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SUFFICIENT OPTIMALITY CONDITIONS AND DUALITY FOR NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS VIA HIGHER ORDER STRONG CONVEXITY

Balendu B. UPADHYAY
Department of Mathematics
National Institute of Technology, Manipur, Imphal-795004, INDIA
bhooshan@nitmanipur.ac.in

Ningthoujam PRIYOBARTA
Department of Mathematics
National Institute of Technology, Manipur, Imphal-795004, INDIA
ningthoujampriyo9@gmail.com

Yumnam S. ROHEN
Department of Mathematics
National Institute of Technology, Manipur, Imphal-795004, INDIA
ymnehor2008@yahoo.com

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Abstract: In this paper, we define some new generalizations of strongly convex functions of order m for locally Lipschitz functions using Clarke subdifferential. Suitable examples illustrating the non emptiness of the newly defined classes of functions and their relationships with classical notions of pseudoconvexity and quasiconvexity are provided. These generalizations are then employed to establish sufficient optimality conditions for a nonsmooth multiobjective optimization problem involving support functions of compact convex sets. Furthermore, we formulate a mixed type dual model for the primal problem and establish weak and strong duality theorems using the notion of strict efficiency of order m. The results presented in this paper extend and unify several known results from the literature to a more general class of functions as well as optimization problems.

Keywords: Nonsmooth Multiobjective Programming, Support Functions, Strict Minimizers, Optimality Conditions, Mixed Duality.

MSC: 90C29, 90C46, 49N15.

1. INTRODUCTION

The nonsmooth phenomena occur naturally and frequently in optimization theory. This led to the introduction of several types of generalized directional derivatives and subdifferentials, for comprehensive overview of these concepts, we refer to Clarke [6], Loffe [16], Michel and Penot [17], Mordukhovich [27] and the references therein. The class of support functions of convex compact sets [28] is one among the few classes of non-differentiable functions, for which the subdifferential can be explicitly expressed. Every sublinear function defined on whole of $\mathbb{R}^n$ may be written as a support function. Due to their direct relation with cost functions, support functions have wider applications in several areas of modern research such as microeconomics and consumer theory, for details, see Zalinescu [32] and the references cited therein. Optimization problems involving support functions have been extensively studied, since, any property related to support functions can easily be translated to the corresponding property of the cost functions. A nonsmooth multiobjective programming problem, in which each component of the objective function contains a support function, was studied by Mond and Schechter [26]. Yang et al. [31] studied Wolfe type and Mond-Weir type dual problems for a class of nonsmooth multiobjective programming problems involving support functions. Recently, these problems have been widely studied by many scholars, see for example, Bae et al. [3], Kim and Bae [13], Kim and Lee [14], Mishra et al. [18-25] and the references cited therein.

The concept of strict local minimizers has been introduced by Cromme [8] in the study of the convergence of iterative numerical procedures. Auslender [2] studied the following optimization problem:

$$\text{Minimize } f(x)$$

$$\text{subject to } x \in S,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz function, S is a closed subset of $\mathbb{R}^n$. He [2] obtained necessary and sufficient optimality conditions for isolated local minimizers of orders 1 and 2. Studniarski [29] extended the results of Auslender [2] to any extended real-valued function f, any arbitrary set $S \subseteq \mathbb{R}^n$ and isolated local minimizers of order m. Ward [30] renamed isolated local minimizers of order m as strict local minimizers of order m and investigated the general conditions for $m = 1, 2$ under which m^{th} derivative test characterizes the strict local minimizer of order m. Jimenez [9] extended the concept of the strict minimality for scalar optimization problem to multiobjective optimization problem and introduced the concept of super-strict efficiency. Jimenez and Novo [10-11] derived first and second order sufficient optimality conditions for smooth and nonsmooth multiobjective optimization problems with an arbitrary feasible set and twice directionally differential objective function.

It is well-known that convexity plays a central role in optimization theory. However, in some real world applications, the notion of convexity does not suffice any longer. To provide a more accurate representation, modeling and solutions of real world problems, several generalizations of the notion of convexity have been introduced. For more expositions and references about generalized convex functions, we refer to the reader Mishra et al. [23-25]. Lin and Fukushima [15] introduced the concept of strongly convex functions of order m, which is a generalization of the notion of strongly convex functions of order 2, given by Karamardian and Schaible [12]. Bhatia [5] defined strong convexity of order m, and its generalizations for Lipschitz functions and obtained necessary and sufficient optimality conditions for a nonsmooth multiobjective optimization problem; also, he established its relation with a variational inequality problem.

Bae and Kim [4] obtained necessary and sufficient optimality conditions for a nonsmooth optimization problem under higher order strong convexity assumptions and established weak and strong duality theorems for a strict minimizer of order m. Recently, Arora et al. [1] have introduced four new generalized classes of strongly convex functions of order m and derived the characterizations of the solution sets of strict minimizers of order m. They have also established sufficient optimality conditions for a multiobjective optimization problem and obtained mixed duality results.

