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HIGHER ORDER DUALITY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEM WITH GENERALIZED CONVEXITY

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Abstract: We have introduced higher order generalized hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invex function. Then, we have estabilished higher order weak, strong and strict converse duality theorems for a multiobjective fractional programming problem with support function in the numerator of the objective function involving higher order generalized hybrid $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invex functions. Our results extend and unify several results from the literature.

Keywords: Multiobjective Fractional Programming, Support Function, Duality, Higher Order $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invex Function.

MSC: 90C26, 90C29, 90C32, 90C46.

1. INTRODUCTION

In the last three decades, several definitions extending the concept of convexity of a function have been introduced by many researchers including Schmitendorf [17], Vial [21], Hanson and Mond [6], Rueda and Hanson [16], Preda [14], and Antczak [1]. A significant generalization of convex function is introduced by Hanson [5] and Cravan [2]. In 1981, Hanson [5] generalized the Karush-Kuhn-Tucker type sufficient optimality conditions with the help of a new class of generalized convex functions for differentiable real valued functions which are defined on R^n. This class of functions was later named by Cravan [2] as the class of "invex" functions due to their property of invariance under convex transformations.

Duality for nonlinear programming was studied by many researchers. Zalmai [22] studied nondifferentiable fractional programming containing arbitrary norms. Husain and Jabeen [3] studied duality for a fractional programming problem involving support function. Yang [19] introduced nondifferentiable multiobjective programming problem where the objective function contains a support function of a compact convex set.

Higher order duality has been studied by many researchers in the last few years, e.g., Zhang [23] obtained higher order duality in multiobjective programming problem; Mangasarian [8] formulated a class of second and higher order dual for nonlinear programming problems involving twice differentiable functions; Mond and Weir [12] also estabilshed higher order duality for generalized convexity; Mond and Zhang [13] obtained duality results for various higher order dual programming problems under higher order invexity assumptions; Mishra and Rueda [9, 11] estabilished duality results under higher order generalized invexity; Yang *et al.* [20] discussed higher order duality results under generalized convexity assumption for multiobjective programming problems involving support function. Kim and Lee [7] also studied higher order duality. Mishra and Giorgi [10] presented several types of invexity and higher order duality in their book Invexity and Optimization.

In this paper, motivated by the earlier works on higher order duality, we first introduce one new generalized invex function, called higher order $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invex function. Further, conditions have been obtained under which a fractional function is higher order $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invex with respect to a differentiable function $G : X \times R^n \to R, (X \subset R^n)$. More precisely, this paper is an extension of the generalized hybrid $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invex function, introduced by Verma [18] to a class of higher-order duality; the sufficient optimality conditions have also been derived and duality results have been estabilished for Schaible type dual of a nondifferentiable multiobjective fractional programming problem.

2. NOTATIONS AND PRELIMINARIES

In this section, we discuss some important notations and definitions.
The following convention for vector inequalities will be used throughout this paper.
The index set K= $\{1, 2, \ldots, k\}$ and $M = \{1, 2, \ldots, m\}$.

If $x, y \in R^n$, then

$(i) x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, 3, \dots, n$;
$(ii) x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, 3, \dots, n$;
$(iii) x \geqq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, 3, \dots, n$;
$(iv) x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Definition 2.1. *A function f is said to be higher order $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invex at $u \in X$ with respect to the function $G(u, p)$, if there exist $b : X \times X \to [0, \infty)$, $\rho : X \times X \to R$ and $\theta : X \times X \to R^n$, where X is a nonempty subset of R^n (n- dimensional Euclidean space) such that for $y \in R^n$ and $\tilde{p}, \tilde{r} \in R$, we have*

$$b(x,u)\left[\frac{1}{\tilde{r}}(e^{\tilde{r}(f(x)-f(u))}-1)\right] \geqq \frac{1}{\tilde{p}}\left(\langle \nabla f(u) + \nabla_p G(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle\right) + G(u,p) - p^T \nabla_p G(u,p) + \rho(x,u)\|\theta(x,u)\|^2, \quad (1)$$

