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SOME REMARKS ON DUALITY AND OPTIMALITY OF A CLASS OF CONSTRAINED CONVEX QUADRATIC MINIMIZATION PROBLEMS

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Abstract: In this paper the duality and optimality of a class of constrained convex quadratic optimization problems have been studied. Furthermore, the global optimality condition of a class of interval quadratic minimization problems has also been discussed.

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MSC: 90C20, 90C26, 90C30.

1. INTRODUCTION

A large number of non-linear programming problems can be formulated as quadratic programming problems e.g., max-clique problem, rank minimization problem, etc.[2, 3, 5, 6, 7, 8, 9, 12, 13]. A global unconstrained quadratic optimization problem is of the form

$$\underset{x\in S}{\text{Minimize}}\ x^T Ax \tag{1}$$

where A is an arbitrary $n \times n$ symmetric matrix and S is the standard simplex in $\mathbb{R}^n$. One of the most significant characteristics of the form (1) is that problems of such form are NP-hard [2]. Note that, without any loss of generality, all the entries of A can be assumed as non-negative[5].

The general constrained quadratic minimization problem consists of a quadratic objective function and a set of linear inequality constraints

$$\begin{aligned} \text{Minimize } & \frac{1}{2}x^T Ax + b^T x \\ \text{Subject to } & Bx \leq c \end{aligned} \tag{2}$$

where b is an n-vector, c is an m-vector, A is an $n \times n$ matrix and B is an $m \times n$ matrix. If the matrix A is positive semidefinite or positive definite, then (2) becomes a convex programming problem and consequently, becomes solvable in polynomial time [11]. Actually, problems like (2) are generalizations of the linear programming problems, specifically in regard to the objective function. Bomze [6] explored the possibilities of applying branch-and-bound techniques to Standard Quadratic problems (StQPs). Anderson, Roos and Terlaky [1] presented an implementation of a primal-dual interior-point method for solving large scale sparse conic quadratic optimization problems based on the work of Nesterov and Todd [14] on self-scaled cones. Man-Cho So, Zhang and Ye [13] studied semidefinite programming (SDP) models for a class of discrete and continuous quadratic optimization problems in the complex Hermitian form.

In this paper, instead of considering linear constraints, we have considered quadratic constraints and characterized the global solution and duality of such class of problems. Suitable examples have also been cited to illustrate the proposed.

2. NOTATION AND PREREQUISITES

The following notations will be used throughout the sequel:

$\{e_j\}_{j=1}^n$:= The canonical basis of $\mathbb{R}^n$.

$e := (1, ..., 1)^T$.

$diag(A)$:= The $n \times n$ diagonal matrix with entries a_{ii} of a given matrix $n \times n$ matrix $A = (a_{ij})$.

Definition 2.1. [5] The symbol " $\succeq$ " denotes a partial order, namely the *Lowner partial order* on the set of matrices, defined as $A \succeq B$ or $A - B \succeq 0$ if and only if $A - B$ is positive semidefinite.

Let X_1 , $X_2 \subseteq \mathbb{R}^n$ be two partially ordered spaces and P_2 be a positive cone in X_2. Let $\phi : X_1 \to X_2$ and $f : X_1 \to X_2$ be strictly convex quadratic functions such that $\phi(x) = (x^T Ax + 2b^T x + q)$ and $f(x) = (x^T Cx + 2d^T x + k)$, where $x \in X_1$. Further, suppose that $h : X_1 \to X_2$ is of the form $h(x) = 2a^T x + p$. Let us consider the following class of convex programming problems:

$$\begin{aligned}\text{Minimize } & \phi(x),\\ \text{Subject } \textit{to } & f(x) \le \theta_{X_2},\\ & h(x) = \theta_{X_2},\\ & x \ge \theta_{X_1},\end{aligned} \tag{QMP}$$

where θ_{X_i} denotes the null vector of $X_i,\ i = 1, 2$.
Henceforth, the above problem will be referred to as QMP.

3. DUALITY AND OPTIMALITY

The Lagrangian function of QMP is

$$L(x, \lambda, \mu) = \phi(x) + \lambda f(x) + \mu h(x). \tag{3}$$

From (3), we get

$$\begin{aligned}L(x, \lambda, \mu) &= x^T Ax + 2b^T x + q + \lambda(x^T Cx + 2d^T x + k) + \mu(2a^T x + p)\\ &= x^T (A + \lambda C)x + 2(b^T + \lambda d^T + \mu a^T)x + q + \lambda k + \mu p.\end{aligned} \tag{4}$$