Motivated by [1, 4-5], we consider a class of nonsmooth multiobjective optimization problem involving support functions of compact convex sets and derive sufficient optimality conditions for efficient minimizers of order m in the framework of some new generalizations of strong convexity of order m. Related to the primal problem, we formulate a mixed dual model and establish weak and strong duality theorems.

We now briefly sketch the contents of the paper. In Section 2, notations and preliminaries are given along with some new generalizations of strong convexity of order m. In Section 3, sufficient optimality conditions have been derived for a class of nonsmooth multiobjective optimization problems involving support functions of compact convex sets. In Section 4, we formulate a mixed dual model for the primal problem and establish weak and strong duality theorems. In Section 5, we give conclusions on our results and suggest future research directions.

2. NOTATIONS AND PRELIMINARIES

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its nonnegative orthant. Let $\langle .,. \rangle$ denotes the Euclidean inner product. Let $X \subseteq \mathbb{R}^n$ be an open convex set equipped with the Euclidean norm $\|.\|$.

Throughout the paper, we adopt the following conventions for vectors in $\mathbb{R}^n$:

$$\begin{aligned}
x &= y \Leftrightarrow x_i = y_i, \forall i = 1, \ldots, n;\\
x &> y \Leftrightarrow x_i > y_i, \forall i = 1, \ldots, n;\\
x &\geqq y \Leftrightarrow x_i \geqq y_i, \forall i = 1, \ldots, n;\\
x &\geq y \Leftrightarrow x_i \geqq y_i, \forall i = 1, \ldots, n \quad but \quad x \neq y.
\end{aligned}$$

Following notions of nonsmooth analysis are from Clarke [6]:

Definition 2.1. *A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz at $z \in X$ if there exist a positive constant M and a neighbourhood N of z such that for any $x, y \in N$,*

$$| f(x) - f(y) | \leq M \| x - y \| .$$

The function f is said to be locally Lipschitz on X if the above condition is satisfied for all $z \in X$.

Definition 2.2. *Let $f : X \to \mathbb{R}$ be locally Lipschitz on X. The Clarke generalized directional derivative of f at $x \in X$ in the direction of a vector $v \in \mathbb{R}^n$, denoted by $f^0(x; v)$, is defined as*

$$f^0(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

Definition 2.3. *Let $f : X \to \mathbb{R}$ be locally Lipschitz on X. Clarke generalized subdifferential of f at $x \in X$, denoted by $\partial^c f(x)$, is defined as*

$$\partial^c f(x) := \{\xi \in \mathbb{R}^n : f^0(x; v) \geq \langle \xi, v \rangle, \ \forall v \in \mathbb{R}^n\}.$$

Proposition 2.4. *Let $f : X \to \mathbb{R}$ be locally Lipschitz on X. Then, for any scalar k, one has*

$$\partial^c (kf)(x) = k\partial^c f(x).$$

Definition 2.5. *Let D be a compact convex set in $\mathbb{R}^n$. The support function of D, denoted by $s(.|D)$, is defined as*

$$s(x|D) := \max\{\langle x, y \rangle : y \in D\}.$$

Furthermore, the set D is uniquely characterized by its support function. For example, the support function of a compact convex set $D = [0, 1]$ is the function

$$s(x|D) = \frac{x + |x|}{2}.$$

The support function $s(.|D)$ is always convex and finite everywhere. The subdifferential of $s(.|D)$ at x, is defined as

$$\partial^c s(x|D) := \{z \in D : \langle z, x \rangle = s(x|D)\}.$$

Definition 2.6. *(Lin and Fukushima [15]) A function $f : X \to \mathbb{R}$ is said to be a strongly convex function of order m if there exists a constant $c > 0$, such that for any $x, y \in X$ and $t \in [0, 1]$, one has*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^m.$$

For $m = 2$, the function is referred to as strongly convex in the ordinary sense, see Karamardian and Schaible [12].

Proposition 2.7. *(Lin and Fukushima [15]) If each $f_i : X \to \mathbb{R}, i = 1, ..., p$ is strongly convex of order m on a convex set X, then for $t_i \geq 0$, $i = 1, \ldots, p$, $\sum_{i=1}^{p} t_i f_i$ and $\max_{1\leq i\leq p} f_i$ are also strongly convex of order m on X.*

The following definitions are nonsmooth extensions of the concepts of pseudoconvex type I (type II) and quasiconvex type I (type II) functions given by Arora et al. [1].

Definition 2.8. *Let $f : X \to \mathbb{R}$ be locally Lipschitz on X, then f is said to be strongly pseudoconvex type I of order m on X, if there exists a constant $c > 0$, such that*

$$\begin{aligned} \langle \xi, x-y \rangle &\geq 0, \;\; for \;\; some \; \xi \in \partial^c f(y) \\ \Rightarrow f(x) &\geq f(y) + c\|x-y\|^m, \forall x, y \in X, \end{aligned}$$

or equivalently, for all $x, y \in X$,

$$f(x) < f(y) + c\|x-y\|^m \Rightarrow \langle \xi, x-y \rangle < 0, \forall \xi \in \partial^c f(y).$$

Remark 2.9. *Every strongly pseudoconvex type I function of order m is pseudoconvex. However, the converse may not be true. For example, the function $f : X = (0, 2) \to \mathbb{R}$, defined by*

$$f(x) := \begin{cases} ln\, x, & 1 < x < 2, \\ 0, & 0 < x \leq 1, \end{cases}$$

is pseudoconvex but not strongly pseudoconvex type I of any order, as for $x = \frac{1}{2}, y = 1$ and $\xi = 0 \in \partial^c f(1) = [0, 1]$, we have $\langle \xi, x-y \rangle = 0$, however, $f(x) \geq f(y) + c\|x-y\|^m$ does not hold for any $c > 0$.