We note that

$$(i)\ b(x,u)\left(f(x) - f(u)\right) \geqq \frac{1}{\tilde{p}}\left(\langle \nabla f(u) + \nabla_p G(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle\right) + G(u,p) - p^T \nabla_p G(u,p) + \rho(x,u)\|\theta(x,u)\|^2 \text{ for } \tilde{p} \neq 0,\ \tilde{r} = 0,$$

$$(ii)\ b(x,u)\left[\frac{1}{\tilde{r}}(e^{\tilde{r}(f(x)-f(u))}-1)\right] \geqq \left(\langle \nabla f(u) + \nabla_p G(u,p), y\rangle\right) + G(u,p) - p^T \nabla_p G(u,p) + \rho(x,u)\|\theta(x,u)\|^2 \text{ for } \tilde{p} = 0,\ \tilde{r} \neq 0,$$

$$(iii)\ b(x,u)\left(f(x) - f(u)\right) \geqq \left(\langle \nabla f(u) + \nabla_p G(u,p), y\rangle\right) + G(u,p) - p^T \nabla_p G(u,p) + \rho(x,u)\|\theta(x,u)\|^2 \text{ for } \tilde{p} = 0,\ \tilde{r} = 0.$$

Definition 2.2. *A function f is said to be strictly higher order $B-(b, \rho, \theta, \tilde{p}, \tilde{r})$-invex at $u \in X$ with respect to the function $G(u, p)$, if there exist $b : X \times X \to [0, \infty)$, $\rho : X \times X \to R$ and $\theta : X \times X \to R^n$ such that for $y \in R^n$ and $\tilde{p}, \tilde{r} \in R$, we have*

$$b(x,u)\left[\frac{1}{\tilde{r}}(e^{\tilde{r}(f(x)-f(u))}-1)\right] > \frac{1}{\tilde{p}}\left(\langle \nabla f(u) + \nabla_p G(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle\right) + G(u,p) - p^T \nabla_p G(u,p) + \rho(x,u)\|\theta(x,u)\|^2.$$

Remark 2.1. *The exponentials appearing on the right hand side of the above inequalities are understood to be taken componentwise and $\mathbf{1} = (1, 1, \dots, 1) \in R^n$.*

Definition 2.3. *Let C be a compact convex set in R^n and support $s(x/C)$ of C is defined by*

$$s(x/C) = max\left\{x^T y^* : y^* \in C\right\}.$$

The support function s(x/C), being convex and everywhere finite, has a subgradient [Rockafellar [15]] at every point x, i.e., there exists $u \in C$ such that,

$$s(y^*/C) \geqq s(x/C) + u^T(y^* - x).$$

And the subdifferential of $s(x/C)$ is given by

$$\partial s(x/C) = \left\{u \in S : u^T x = s(x/C)\right\}.$$

For any set $C \subset R^n$, the normal cone to C at any point $x \in C$ is defined by

$$N_c(x) = \left\{y^* \in R^n : {y^*}^T(u - x) \leqq 0 \, for\ all\ u \in C\right\}.$$

It could be verified that for a compact convex set C, $y^* \in N_c\,(x)$ if and only if $s(y^*/C) = x^T y^*$, or equivalently, x is in the subdifferential of s at y^*.

In this paper, we consider the following nondifferentiable multiobjective fractional programming problem :

$$(\mathbf{P}) \quad Minimize\ F(x) = \left(\frac{f_1(x)+s(x/D_1)}{g_1(x)}, \frac{f_2(x)+s(x/D_2)}{g_2(x)}, \ldots, \frac{f_k(x)+s(x/D_k)}{g_k(x)}\right)$$

$$\text{subject to } x \in X_0 = \left\{x \in X : h_j(x) + s(x/E_j) \leqq 0, j \in M\right\}, \qquad (2)$$

where X_0 denotes the set of all feasible solutions for (**P**).

