, y)$ and using (15), we get

$$\left\langle \sum_{j=1}^{p} \tau_j \left(\xi_j + z_j\right), \eta(x, y)\right\rangle \geq 0,$$

for each $x \in X$ and some $\xi_j \in \partial_c f_j(y)$.
Thus, from definition of pseudo-invexity, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \langle z_j, x\rangle\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(y) + \langle z_j, y\rangle\right],$$

which contradicts (14). This completes the proof.

The following corollary is a direct consequence of Remark 2.1 and Theorem 4.1.

Corollary 4.1. *Let x be feasible for (MOSIP) and $(y, \tau, \lambda, z_1, ..., z_p)$ be feasible for (MOSID). If $\tau_j(f_j(\cdot) + \langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ and $\lambda_i g_i(\cdot), i \in I$ are invex at y with respect to the same η. Then the following cannot hold:*

$$f_j(x) + S(x|C_j) < f_j(y) + \langle z_j, y\rangle \quad \text{for all} \quad j = 1, ..., p.$$

The following example shows that the generalized invexity imposed in the above theorem is essential.

Example 4.1. *We consider the following problem:*

$$\text{(MOSIP)} \qquad \min\left(f_1(x) + S(x|C_1), f_2(x) + S(x|C_2)\right)$$

$$\text{subject to} \quad g_i(x) \leq 0, \quad i \in I$$

$$x \in \mathbb{R},$$

where, $I := \mathbb{N}$ and f_1, f_2, $S(x|C_1)$, $S(x|C_2)$ are functions defined as:
$f_1(x) = -2x, f_2(x) = x^2, S(x|C_1) = S(x|C_2) = |x|$ for $C_1 = C_2 = [-1, 1]$ and

$$g_i(x) = -i|x|, \text{ for } i \in I.$$

The feasible solution for problem (MOSIP) is $M := \mathbb{R}$ and let set $\bar{x} = 1 \in M$.
The Mond-Weir dual for (MOSIP) can be given as:

$$\text{(MOSID)} \qquad \max\left(-2y + z_1, y^2 + z_2 y\right)$$

$$0 \in \sum_{j=1}^{2} \tau_j\left(\partial f_j(y) + z_j\right) + \sum_{i\in I} \lambda_i \partial g_i(y),$$

$$\sum_{i\in I} \lambda_i g_i(y) \geq 0,$$

$y \in \mathbb{R}^n, \tau_j \geq 0, \sum_{j=1}^{2} \tau_j = 1$, $\lambda_i \geq 0$ with $\lambda = (\lambda_i)_{i\in I} \neq 0$ for finitely many indices $i \in \mathbb{N}$ and $z_j \in C_j$, for $j = 1, 2$.
By choosing $\bar{y} = 0, \tau_1 = \tau_2 = \frac{1}{2}, z_1 = 1, z_2 = 0$ and $\lambda = (1, 0, ...)$. We have $(y, \tau, \lambda, z_1, z_2)$ as a feasible point of (MOSID). Observe that $\lambda_i g_i(\cdot)$ at y is not quasi-invex with respect to $\eta(y, \bar{y}) = y - \bar{y}$ and that $f_1(\bar{x}) + S(\bar{x}|C_1) = -1 < f_1(y) + \langle z_1, y\rangle = -y + y = 0$ holds. This means that quasi-invexity and pseudo-invexity are essential for weak duality as described in Theorem 4.1 .

The following theorem gives strong duality relation between the primal problem (MOSIP) and the dual problem (MOSID).

Theorem 4.2. *[Strong Duality] Let x be a weakly efficient solution for (MOSIP) at which a suitable constraints from Definition 2.10 holds at x^* and cone(G(x)) is closed. If the pseudo-invexity and quasi-invexity assumptions of the weak duality theorem are satisfied, then there exists $(\tau, \lambda, z_1, ..., z_p)$ such that $(x, \tau, \lambda, z_1, ..., z_p)$ is a weakly efficient solution for (MOSID) and the respective objective values are equal.*

Proof: Since x is a weakly efficient solution for (MOSIP) at which the suitable constraints qualification holds and cone(G(x)) is closed, from the Kuhn-Tucker necessary conditions, there exists $(\tau, \lambda, z_1, ..., z_p)$ such that $(x, \tau, \lambda, z_1, ..., z_p)$ is feasible for (MOSID).
On the other hand by weak duality theorem (4.1), the following cannot hold for any feasible y for (MOSID):

$$f_j(x) + S(x|C_j) < f_j(y) + \langle z_j, y\rangle \quad \text{for} \quad j = 1, ..., p.$$

Since $\langle z, x\rangle \leq S(x|C)$, we have

$$f_j(x) + \langle z_j, x\rangle < f_j(y) + \langle z_j, y\rangle \quad \text{for} \quad j = 1, ..., p.$$

Thus, $(x, \tau, \lambda, z_1, ..., z_p)$ is a weakly efficient solution for (MOSID) and the objective values of (MOSIP) and (MOSID) are equal at x.