Definition 2.10. *Let $f :\to \mathbb{R}$ be a locally Lipschitz function on X, then f is said to be strongly pseudoconvex type II (strictly strongly pseudoconvex type II) of order m on X, if there exists a constant $c > 0$, such that*

$$\begin{aligned} \langle \xi, x-y \rangle + c\|x-y\|^m &\geq 0 \;\; for \;\; some \; \xi \in \partial^c f(y) \\ \Rightarrow f(x) &\geq (>) f(y), \forall x, y \in X, \end{aligned}$$

or equivalently, for all $x, y \in X$,

$$\begin{aligned} f(x) &< (\leq) f(y) \\ \Rightarrow \langle \xi, x-y \rangle + c\|x-y\|^m &< 0, \forall \xi \in \partial^c f(y). \end{aligned}$$

Remark 2.11. *Every strongly pseudoconvex type II function of order m is pseudoconvex, but the converse may not be true. For example, the function* $f : X = (-2, 2) \to \mathbb{R}$*, defined by*

$$f(x) := \begin{cases} -x^2, & x \leq 0, \\ x, & x > 0, \end{cases}$$

is pseudoconvex but not strongly pseudoconvex type II of any order, as for $x = -1, y = 0$*, we have* $f(x) < f(y)$*, but,*$\langle \xi, x - y \rangle + c\|x - y\|^m > 0$*, for* $\xi = 0 \in \partial^c f(0) = [0, 1]$ *and any* $c > 0$*.*

Definition 2.12. *Let* $f : X \to \mathbb{R}$ *be a locally Lipschitz function on X, then f is said to be strongly quasiconvex type I of order m on X, if there exists a constant* $c > 0$*, such that*

$$\begin{aligned} f(x) &\leq f(y) \\ \Rightarrow \langle \xi, x - y \rangle + c\|x - y\|^m &\leq 0, \forall x, y \in X \text{ and } \xi \in \partial^c f(y). \end{aligned}$$

Remark 2.13. *Every strongly quasiconvex type I function of order m is quasiconvex, but the converse may not be true. For example, the function* $f : X = \mathbb{R} \to \mathbb{R}$*, defined by*

$$f(x) := \begin{cases} x, & x \leq 0, \\ 2x, & 0 < x < 1, \\ 2 & x \geq 1, \end{cases}$$

is quasiconvex, but not strongly quasiconvex type I of any order, as for $x = 2, y = 1$*, we have* $f(x) < f(y)$*, but for* $\xi = 2 \in \partial^c f(1) = [0, 2]$*, we have ,* $\langle \xi, x - y \rangle + c\|x - y\|^m > 0$*, for all* $c > 0$ *and any m.*

Definition 2.14. *Let* $f : X \to \mathbb{R}$ *be a locally Lipschitz function on X, then f is said to be strongly quasiconvex type II of order m on X, if there exists a constant* $c > 0$*, such that*

$$\begin{aligned} f(x) &\leq f(y) + c\|x - y\|^m \\ \Rightarrow \langle \xi, x - y \rangle &\leq 0, \forall x, y \in X \text{ and } \xi \in \partial^c f(y). \end{aligned}$$

Remark 2.15. *Every strongly quasiconvex type II function of order m is quasiconvex, but the converse may not be true. For example, the function* $f : X = \mathbb{R} \to \mathbb{R}$*, defined by*

$$f(x) := \begin{cases} x, & x \leq 0, \\ 0, & 0 < x < 1, \\ x - 1, & x > 1, \end{cases}$$

is quasiconvex, but not strongly quasiconvex type II of any order, as for $x = \frac{1}{2}, y = 0$ *and for all* $c > 0$*, we have* $f(x) \leq f(y) + c\|x - y\|^m$ *but for* $\xi = 1 \in \partial^c f(0) = [0, 1]$*, we have* $\langle \xi, x - y \rangle > 0$ *.*

3. OPTIMALITY CONDITIONS

In this section, we establish the sufficient optimality conditions for the following nonsmooth multiobjective optimization problem (NMOP) involving support functions: (NMOP)

$$\begin{aligned} &Minimize\ (f_1(x) + s(x|D_1), \ldots, f_p(x) + s(x|D_p)) \\ &\qquad Subject\ to\ g_j(x) \leq 0, j = 1, \ldots, q, \end{aligned}$$

where $f_i, g_j : X \to \mathbb{R}, i = 1, ..., p, j = 1, ..., q$ are locally Lipschitz functions on X and D_i, for each $i = 1, ..., p$ is a compact convex subset of $\mathbb{R}^n$. Let $S = \{x \in X : g_j(x) \leq 0, j = 1, ..., q\}$ and $J(x^0) = \{j | g_j(x^0) = 0, j = 1, ..., q\}$ denote the set of all feasible solutions for (NMOP) and the set of active restrictions at x^0, respectively.