The functions $f = (f_1, f_2, \ldots, f_k) : X \to R^k$, $g = (g_1, g_2, \ldots, g_k) : X \to R^k$ and $h = (h_1, h_2, \ldots, h_m) : X \to R^m$ are differentiable on X, $f_i(x) + s(x/D_i) \geqq 0$ and $g_i(x) > 0$ $(i \in K)$ for $x \in X$.
Let $G_i : X \times R^n \to R$ $(i \in k)$ be differentiable functions on X, $D_i (i \in K)$, and $E_j (j \in M)$ are compact convex sets in R^n, $z = (z_1, z_2, \ldots, z_k)$, and $w = (w_1, w_2, \ldots, w_m)$, where $z_i \in D_i$ $(i \in K)$ and $w_j \in E_j$ $(j \in M)$.

It is very rare to get ideal solution in multiobjective programming problems, i.e., a point at which all objective functions are optimised, due to conflict of objectives in multiobjective programming problems, optimal solution of one objective is different from optimal solution of another. Thus, we choose an optimal solution from the set of the efficient solutions.

Definition 2.4. *A point $u \in X_0$ is said to be an efficient solution of (**P**), if there is no $x \in X_0$ such that $F(x) \leq F(u)$.*

3. SUFFICIENT OPTIMALITY CONDITIONS

The following result gives the conditions for a fractional function to be higher order $B - (b, \rho, \theta, \tilde{p}, \tilde{r})$-invex. We use it to obtain Karush-Kuhn-Tucker type sufficient optimality conditions.

Theorem 3.1. *Suppose that for some α, if $f_\alpha(.) + (.)^T z_\alpha$ and $-g_\alpha(.)$ are higher order $B-(b_\alpha, \rho_\alpha, \theta_\alpha, \tilde{p}, \tilde{r})$-invex at $u \in X$ with respect to the function $G_\alpha(u,p)$ for same $y \in R^n$, then the fractional function $\frac{f_\alpha(.)+(.)^T Z_\alpha}{g_\alpha(.)}$ is higher order $B-\left(\bar{b}_\alpha, \rho_\alpha, \bar{\theta}_\alpha, \tilde{p}, \tilde{r}\right)$-invex at u with respect to the function $\bar{G}_\alpha(u,p)$, where*

$\bar{b}_\alpha(x,u) = \frac{g_\alpha(x)}{g_\alpha(u)} b_\alpha(x,u)$,

$\bar{\theta}_\alpha(x,u) = \left(\frac{1}{g_\alpha(u)} + \frac{f_\alpha(u)+u^T z_\alpha}{g_\alpha^2(u)}\right)^{\frac{1}{2}} \theta_\alpha(x,u)$,

$\bar{G}_\alpha(u,p) = \left(\frac{1}{g_\alpha(u)} + \frac{f_\alpha(u)+u^T z_\alpha}{g_\alpha^2(u)}\right) G_\alpha(u,p)$.

Proof

Since $\left(\frac{f_\alpha(x)+x^T z_\alpha}{g_\alpha(x)} - \frac{f_\alpha(u)+u^T z_\alpha}{g_\alpha(u)}\right) = \frac{f_\alpha(x)+x^T z_\alpha - f_\alpha(u) - u^T z_\alpha}{g_\alpha(x)} - \frac{f_\alpha(u)+u^T z_\alpha}{g_\alpha(x) g_\alpha(u)}(g_\alpha(x) - g_\alpha(u))$.

Using the definition of higher order invex, we get

$$\begin{aligned}
& b_\alpha(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_\alpha(x)+x^T z_\alpha}{g_\alpha(x)} - \frac{f_\alpha(u)+u^T z_\alpha}{g_\alpha(u)}\right)} - 1\right)\right] \\
& \geqq \frac{1}{g_\alpha(x)}\left(\frac{1}{\tilde{p}}\left(\langle \nabla f_\alpha(u) + z_\alpha + \nabla_p G_\alpha(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle\right)\right. \\
& \quad \left. + G_\alpha(u,p) - p^T \nabla_p G_\alpha(u,p) + \rho_\alpha(x,u)\|\theta_\alpha(x,u)\|^2\right) \\
& \quad + \frac{f_\alpha(u) + u^T z_\alpha}{g_\alpha(x) g_\alpha(u)}\left(\frac{1}{\tilde{p}}\left(\langle -\nabla g_\alpha(u) + \nabla_p G_\alpha(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle\right)\right. \\
& \quad \left. + G_\alpha(u,p) - p^T \nabla_p G_\alpha(u,p) + \rho_\alpha(x,u)\|\theta_\alpha(x,u)\|^2\right).
\end{aligned}$$