The following corollary is a direct consequence of Remark 2.1 and Theorem 4.2.

Corollary 4.2. *Let x be a weakly efficient solution for (MOSIP) at which the suitable constraints qualification from Definition 2.10 holds at x^* and cone(G(x)) is closed. If the invexity assumptions of the weak duality theorem are satisfied, then there exists $(\tau, \lambda, z_1, ..., z_p)$ such that $(x, \tau, \lambda, z_1, ..., z_p)$ is a weakly efficient solution for (MOSID) and the respective objective values are equal.*

The following theorem gives strict converse duality relation between the primal problem (MOSIP) and the dual problem (MOSID).

Theorem 4.3. *[Strict converse duality] Let x^* be a weakly efficient solution for (MOSIP) at which a suitable constraint from Definition 2.10 holds at x^* and cone(G (x^*)) is closed. Let $\tau_j(f_j(\cdot) + \langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ be pseudo-invex and $\lambda_i g_i(\cdot), i \in I$ be quasi-invex with respect to the same η. If $(\overline{x}, \tau, \lambda, z_1, ..., z_p)$ is a weak efficient solution for (MOSID) and $\tau_j(f_j(\cdot) + \langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ are strictly pseudo-invex at $\overline{x}$, then $\overline{x} = x^*$.*

Proof: We prove the result of theorem by contradiction. Assume that $\overline{x} \neq x^*$. Then by strong duality Theorem (4.2) there exists $(\tau, \lambda, z_1, ..., z_p)$ such that $(x^*, \tau, \lambda, z_1, ..., z_p)$ is a weakly efficient solution for (MOSIP) and

$$f_j(x^*) + S(x^*|C_j) = f_j(\overline{x}) + \langle z_j, \overline{x}\rangle \quad for \quad j = 1, ..., p.$$

Using $\langle z_j, x^*\rangle = S(x^*|C_j), j = 1, ..., p$, we have

$$\sum_{j=1}^{p} f_j(x^*) + \langle z_j, x^*\rangle = \sum_{j=1}^{p} f_j(\overline{x}) + \langle z_j, \overline{x}\rangle \quad \text{for} \quad j = 1, ..., p. \tag{16}$$

As x^* is a weakly efficient solution for (MOSIP), $\lambda_i \geq 0$ and $(\overline{x}, \tau, \lambda, z_1, ..., z_p)$ is a weakly efficient solution for (MOSIP), we have

$$\sum_{i\in I} \lambda_i g_i(x^*) \leq \sum_{i\in I} \lambda_i g_i(\overline{x}).$$

From definition of quasi-invexity of $\lambda_i g_i(.)$,$i \in I$

$$\left\langle \sum_{i\in I} \lambda_i \zeta_i, \eta(x^*, \overline{x}) \right\rangle \leq 0, \tag{17}$$

for every $x^* \in X$ and every $\zeta_i \in \partial_c g_i(\overline{x})$.

Now from (12) and(17), we have

$$\left\langle \sum_{j=1}^{p} \tau_j \left(\xi_j + z_j\right), \eta(x^*, \overline{x}) \right\rangle \geq 0.$$

for each $x^* \in X$ and some $\xi_j \in \partial_c f_j(\overline{x})$.
Thus from strict pseudo-invexity of $\tau_j(f_j(\cdot) + \langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ at $\overline{x}$, we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x^*) + \langle z_j, x^*\rangle\right] > \sum_{j=1}^{p} \tau_j \left[f_j(\overline{x}) + \langle z_j, \overline{x}\rangle\right], \tag{18}$$

which contradicts (16). Therefore, $x^* = \overline{x}$.

The following corollary is a direct consequence of Remark 2.1 and Theorem 4.3.