Motivated by Chandra et al. [7], Bae and Kim [4] introduced the following regularity conditions for (NMOP)

Definition 3.1. *Let x^0 be a feasible solution for (NMOP). The basic regularity condition (BRC) is said to be satisfied at x^0, if there exists $r \in \{1, ..., p\}$ such that the scalars $\lambda_i^0 \geq 0$, $w_i \in D_i$, $i = 1, ..., p$, $i \neq r$, $\mu_j^0 \geq 0$, $j \notin J(x^0)$, which satisfy*

$$0 \in \sum_{i=1, i\neq r}^{p} \lambda_i^0 (\partial^c f_i(x^0) + w_i) + \sum_{j=1}^{q} \mu_j^0 \partial^c g_j(x^0)$$

are $\lambda_i^0 = 0, \forall i = 1, ..., p, i \neq r$ and $\mu_j^0 = 0, j = 1, ..., q$.

Definition 3.2. *(Arora et al. [1]) Let $m \geq 1$ be an integer. A point $x^0 \in S$ is said to be an efficient minimizer of order m for (NMOP) if there exists a constant $c \in int\mathbb{R}^p_+$, such that for all $x \in S$,*

$$f_i(x) + s(x|D_i) \nleq f_i(x^0) + s(x^0|D_i) + c_i \|x - x^0\|^m, i = 1, ..., p, i \neq k$$

$$f_k(x) + s(x|D_k) \nless f_k(x^0) + s(x^0|D_k) + c_k \|x - x^0\|^m, \quad for\ some\ k.$$

Definition 3.3. *(Bae and Kim, [4]) Let $m \geq 1$ be an integer. A point $x^0 \in S$ is said to be a strict minimizer of order m for (NMOP) if there exists a constant $c \in int\mathbb{R}^p_+$, such that for all $x \in S$,*

$$f_i(x) + s(x|D_i) \nless f_i(x^0) + s(x^0|D_i) + c_i \|x - x^0\|^m, \forall i = 1, ...p.$$

It is evident by the definitions that every efficient minimizer of order m for (NMOP) is also strict minimizer of order m.

Now, we state the following necessary optimality conditions for (NMOP) established by Bae and Kim [4].

Theorem 3.4. *(Karush-Kuhn-Tucker type necessary optimality conditions) Let $x^0 \in S$ be a strict minimizer of order m and the functions $f_i, i = 1, ..., p$ and $g_j, j = 1, ..., q$ are locally Lipschitz at x^0. Assume that the basic regularity condition (BRC) holds at x^0, then there exist $\lambda^0 \in \mathbb{R}^p_+$, $w_i^0 \in D_i, i = 1, ..., p$, $\mu^0 \in \mathbb{R}^q_+$ such that*

$$\begin{aligned} 0 &\in \sum_{i=1}^{p} \lambda_i^0 \partial^c f_i(x^0) + \sum_{i=1}^{p} \lambda_i^0 w_i^0 + \sum_{j=1}^{q} \mu_j^0 \partial^c g_j(x^0), && (8) \\ \langle x^0, w_i^0 \rangle &= s(x^0|D_i), i = 1, ..., p, && (9) \\ \mu_j^0 g_j(x^0) &= 0, j = 1, ..., q, && (10) \\ \langle \lambda^0, e \rangle &= 1, where\ \ e = (1, ..., 1) \in R^p. && (11) \end{aligned}$$

Now, we prove the following sufficient optimality conditions for (NMOP) using strong quasiconvexity type II and strong pseudoconvexity type II assumptions.

Theorem 3.5. *Let the conditions (8)-(11) be satisfied at $x^0 \in S$. Assume that $(f_i(.) + \langle ., w_i \rangle), i = 1, ..., p$ be strongly quasiconvex type II of order m on X and $\sum_{j=1}^{q} \mu_j^0 g_j(.)$ be strictly strongly pseudoconvex type II of order m on X, then x^0 is an efficient minimizer of order m for (NMOP).*

Proof: *We suppose on contrary that x^0 is not an efficient minimizer of order m for (NMOP). Then for every $c_i > 0, i = 1, ..., p$, we have*

$$\begin{aligned} f_i(x) + s(x|D_i) &\leq f_i(x^0) + s(x^0|D_i) + c_i \|x - x^0\|^m, \forall i = 1, ..., p, i \neq k \\ f_k(x) + s(x|D_k) &< f_k(x^0) + s(x^0|D_k) + c_k \|x - x^0\|^m, for\ \ some\ \ k. \end{aligned}$$