Since inner product on R^n is symmetric, we obtain

$$\begin{aligned}
& b_\alpha(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_\alpha(x)+x^T z_\alpha}{g_\alpha(x)} - \frac{f_\alpha(u)+u^T z_\alpha}{g_\alpha(u)}\right)} - 1\right)\right] \\
& \geqq \frac{1}{g_\alpha(x)}\left(\frac{1}{\tilde{p}}(e^{\tilde{p}y} - \mathbf{1})^T \left(\nabla f_\alpha(u) + z_\alpha + \nabla_p G_\alpha(u,p)\right)\right. \\
& \quad \left. + G_\alpha(u,p) - p^T \nabla_p G_\alpha(u,p) + \rho_\alpha(x,u)\|\theta_\alpha(x,u)\|^2\right) \\
& \quad + \frac{f_\alpha(u) + u^T z_\alpha}{g_\alpha(x) g_\alpha(u)}\left(\frac{1}{\tilde{p}}(e^{\tilde{p}y} - \mathbf{1})^T \left(-\nabla g_\alpha(u) + \nabla_p G_\alpha(u,p)\right)\right. \\
& \quad \left. + G_\alpha(u,p) - p^T \nabla_p G_\alpha(u,p) + \rho_\alpha(x,u)\|\theta_\alpha(x,u)\|^2\right)
\end{aligned}$$

$$= \frac{g_\alpha(u)}{\tilde{p}g_\alpha(x)}\left[(e^{\tilde{p}y}-\mathbf{1})^T\left(\frac{\nabla f_\alpha(u)+z_\alpha}{g_\alpha(u)}+\frac{\nabla_p G_\alpha(u,p)}{g_\alpha(u)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha^2(u)}(-\nabla g_\alpha(u)+\nabla_p G_\alpha(u,p))\right)\right]$$
$$+\left(\frac{1}{g_\alpha(x)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(x)g_\alpha(u)}\right)\left(G_\alpha(u,p)-P^T\nabla_p G_\alpha(u,p)\right)$$
$$+\rho_\alpha(x,u)\left(\frac{1}{g_\alpha(x)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(x)g_\alpha(u)}\right)\|\theta_\alpha(x,u)\|^2$$

$$= \frac{g_\alpha(u)}{\tilde{p}g_\alpha(x)}\left(\langle\nabla\left(\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(u)}\right)+\left(\frac{1}{g_\alpha(u)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha^2(u)}\right)\nabla_p G_\alpha(u,p),e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\left(\frac{1}{g_\alpha(x)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(x)g_\alpha(u)}\right)\left(G_\alpha(u,p)-P^T\nabla_p G_\alpha(u,p)\right)$$
$$+\rho_\alpha(x,u)\left(\frac{1}{g_\alpha(x)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(x)g_\alpha(u)}\right)\|\theta_\alpha(x,u)\|^2,$$

which implies

$$\frac{g_\alpha(x)}{g_\alpha(u)}b_\alpha(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_\alpha(x)+x^Tz_\alpha}{g_\alpha(x)}-\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(u)}\right)}-1\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle\nabla\left(\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(u)}\right)+\left(\frac{1}{g_\alpha(u)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha^2(u)}\right)\nabla_p G_\alpha(u,p),e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\left(\frac{1}{g_\alpha(u)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha^2(u)}\right)\left(G_\alpha(u,p)-p^T\nabla_p G_\alpha(u,p)\right)$$
$$+\rho_\alpha(x,u)\left(\frac{1}{g_\alpha(u)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha^2(u)}\right)\|\theta_\alpha(x,u)\|^2$$

it follows that,

$$\bar{b}_\alpha(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_\alpha(x)+x^Tz_\alpha}{g_\alpha(x)}-\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(u)}\right)}-1\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle\nabla\left(\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha(u)}\right)+\nabla_p\bar{G}_\alpha(u,p),e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\bar{G}_\alpha(u,p)-p^T\nabla_p\bar{G}_\alpha(u,p)+\rho_\alpha(x,u)\|\bar{\theta}_\alpha(x,u)\|^2.$$