Let us consider the following problem:

$$\underset{x}{\text{Minimize}} \quad L(x, \lambda, \mu). \tag{5}$$

Since the function is convex and differentiable in x, the minimum is given by

$$\nabla_x L(x, \lambda, \mu) = 0. \tag{6}$$

From (6), we get

$$(A + \lambda C)x + (b + \lambda d + \mu a) = 0. \tag{7}$$

Since A and C are positive definite, they are non-singular and hence, $(A + \lambda C)$ is non-singular. This implies that equation (6) has a unique solution which is the minimizer of $L(x, \lambda, \mu)$ and consequently, the minimizer is given by

$$x = -(A + \lambda C)^{-1}(b + \lambda d + \mu a). \tag{8}$$

Substituting the value of x in (4) for the Lagrangian function, we get the dual objective as to maximize the following function:

$$L(\lambda) = \ - (b^T + \lambda d^T + \mu a^T)(A + \lambda C)^{-1}(b + \lambda d + \mu a) + (q + \lambda k + \mu p). \tag{9}$$

We rewrite the dual problem by introducing a new variable $t = \ (A + \lambda C)^{-1}(b + \lambda d + \mu a)$, as follows:

$$\begin{aligned}\text{Maximize } & - t^T (A + \lambda C)t + (q + \lambda k + \mu p)\\ \text{Subject to } & (A + \lambda C)t = \ (b + \lambda d + \mu a)\\ & \lambda \ge 0, \mu \ge 0.\end{aligned} \tag{10}$$

Let us now compute the difference Δ between the values of the primal objective at the primal feasible solution x and the dual objective at the dual feasible solution (λ, μ, t), which is given by

$$\Delta = \ x^T Ax + 2b^T x + q - (q + \lambda k + \mu p - t^T (A + \lambda C)t).$$

Since $b = (A + \lambda C)t - \lambda d - \mu a$, we have,

$$\begin{aligned}\Delta = &\ x^T Ax + t^T (A + \lambda C)t + 2[t^T (A + \lambda C) - \lambda d^T - \mu a^T]x - (\lambda k + \mu p) \\ &= (x + t)^T (A + \lambda C)(x + t) - \lambda(x^T Cx + 2d^T x + k) \\ &= -\lambda(x^T Cx + 2d^T x + k).\end{aligned} \tag{11}$$

The immediate consequences of (11) are the following:

Proposition 3.1.(Weak Duality) Since $\lambda > 0$ and $(x^T Cx + 2d^T x + k) \leq \theta_{X_1}$, for any feasible solution x of (QMP), we always get $\Delta \geq 0$. Thus, from any feasible dual solution, one can obtain a lower bound on the value of the primal. Conversely, primal feasible solution gives upper bounds on the value of the dual.

Proposition 3.2. (Strong Duality) If both primal and dual problems achieve exactly the same optimal value, then the duality gap $\Delta = 0$. Since $\lambda \geq 0$, from (11), we conclude that strong duality holds if and only if either $\lambda = 0$ or $(x^T Cx + 2d^T x + k) = 0$.

Example 3.1.

$$\begin{aligned}\text{Minimize } \phi(x) = &\ 6(x_1 - 10)^2 + 4(x_2 - 12.5)^2 \\ \text{Subject to } f_1(x) = &\ {x_1}^2 + (x_2 - 5)^2 - 50 \leq \theta \\ f_2(x) = &\ {x_1}^2 + 3{x_2}^2 - 200 \leq \theta \\ f_3(x) = &\ (x_1 - 6)^2 + {x_2}^2 - 37 \leq \theta \\ x \geq &\ \theta\end{aligned}$$

The solution of the problem $x = -(A + \lambda B + \delta C + \mu D)^{-1}(a + \lambda b + \delta c + \mu d)$ and $\lambda \geq 0,\ \delta \geq 0,\ \mu \geq 0$.
where $A = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
$a = \begin{pmatrix} -60 \\ -50 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$, $c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$
$\therefore\ \ x = -\begin{pmatrix} 6 + \lambda + \delta + \mu & 0 \\ 0 & 4 + \lambda + 3\delta + \mu \end{pmatrix}^{-1} \begin{pmatrix} -60 - 6\mu \\ -50 - 5\lambda \end{pmatrix}$
Using KKT conditions, one can verify that for $\lambda = 2,\ \delta = 0,\ \mu = 4$, the optimal solution x is given by $(7, 6)$.