Since $(f_i(.) + \langle ., w_i \rangle), i = 1, ..., p$ is strongly quasiconvex type II of order m at x^0, from the above inequalities, it follows that

$$\begin{aligned} \langle \xi_i + w_i, x - x^0 \rangle &\leq 0, \forall i = 1, ..., p, i \neq k \\ \langle \xi_k + w_k, x - x^0 \rangle &< 0, for\ \ some\ \ k, \forall \xi_i \in \partial^c f_i(x^0)\ and\ w_i \in D_i. \end{aligned}$$

As $\langle \lambda^0, e \rangle = 1$, we get

$$\left\langle \sum_{i=1}^{p} \lambda_i^0 \xi_i + \sum_{i=1}^{p} \lambda_i^0 w_i, x - x^0 \right\rangle \leq 0, \forall \xi_i \in \partial^c f_i(x^0)\ and\ w_i \in D_i.$$

Then, from (8), we have

$$\left\langle \sum_{j=1}^{q} \mu_j^0 \zeta_j, x - x^0 \right\rangle \geq 0, \forall \zeta_j \in \partial^c g_j(x^0).$$

The above inequality implies that

$$\left\langle \sum_{j=1}^{q} \mu_j^0 \zeta_j, x - x^0 \right\rangle + c\|x - x^0\|^m \geq 0, \forall c > 0\ and\ \zeta_j \in \partial^c g_j(x^0).$$

Since $\sum_{j=1}^{q} \mu_j^0 g_j(.)$ *is strictly strongly pseudoconvex type II of order* m*, from the above inequality, we get*

$$\sum_{j=1}^{q} \mu_j^0 g_j(x) \quad > \quad \sum_{j=1}^{q} \mu_j^0 g_j(x^0) = 0,$$

which is not possible.
Hence, x^0 *is an efficient minimizer of order* m *for (NMOP).*

Remark 3.6. *In view of the definitions of strictly strong pseudoconvexity type II of order* m*, strong quasiconvexity type II of order* m *and the formulation of the problem (NMOP), it is clear that our results on optimality extend and generalize corresponding results by Arora et al. [1], Bae and Kim [4] and Bhatia [5].*

The following example justifies the significances of the Theorems 3.4 and 3.5.

Example 3.7. *We consider the following multiobjective optimization problem involving support functions:*
(P)

$$\text{Minimize } (f_1(x) + s(x|D_1), f_2(x) + s(x|D_2))$$
$$\text{Subject to } g_j(x) \quad \leq \quad 0, j = 1, 2,$$

where $D_1 = [-1, 1]$*,* $D_2 = [1, 3]$ *are compact convex sets and let* $f_i, g_j : X = (-3, 3) \to \mathbb{R}, i, j = 1, 2$ *are functions defined by*

$$f_1(x) = x^2,$$

$$f_2(x) = \begin{cases} x, & x \geq 0, \\ -x^2 + x, & x < 0, \end{cases} \quad s(x|D_1) = \mid x \mid, \; s(x|D_2) = 2x + \mid x \mid,$$

$$g_1(x) = \begin{cases} x^2 + x - 1, & x > 0; \\ x^2 - 1, & x \leq 0, \end{cases} \text{ and } g_2(x) = \begin{cases} e^x - 1, & x > 0; \\ x^3, & x \leq 0. \end{cases}$$

It is clear that the functions $f_i, g_j, i, j = 1, 2$ *are locally Lipschitz functions on* X*. Then, we have*

$$f_1(x) + s(x|D_1) = \begin{cases} x^2 + x, & x > 0; \\ x^2 - x, & x \leq 0, \end{cases} \text{ and } f_2(x) + s(x|D_2) = \begin{cases} 4x, & x \geq 0, \\ -x^2 + 2x, & x < 0. \end{cases}$$

Now, one can easily evaluate that

$$\partial^c(f_1(x) + s(x|D_1)) = \begin{cases} 2x + 1, & x > 0, \\ [-1,1], & x = 0, \\ 2x - 1, & x < 0, \end{cases} \quad \partial^c(f_2(x) + s(x|D_2)) = \begin{cases} 4, & x > 0, \\ [2,4], & x = 0, \\ -2x + 2, & x < 0, \end{cases}$$

$$\partial^c g_1(x) = \begin{cases} 2x+1, & x>0, \\ [0,1], & x=0, \\ 2x, & x<0, \end{cases} \text{ and } \partial^c g_2(x) = \begin{cases} e^x, & x>0, \\ [0,1], & x=0, \\ 3x^2, & x<0. \end{cases}$$