Therefore, $\frac{f_\alpha(.)+(.)^Tz_\alpha}{g_\alpha(.)}$ is higher order $B-\left(\bar{b}_\alpha,\rho_\alpha,\bar{\theta}_\alpha,\tilde{p},\tilde{r}\right)$-invex at u with respect to function $\bar{G}_\alpha(u,p)$. □

Remark 3.1. *It can also be shown that* $\frac{f_\alpha(.)+(.)^Tz_\alpha}{g_\alpha(.)}$ *is higher order* $B-\left(\bar{b}_\alpha,\bar{\rho}_\alpha,\theta_\alpha,\tilde{p},\tilde{r}\right)$*-invex at* u *with respect to* $\bar{G}_\alpha(u,p)$*, where* $\bar{b}_\alpha(x,u)$*,* $\bar{G}_\alpha(u,p)$ *are as the above and* $\bar{\rho}_\alpha(x,u)=\left(\frac{1}{g_\alpha(u)}+\frac{f_\alpha(u)+u^Tz_\alpha}{g_\alpha^2(u)}\right)\rho_\alpha(x,u)$.

Remark 3.2. *If* $-g_\alpha(.)$ *is strictly higher order* $B-(b_\alpha,\rho_\alpha,\theta_\alpha,\tilde{p},\tilde{r})$*- invex at* u *with respect to* $G_\alpha(u,p)$ *and* $f_\alpha(.)+(.)^T z_\alpha>0$ *or* $f_\alpha(.)+(.)^T z_\alpha$ *is strictly higher order* $B-(b_\alpha,\rho_\alpha,\theta_\alpha,\tilde{p},\tilde{r})$*- invex at* u *with respect to* $G_\alpha(u,p)$*, then* $\frac{f_\alpha(.)+(.)^T z_\alpha}{g_\alpha(.)}$ *is strictly higher order* $B-\left(\bar{b}_\alpha,\rho_\alpha,\bar{\theta}_\alpha,\tilde{p},\tilde{r}\right)$*-invex at* u *with respect to function* $\bar{G}_\alpha(u,p)$*.*

Theorem 3.2. *(Karush-Kuhn-Tucker type sufficient optimality conditions) Let* u *be a feasible solution of (**P**). Let there exist* $\lambda>0$*,* $\lambda\in R^k$*,* $\mu_j\geqq 0$*,* $\mu\in R^m$*,* $z_i\in R^n$ $(i\in K)$*,* $w_j\in R^n$ $(j\in M)$*, such that*

$$\sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^T z_i}{g_i(u)}\right)+\sum_{j=1}^{m}\mu_j\left(\nabla h_j(u)+w_j\right)=0, \tag{3}$$

$$\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^T w_j\right)=0, \tag{4}$$

$$u^T z_i=s(u/D_i),\ \ z_i\in D_i,\ \ i\in K,$$

$$u^T w_j=s(u/E_j),\ \ w_j\in E_j,\ \ j\in M.$$

If
(i) $\rho_i^1(x,u)\geqq 0$ $(i\in K)$, $\rho^2(x,u)\geqq 0$ *for the same* $y\in R^n$*,*
(ii) $f_i(.)+(.)^T z_i$ *and* $-g_i(.)$ *are higher order* $B-\left(b_i^1,\rho_i^1,\theta_i^1,\tilde{p},\tilde{r}\right)$*-invex at* u *with respect to* $G_i(u,p), i\in K$*,*
(iii) $\sum_{j=1}^{m}\mu_j\left(h_j(.)+(.)^T w_j\right)$ *is higher order* $B-(b^2,\rho^2,\theta^2,\tilde{p},\tilde{r})-$*invex at* u *with respect to* $-\sum_{i=1}^{k}\lambda_i\bar{G}_i(u,p)$*,*
(iv) $b_i^{-1}(x,u)>0$ *for at least one* $i\in K$*,*