4. GLOBAL OPTIMALITY OF A CLASS OF QMP

Let X_1 , $X_2 \subseteq \mathbb{R}^n$ be two partially ordered spaces and P_2 is a positive cone in X_2. Let $\phi : X_1 \to X_2$ and $f_i : X_1 \to X_2$, where $i \in \{1, 2, ..., n\}$ be strictly convex quadratic functions such that $\phi(x) = (x^T Ax+2b^T x+c)$ and $f_i(x_i) = (x_i^2-1)$, where $x_i \in X_1$. Let us Consider the following class of convex quadratic minimization problems:

$$\begin{aligned} &\text{Minimize } \phi(x) \\ \text{Subject to } &f_i(x) \leq \theta_{X_1}. \end{aligned} \tag{12}$$

The Lagrangian function of the above problem is

$$\begin{aligned} L(x,y) =& \ x^T Ax + 2b^T x + c + \sum_{i=1}^{n} y_i(x_i^2 - 1) \\ =& \ x^T(A+Y)x + 2b^T x + c - e^T y, \end{aligned} \tag{13}$$

where $Y = \ diag(y_i), \ i = 1, 2, ..., n$.
The dual problem corresponding to (13) is defined by

$$sup\{h(y) : y \in \mathbb{R}^n \cap domh\}, \tag{14}$$

where h is the dual functional given by

$$h(y) := \ inf\{L(x,y) : x \in \mathbb{R}^n\} \tag{15}$$

$$domh = \ \{y \in \mathbb{R}^n : \ h(y) > -\infty\}. \tag{16}$$

From weak duality relationship, we get

$$\phi(x) \geq h(y), \ \forall x \in F, \forall y \in \mathbb{R}^n \cap domh. \tag{17}$$

We recall the following useful results that follow from basic duality theory.

Lemma 4.1. [3] If there exists $\overline{x} \in F$, the feasible set of (12) and $\overline{y} \in \mathbb{R}^n \cap domh$ such that $\phi(\overline{x}) = h(\overline{y}) = \underset{x}{inf}(L(x, \overline{y}))$, then $\overline{x}$ is a global optimal solution of (12).

Lemma 4.2. [3] Let A be an $n \times n$ symmetric matrix and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function $\phi(x) = \ x^T Ax + 2b^T x + c$, where $b \in \mathbb{R}^n$. Then $inf\{\phi(x) : x \in \mathbb{R}^n\} > -\infty$ if and only if the following two conditions hold
(i) there exists $x \in \mathbb{R}^n$ such that $Ax + b = \ 0$.
(ii) the matrix A is positive semidefinite.

We now establish the following global optimality criteria:
Theorem 4.1. Let Y be an $n \times n$ diagonal matrix with diagonal elements $y_i, \ i = 1, 2, ..., n$. Let F be the feasible set of problem (12). If $x = -(A + Y)^{-1}b \in F$, then x is a global optimal solution of (12).
Proof. From the above lemma, we have $inf\{L(x, y) : x \in \mathbb{R}^n\} > -\infty$ if and only if the following conditions hold
(i) there exists $x \in \mathbb{R}^n$ such that $(A + Y)x + b = \ 0$.
(ii) $(A + Y) \succeq 0$.
Let x be any feasible solution of (12). For $x = -(A + Y)^{-1}b \in F$, let

$$e^T y = -x^T Ax - b^T x. \tag{18}$$

Hence, we get

$$(A+Y)x + b = (A+Y)(-(A+Y)^{-1}b) + b = 0.$$

Using (15), we rewrite the dual objective h as

$$\begin{aligned} h(y) &= \underset{x\in\mathbb{R}^n}{inf}\ \{x^T(A+Y)x + 2b^Tx + c - e^Ty\} \\ &= x^T(A+Y)x + 2b^Tx + c - e^Ty \\ &= -b^Tx + 2b^Tx + c + x^TAx + b^Tx \\ &= x^TAx + 2b^tx + c \\ &= \phi(x). \end{aligned} \tag{19}$$

Since (12) is convex, $(A+Y) \succeq 0$, implying y is feasible for (14). Hence, from lemma (4.1) and (4.2), the result follows.

Example 4.1.

$$\begin{aligned} \text{Minimize } \phi(x) = &\ x_1^2 + 6x_1x_2 + x_2^2 + 9x_1 + 3 \\ \text{Subject to } f_1(x) = &\ {x_1}^2 - 1 \leq 0 \\ f_2(x) = &\ {x_2}^2 - 1 \leq 0 \end{aligned}$$

The solution of the problem is $x = -(A+Y)^{-1}b$, where $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} \frac{9}{2} \\ 0 \end{pmatrix}$.

Therefore, $x = -\begin{pmatrix} 1+y_1 & 3 \\ 3 & 1+y_2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{9}{2} \\ 0 \end{pmatrix}$.

Using KKT conditions, one can verify that for $y_1 = \dfrac{13}{2}$ and $y_2 = 2$, the optimal solution x is given by $(-1, 1)$.

5. CONCLUSION

Theorem 4.1. provides with the global optimality criteria for convex minimization problems expressed in the form given by (12). Analogical results can be investigated for non-convex problems in future.

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