The set of feasible solutions for the problem (P) is $S = \{x \in X | 0 \leq x < 3\}$. It is evident that basic regularity conditions (BRC) are satisfied at the feasible point $x^0 = 0$. Moreover, there exist $r = 1 \in \{1,2\}$ and the scalars $\lambda_i^0 \geqq 0$, $w_i \in D_i$, $i = 1,2$, $i \neq r$, $\mu_j^0 \geqq 0$, $j \notin J(x^0)$, which satisfy

$$0 \in \sum_{i=1, i\neq r}^{p} \lambda_i^0 \left(\partial^c f_i(x^0) + w_i \right) + \sum_{j=1}^{q} \mu_j^0 \partial^c g_j(x^0)$$

are $\lambda_i^0 = 0, \forall i = 2, i \neq r$ and $\mu_j^0 = 0$, $j = 1$ with $J(x^0) = \{2\}$. Therefore, it is clear that there exist $\lambda = (1,0) \in \mathbb{R}^2_+$, $\mu = (1,0) \in \mathbb{R}^2_+$ such that the basic regularity conditions (BRC) and optimality conditions (8)-(11) are satisfied at $x^0 = 0$. It is easy to see that the functions $f_i(x) + s(x|D_i), i = 1,2$ are strongly quasiconvex type II of order 2 with $c_i = 1, i = 1,2$. For $\mu = (0,1) \in \mathbb{R}^2_+$, the function $\sum\limits_{j=1}^{q} \mu_j^0 g_j(.)$ is strictly strongly pseudoconvex type II of order 2 on X, with $c = 1$. Hence, the assumptions of the Theorem 3.5 are also satisfied. One can easily check that $x^0 = 0$ is an efficient minimizer of order 2 with $c = (1,1)$.

4. MIXED DUALITY

In this section, we derive the relationship between (NMOP) and its mixed type dual under the assumptions of some new generalizations of strong convexity of order m.

Let the index set $Q = \{1, ..., q\}$ be partitioned into two disjoint subsets K and J, such that $Q = K \cup J$. We formulate mixed type dual model for (NMOP) as follows: (NMOD)

$$Maximize \left(f_1(u) + \langle u, w_1 \rangle + \sum_{j \in J} \mu_j g_j(u), ..., f_p(u) + \langle u, w_p \rangle + \sum_{j \in J} \mu_j g_j(u) \right)$$

$$\begin{aligned} Subject\ to\ 0 &\in \sum_{j=1}^{p} \partial^c \lambda_i f_i(u) + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{q} \partial^c \mu_j g_j(u), \\ \mu_k g_k &\geq 0, k \in K, \\ \mu &\geq 0, w_i \in D_i, i = 1, ..., p, \\ \lambda &= (\lambda_1, ..., \lambda_p) \in \wedge^+, \end{aligned}$$

where $\wedge^+ = \{\lambda \in \mathbb{R}^p : \lambda \geq 0, \langle \lambda, e \rangle = 1, e = (1, ..., 1) \in \mathbb{R}^p\}$. Let S_D denote the set of feasible solutions for (NMOD).

Theorem 4.1. *(Weak duality) Let x and (u, λ, w, μ) be feasible solutions for (NMOP) and (NMOD), respectively. Suppose that $\left(\sum_{i=1}^{p} \lambda_i(f_i(.) + \langle ., w_i\rangle) + \sum_{j\in J} \mu_j g_j(.)\right)$, is strongly pseudoconvex type I of order m at u and $\sum_{k\in K} \mu_k g_k(.)$ is strongly quasiconvex type I of order m at u, then the following cannot hold:*

$$f_i(x) + \langle x, w_i\rangle \quad < \quad f_i(u) + \langle u, w_i\rangle + \sum_{j\in J} \mu_j g_j(u), \forall i = 1, \dots, p.$$

Proof. Since (u, λ, w, μ) is a feasible solution for (NMOD). Therefore, we get

$$\sum_{i=1}^{p} \xi_i + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{q} \zeta_j \quad = \quad 0 \tag{12}$$

and

$$\mu_k g_k(u) \quad \geq \quad 0, k \in K,$$

for some $\xi_i \in \partial^c \lambda_i f_i(u), i = 1, ..., p, w_i \in D_i$ and $\zeta_j \in \partial^c \mu_j g_j(u), j = 1, ..., q$.
Again, as x is feasible for (NMOP), $g_k(x) \leq 0, k \in K$ and also $\mu_k \geq 0, k \in K$, we have

$$\sum_{k\in K} \mu_k g_k(x) \quad \leq \quad \sum_{k\in K} \mu_k g_k(u).$$

Since $\sum_{k\in K} \mu_k g_k(.)$ is strongly quasiconvex type I of order m, therefore, there exists constant $c > 0$, such that

$$\left\langle \sum_{k\in K} \zeta_k, x - u \right\rangle + c||x - u||^m \quad \leq \quad 0, \forall \zeta_k \in \partial^c \mu_k g_k(u), k \in K.$$

Using (12) and the above inequality, we have

$$\left\langle \sum_{i=1}^{p} \xi_i + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j\in J} \zeta_j, x - u \right\rangle - c||x - u||^m \quad \geq \quad 0, \tag{13}$$

for some $\xi_i \in \partial^c \lambda_i f_i(u), w_i \in D_i$ and $\zeta_j \in \partial^c \mu_j g_j(u), j \in J$.
From (13), it follows that

$$\left\langle \sum_{i=1}^{p} \xi_i + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j\in J} \zeta_j, x - u \right\rangle \quad \geq \quad 0.$$