then u *is an efficient solution of (P).*

Proof. Suppose u is not an efficient solution of (P). Then, there exists $x\in X$ such that

$$\left(\frac{f_1(x)+s(x/D_1)}{g_1(x)},\ldots,\frac{f_k(x)+s(x/D_k)}{g_k(x)}\right)\leq\left(\frac{f_1(u)+s(u/D_1)}{g_1(u)},\ldots,\frac{f_k(u)+s(u/D_k)}{g_k(u)}\right).$$

Therefore,

$$\begin{aligned}&\left(\frac{f_1(x)+x^Tz_1}{g_1(x)},\ldots,\frac{f_k(x)+x^Tz_k}{g_k(x)}\right)\\ &\quad\leqq\left(\frac{f_1(x)+s(x/D_1)}{g_1(x)},\ldots,\frac{f_k(x)+s(x/D_k)}{g_k(x)}\right)\\ &\quad\leq\left(\frac{f_1(u)+s(u/D_1)}{g_1(u)},\ldots,\frac{f_k(u)+s(u/D_k)}{g_k(u)}\right)\\ &\quad=\left(\frac{f_1(u)+u^Tz_1}{g_1(u)},\ldots,\frac{f_k(u)+u^Tz_k}{g_k(u)}\right),\ using\ u^Tz_i=s(u/D_i), i\in K.\end{aligned}$$

From hypothesis (*iv*) and using $\lambda > 0$, we obtain

$$\sum_{i=1}^{k}\lambda_i b_i^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_i(x)+x^Tz_i}{g_i(x)}-\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)}-1\right)\right]<0. \tag{5}$$

In view of hypothesis (ii) and Theorem (3.1), we have

$$\begin{aligned}&b_1^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_1(x)+x^Tz_1}{g_1(x)}-\frac{f_1(u)+u^Tz_1}{g_1(u)}\right)}-1\right)\right],\ldots,\\ &\quad b_k^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_k(x)+x^Tz_k}{g_k(x)}-\frac{f_k(u)+u^Tz_k}{g_k(u)}\right)}-1\right)\right]\\ &\quad\geqq\frac{1}{\tilde{p}}\left(\langle\nabla\left(\frac{f_1(u)+u^Tz_1}{g_1(u)}\right)+\nabla_p\bar{G}_1(u,p),e^{\tilde{p}y}-\mathbf{1}\rangle\right)\\ &\quad+\bar{G}_1(u,p)-p^T\nabla_p\bar{G}_1(u,p)+\rho_1^1(x,u)\|\theta_1^{-1}(x,u)\|^2,\ldots,\\ &\quad\frac{1}{\tilde{p}}\left(\langle\nabla\left(\frac{f_k(u)+u^Tz_k}{g_k(u)}\right)+\nabla_p\bar{G}_k(u,p),e^{\tilde{p}y}-\mathbf{1}\rangle\right)\\ &\quad+\bar{G}_k(u,p)-p^T\nabla_p\bar{G}_k(u,p)+\rho_k^1(x,u)\|\theta_k^{-1}(x,u)\|^2\end{aligned}$$

or

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i b_i^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_i(x)+x^Tz_i}{g_i(x)}-\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)}-1\right)\right]\\ &\quad\geqq\frac{1}{\tilde{p}}\sum_{i=1}^{k}\lambda_i\left(\langle\nabla\left(\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)+\nabla_p\bar{G}_i(u,p),e^{\tilde{p}y}-\mathbf{1}\rangle\right)\\ &\quad+\sum_{i=1}^{k}\lambda_i\left(\bar{G}_i(u,p)-p^T\nabla_p\bar{G}_i(u,p)\right)+\sum_{i=1}^{k}\lambda_i\rho_i^1(x,u)\|\theta_i^{-1}(x,u)\|^2.\end{aligned} \tag{6}$$