Corollary 4.3. *Let x^* be a weakly efficient solution for (MOSIP) at which a suitable constraint qualification from Definition 2.10 holds at x^* and cone(G (x^*)) is closed. Let $\tau_j(f_j(\cdot) + \langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ be pseudo-invex and $\lambda_i g_i(\cdot), i \in I$ be quasi-invex with respect to the same η. If $(\overline{x}, \tau, \lambda, z_1, ..., z_p)$ is a weak efficient solution for (MOSID) and $\tau_j(f_j(\cdot) + \langle z_j, \cdot\rangle)$ for $j = 1, 2, ..., p$ are strictly pseudo-invex at $\overline{x}$, then $\overline{x} = x^*$.*

5. SPECIAL CASES

Special cases of our problem and its dual problem are as follows:

- If $C_j = \{0\}, j = 1, 2, ..., p$ then (MOSIP) reduces to the problem considered by Kanzi and Nobakhtain [19].
- If $C_j, j = 1, 2, ..., p$ are compact convex sets given by $C_j = \{B_j z_j : \langle z_j, B_j z_j\rangle \leq 1, j = 1, ..., p\}$, where $B_j, j = 1, 2, ..., p$ are positive semi-definite matrices, then we may write, $S(x|C_j) = \langle x, B_j x\rangle^{1/2}, j = 1, 2, ...p$. If f_j for $j = 1, 2, ..., p$ and $g_i, i \in I$ are differentiable, then the problems (MOSIP)and (MOSID) reduce to the problems studied by Mishra *et al.* [25].
- Let $\overline{p}$ and $\overline{q}$ be conjugate exponents; i.e. $\frac{1}{\overline{p}} + \frac{1}{\overline{q}} = 1, \overline{p} \geq 1, \overline{q} \geq 1$. Let B be matrix of appropriate dimension and $\|y\|_{\overline{p}} = [\sum_i |y_i|^{\overline{p}}]^{\frac{1}{\overline{p}}}$. If we take j=1

in objective of primal problem and $C_1 = \{B^T z : ||z||_{\overline{q}} \leq 1\}$, then following Mond and Schechter [29], we may write $S(x|C_1) = ||Bx||_{\overline{p}}$. Furthermore, if I is finite set, then the problems (MOSIP) and (MOSID) reduce to the problems considered by Mond and Schechter [28].

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REFERENCES

[1] Antczak, T., Singh, V., " Optimality and duality for minmax fractional programming with support functions under B-(p, r)-Type I assumptions", *Mathematical and Computer Modelling*, 57 (2013) 1083-1100.

[2] Bae, K. D., Kim, D. S.,"Optimality and duality theorems in nonsmooth multiobjective optimization", *Fixed point theory and applications*, 42 (2011) doi:10.1186/1687-1812-2011-42.

[3] Caristi, G. Ferrara, M., Stefanescu, A., " Semi-infinite multiobjective programming with generalized invexity", *Mathematical Report*, 12 (2010) 217-233.

[4] Craven, B. D., " Invex functions and constrained local minima",*Bulletin of the Australian Mathematical Society*, 24 (1981) 357-366.

[5] Clarke, F. H., *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[6] Chandra, S., Craven, B. D., Mond, B., " Generalized concavity and duality with a square root term", Optimization, 16 (1985) 653-662.

[7] Chuong, T. D., Kim, S. D., " Nonsmooth semi-infinite multiobjective optimization problems", *Journal of Optimization Theory and Applications*, 160 (2014) 748-762.

[8] Goberna, M. A., Lopez, M. A., *Linear Semi-Infinite Optimazation*, Wiley Chichester, 1998.

[9] Goberna, M. A., Lopez, M. A., "Linear semi-infinite programming theory, An updated survey", *European Journal of Operational Research*, 143 (2002) 390-405.

[10] Hanson, M. A., "On sufficiency of the Kuhn-Tucker conditions", *Journal of Mathematical Analysis and Applications*, 80 (1981) 545-550.

[11] Hettich, R., Haaren, E., Ries, M., Still, G., "Accurate numerical approximations of eigenfrequencies and eigenfunctions of elliptic membranes", *Journal of Applied Mathematics and Mechanic*, 67 (1987) 589-597.

[12] Hettich, R., Kortanek, K. O., "Semi-infinite programming: theory, methods and applications", SIAM Review, 35 (1993) 380-429.

[13] Husain, I., Abha, Jabeen,Z., "On nonlinear programming with support functions", *Journal of Applied Mathematics and Computing*, 10 (2002) 83-99.