Since $\left(\sum_{i=1}^{p} \lambda_i(f_i(.) + \langle ., w_i\rangle) + \sum_{j\in J} \mu_j g_j(.)\right)$ is strongly pseudoconvex type I of order m at u, from the above inequality, we get

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + \langle x, w_i\rangle) + \sum_{j\in J} \mu_j g_j(x) \quad \geq \quad \sum_{i=1}^{p} \lambda_i(f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u) + c||x - u||^m. \tag{14}$$

From (14), it follows that

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + \langle x, w_i\rangle) + \sum_{j\in J} \mu_j g_j(x) \quad \geq \quad \sum_{i=1}^{p} \lambda_i(f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u).$$

Since $g_j(x) \leq 0, \mu_j \geq 0, j \in J$, we have

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + \langle x, w_i\rangle) \quad \geq \quad \sum_{i=1}^{p} \lambda_i(f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u),$$

which is equivalent to

$$\sum_{i=1}^{p} \lambda_i((f_i(x) + \langle x, w_i\rangle) - (f_i(u) + \langle u, w_i\rangle)) \quad \geq \quad \sum_{j\in J} \mu_j g_j(u).$$

Therefore,

$$(f_i(x) + \langle x, w_i\rangle) \quad < \quad (f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u), \forall i = 1, ..., p,$$

cannot hold.
Hence, the result follows.

Now, we prove the weak duality theorem for (NMOD) using strong pseudoconvex type II and strong quasiconvex type II functions. □

Theorem 4.2. *(Weak duality) Let x and (u, λ, w, μ) be feasible solutions for (NMOP) and (NMOD), respectively. Assume that $\left(\sum_{i=1}^{p} \lambda_i(f_i(.) + \langle ., w_i\rangle) + \sum_{j\in J} \mu_j g_j(.)\right)$, is strongly pseudoconvex type II of order m at u and $\sum_{k\in K} \mu_k g_k(.)$ is strongly quasiconvex type II of order m at u, then the following cannot hold:*

$$f_i(x) + \langle x, w_i\rangle \quad < \quad f_i(u) + \langle u, w_i\rangle + \sum_{j\in J} \mu_j g_j(u), \forall i = 1, ..., p.$$

Proof. Since (u, λ, w, μ) is feasible solution for (NMOD). Therefore, we get

$$\sum_{i=1}^{p} \xi_i + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{q} \zeta_j \quad = \quad 0, \tag{15}$$

for some $\xi_i \in \partial^c \lambda_i f_i(u), i = 1, ..., p, w_i \in D_i$ and $\zeta_j \in \partial^c \mu_j g_j(u), j = 1, ..., q$.
Again, as x is feasible for (NMOP), $g_k(x) \leq 0, k \in K$ and also $\mu_k \geq 0, k \in K$, hence

$$\sum_{k\in K} \mu_k g_k(x) \quad \leq \quad \sum_{k\in K} \mu_k g_k(u).$$

Then, for all $c > 0$, we have

$$\sum_{k\in K} \mu_k g_k(x) \quad \leq \quad \sum_{k\in K} \mu_k g_k(u) + c\|x-u\|^m. \tag{16}$$

Since $\sum_{k\in K} \mu_k g_k(.)$ is strongly quasiconvex type II of order m, from (16), we get

$$\left\langle \sum_{k\in K} \zeta_k, x-u \right\rangle \quad \leq \quad 0, \forall \zeta_k \in \partial^c \mu_k g_k(u), k \in K. \tag{17}$$

Using (13) and (17), we have

$$\left\langle \sum_{i=1}^{p} \xi_i + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j\in J} \zeta_j, x-u \right\rangle + c\|x-u\|^m \quad \geq \quad 0,$$

for all $c > 0$ and for some $\xi_i \in \partial^c \lambda_i f_i(u), w_i \in D_i, i = 1, ..., p$ and $\zeta_j \in \partial^c \mu_j g_j(u), j \in J$. Since $\left(\sum_{i=1}^{p} \lambda_i(f_i(.) + \langle ., w_i\rangle) + \sum_{j\in J} \mu_j g_j(.)\right)$ is strongly pseudoconvex type II of order m at u, we get

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + \langle x, w_i\rangle) + \sum_{j\in J} \mu_j g_j(x) \quad \geq \quad \sum_{i=1}^{p} \lambda_i(f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u). \tag{18}$$

From (18), we have

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + \langle x, w_i\rangle) + \sum_{j\in J} \mu_j g_j(x) \quad \geq \quad \sum_{i=1}^{p} \lambda_i(f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u).$$

Since $g_j(x) \leq 0, \mu_j \geq 0, j \in J$, we get

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + \langle x, w_i\rangle) \quad \geq \quad \sum_{i=1}^{p} \lambda_i(f_i(u) + \langle u, w_i\rangle) + \sum_{j\in J} \mu_j g_j(u),$$

which is equivalent to

$$\sum_{i=1}^{p} \lambda_i((f_i(x) + \langle x, w_i\rangle) - (f_i(u) + \langle u, w_i\rangle)) \quad \geq \quad \sum_{j\in J} \mu_j g_j(u).$$