[14] Husain, I., Jabeen, Z., " On fraction programming containing support functions", *Journal of Applied Mathematics and Computing*, 18 (2005) 361-376.

[15] Jayakumar, V., Mond, B., "On generalized convex mathematical programming problem", *Journal of the Australian Mathematical Society. Series B.*, 34 (1992) 43-53.

[16] Kropat, E., Pickl, S., Rößler, A., Weber, G. W., "A new algorithm from semi-infinite optimization for a problem of time-minimal control", *Computational Technologies*, 5 (4)(2000) 67-81.

[17] Kanzi, N., "Neccessary optimality conditions for nonsmooth semi-infinite programming problems", *Journal of Global Optimization*, 49 (2011) 713-725.

[18] Kanzi, N., Nobakhtian, S., "Optimality conditions for nonsmooth semi-infinite programming", *Optimization*, 53 (2008) 717-727.

[19] Kanzi, N., Nobakhtian, S., " Optimality conditions for nonsmooth semi-infinite multiobjective programming", *Optimization Letters*, 8 (2014) 1517-1528.

[20] Kostyukova, O. I., Tchemisova, T. V., " Sufficient optimality conditions for convex semi-infinite programming", *Optimization Methods and Software*, 25 (2)(2010) 279-297.

[21] Kim, D. S., Bae, K. D., "Optimality conditions and duality for a class of nondifferentiable multiobjective programming problems", *Taiwanese Journal of Mathematics*, 13 (2)(2009) 789-804.

[22] Kim, D. S., Lee, H. J.," Optimiality conditions and duality in non-smooth multiobjective programs", *Journal of Inequalities and Applications*, (2010) doi:10.1155/2010/939537.

[23] Lopez, M. A., Still, G." Semi-infinite programming", *European Journal of Operational Research*, 180 (2007) 491-518.

[24] Mishra, S. K., Jaiswal, M., Le Thi, H. A., "Duality for nonsmooth semi-infinite programming problems", *Optimization Letters*, 6 (2012) 261-271.

[25] Mishra, S. K., Jaiswal, M., " Optimality conditions and duality for nondifferential multiobjective semi-infinite programming", *Vietnam Journal of Mathematics*, 402&3 (2012) 331-343.

[26] Mishra, S. K., Giorgi, G., *Invexity and Optimization*, Springer-Verlag Berlin Heidelberg, 2008.

[27] Mishra, S. K., Wang, S. Y., Lai, K. K., *Generalized Convexity and Vector Optimization*, Springer-Verlag Berlin Heidelberg, 2009.

[28] Mond, B., Schechter, M., "A programming problem with an L_p norm in the objective function", *Journal of the Australian Mathematical Society. Series B.*, 19 (1976) 333-342.

[29] Mond, B., Schechter, M., "A duality theorem for a homogeneous fractional programming problem",*Journal of Optimization Theory and Applications*, 25 (1978) 349-359.

[30] Mond, B., Schechter, M., "Nondifferentiable symmetric duality", *Bulletin of the Australian Mathematical Society*, 53 (2)(1996) 177-188.

[31] Mond, B., "A class of nondifferentiable mathematical programming problems", *Journal of Mathematical Analysis and Applications*, 46 (1974) 169-174.

[32] Preda,V., "On generalized convexity and duality with a square root term", *Zeitschrift für Operations Research*, 36 (1992) 547-563.

[33] Preda, V., Köller, E., "Duality for a nondifferentiable programming problem with a square root term", *Revue Roumaine de Mathmatiques Pures et Appliques*, 45 (2000) 873-882.

[34] Reemtsen, R., Rückmann, J. J.,*Semi-infinite programming nonconvex optimization and its applications*, Kluwer Academic Publishers, Boston, 1998.

[35] Singh, C., "Nondifferentiable fractional programming with Hanson-Mond classes of functions", *Journal of Global Optimization*, 49 (1986) 431-447.

[36] Son, T. Q., Strodiot, J. J., Nguyen, V. H., "ε-optimality and ε-Lagrangian duality for a nonconvex programming problem with an infinite number of constraints", *Journal of Global Optimization*, 141 (2009) 389-409.

[37] Son, T. Q., Kim, D. S., "ε-mixed type duality for nonconvex multiobjective programs with an infinite number of constraints", *Journal of Global Optimization*, 57 (2013) 447-465.

[38] Zhang, J., Mond, B., "Duality for class of nondifferentiable fractional programming problems", *International Journal of Management* & *Information Systems*, 14 (1) (1998) 71-88.