Therefore,

$$f_i(x) + \langle x, w_i\rangle \quad < \quad f_i(u) + \langle u, w_i\rangle + \sum_{j\in J} \mu_j g_j(u), \forall i = 1, ...p,$$

cannot hold. Hence, the result follows. □

Definition 4.3. *Let $m \geq 1$ be an integer. A point $(u^0, \lambda^0, w^0, \mu^0) \in S_D$ is said to be a strict maximizer of order m for (NMOD) if there exists an $c \in int\mathbb{R}^p_+$, such that for all $(u, \lambda, w, \mu) \in S_D$ and $i = 1, ..., p$,*

$$f_i(x^0) + s(x^0|D_i) + \sum_{j\in J} \mu_j g_j(x^0) + c_i\|x - x^0\|^m \quad \nless \quad f_i(x) + s(x|D_i) + \sum_{j\in J} \mu_j g_j(x)$$

Theorem 4.4. *(Strong duality) If x^0 is a strict minimizer of order m for (NMOP) and the basic regularity condition (BRC) holds at x^0, then there exist $\lambda^0 \in \mathbb{R}^p_+, w_i \in D_i, i = 1, ..., p$ and $\mu^0 \in \mathbb{R}^q_+$, such that $(x^0, w^0, \lambda^0, \mu^0)$ is a feasible solution for (NMOD) and $\langle x^0, w^0_i\rangle = s(x^0|D_i), i = 1, ..., p$. Moreover, if the assumptions of weak duality theorem(either Theorem 4.1 or Theorem 4.2) are satisfied, then $(x^0, w^0, \lambda^0, \mu^0)$ is a strict maximizer of order m for (NMOD).*

Proof. Let x^0 is a strict minimizer of order m for (NMOP) and the basic regularity condition (BRC) holds at x^0. Then by Theorem 3.1, there exist $\lambda^0 \in \mathbb{R}^p_+, \langle \lambda^0, e\rangle = 1, w^0_i \in D_i, i = 1, ..., p, \mu^0 \in \mathbb{R}^q_+$, such that Karush-Kuhn-Tucker optimality conditions (8)-(11) are satisfied at x^0 . From which, we conclude that $(x^0, w^0, \lambda^0, \mu^0)$ is a feasible solution for (NMOD). Applying Theorem 4.1 (Theorem 4.2), the following cannot hold:

$$f_i(x^0) + \langle x^0, w^0_i\rangle \quad < \quad f_i(u) + \langle u, w_i\rangle + \sum_{j\in J} \mu_j g_j(u), \forall i = 1, ..., p,$$

where (u, w, λ, μ) is any feasible solution for (NMOD).
Therefore, for any $c \in int\mathbb{R}^p_+$ and any $(u, w, \lambda, \mu) \in S_D$, we get

$$f_i(x^0) + \langle x^0, w^0_i\rangle + \sum_{j\in J} \mu_j g_j(x^0) + c_i\|u - x^0\|^m \quad \nless \quad f_i(u) + \langle u, w_i\rangle + \sum_{j\in J} \mu_j g_j(u), \forall i = 1, ..., p.$$

Consequently, $(x^0, w^0, \lambda^0, \mu^0)$ is a strict maximizer of order m for (NMOD).
Hence, the result follows. □

Remark 4.5. *If $D_i = \{0\}, i = 1, ..., k$ and the functions $f_i, g_j; i = 1, ..., p, j = 1, ..., q$ are differentiable, our results on duality reduce to the one of Arora et al. [1]*

5. CONCLUSION AND FUTURE DIRECTIONS

In this paper, we have defined some new generalizations of strong convexity of order m for locally Lipschitz functions using Clarke subdifferential. Relationships between newly defined functions and the existing classical functions are analyzed through suitable examples. We consider a nonsmooth multiobjective optimization problem involving support functions (NMOP) and establish sufficient optimality conditions using these generalizations. Furthermore, we formulate a mixed type dual model (NMOD) related to the primal problem (NMOP) and establish weak and strong duality theorems using the generalizations of the strongly convex functions of order m and the notion of strict efficiency of order m. The results of the paper extend and unify several results of Arora et al. [1], Bae and Kim [4] and Bhatia [5], on optimality and duality into the nonsmooth case as well as to a more general class of optimization problems. However, for the sake of simplicity and easy understanding, we have focused on nonsmooth multiobjective optimization problem involving locally Lipschitz functions and using Clarke subdifferential. One can attempt to carry out the similar study for a more general class of nonsmooth vector optimization problems involving only lower semicontinuous functions.It is known that the Mordukhovich limiting subdifferential [27] has much better Lagrange multiplier rule than Clarke subdifferential and for locally Lipschitz functions, it is nonempty and bounded by the rank of Lipschitzia. One can obtain simillar results in more general space setting, such as Banach or Asplund spaces using limiting subdifferential. Furthermore, the results of this paper could be used to establish the relationship between nonsmooth vector variational inequality problems and nonsmooth vector optimization problems using Michel-Penot subdifferentials [17] or Mordukhovich limiting subdifferentials [27].

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