From hypothesis (*iii*), it yields

$$b^2(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\sum_{j=1}^{m}\mu_j\left(h_j(x)+x^Tw_j-h_j(u)-u^Tw_j\right)}-1\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle \sum_{j=1}^{m}\mu_j(\nabla h_j(u)+w_j)-\sum_{i=1}^{k}\lambda_i\nabla_p\bar{G}_i(u,p), e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$-\sum_{i=1}^{k}\lambda_i\left(\bar{G}_i(u,p)-p^T\nabla_p\bar{G}_i(u,p)\right)+\rho^2(x,u)\|\theta^2(x,u)\|^2. \quad (7)$$

Adding equations (6) and (7), we get

$$\sum_{i=1}^{k}\lambda_i b_i^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_i(x)+x^Tz_i}{g_i(x)}-\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)}-1\right)\right]$$
$$+b^2(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\sum_{j=1}^{m}\mu_j\left(h_j(x)+x^Tw_j-h_j(u)-u^Tw_j\right)}-1\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle \sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)+\sum_{i=1}^{k}\lambda_i\nabla_p\bar{G}_i(u,p), e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\frac{1}{\tilde{p}}\left(\langle \sum_{j=1}^{m}\mu_j(\nabla h_j(u)+w_j)-\sum_{i=1}^{k}\lambda_i\nabla_p\bar{G}_i(u,p), e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\sum_{i=1}^{k}\lambda_i\rho_i^1(x,u)\|\theta_i^{-1}(x,u)\|^2+\rho^2(x,u)\|\theta^2(x,u)\|^2.$$

Using (4) and simplifying it, we get

$$\sum_{i=1}^{k}\lambda_i b_i^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_i(x)+x^Tz_i}{g_i(x)}-\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)}-1\right)\right]+b^2(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\sum_{j=1}^{m}\mu_j\left(h_j(x)+x^Tw_j\right)}-1\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle \sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)+\sum_{j=1}^{m}\mu_j(\nabla h_j(u)+w_j), e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\sum_{i=1}^{k}\lambda_i\rho_i^1(x,u)\|\theta_i^{-1}(x,u)\|^2+\rho^2(x,u)\|\theta^2(x,u)\|^2.$$

Using (3) and hypothesis (i), we get

$$\sum_{i=1}^{k}\lambda_i b_i^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_i(x)+x^Tz_i}{g_i(x)}-\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)}-1\right)\right]$$
$$+b^2(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\sum_{j=1}^{m}\mu_j\left(h_j(x)+x^Tw_j\right)}-1\right)\right]\geqq 0.$$

From the primal constraint (2) and the facts that $\mu_j \geqq 0$, $x^Tw_j \leqq s(x/E_j)$, $j \in M$, $b^2(x,u) \geq 0$, it yields

$$\sum_{i=1}^{k}\lambda_i b_i^{-1}(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\left(\frac{f_i(x)+x^Tz_i}{g_i(x)}-\frac{f_i(u)+u^Tz_i}{g_i(u)}\right)}-1\right)\right]\geqq 0.$$

Which contradicts (5). Hence, we have the result. □

Remark 3.3. *If we assume* $\lambda \geq 0$ *in Theorem 3.2, then in hypothesis* (ii) $f_i(.) + (.)^T z_i$ *are required to be strictly higher order* $B-(b_i^1, \rho_i^1, \theta_i^1, \tilde{p}, \tilde{r})$ *invex at* u *with respect to* $G_i(u,p), i \in K$.

4. DUALITY

In this section, we consider the following Schaible type dual for **(P)** and estabilish weak, strong and strict converse duality Theorems assuming $B-(b,\rho,\theta,\tilde{p},\tilde{r})$-invexity:

(SD) Maximize $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$

$$\text{subject to } \nabla\left(\sum_{i=1}^{k}\lambda_i(f_i(u)+u^Tz_i-\nu_i g_i(u))+\sum_{j=1}^{m}\mu_j(h_j(u)+u^Tw_j)\right)=0, \tag{8}$$

$$f_i(u)+u^Tz_i-\nu_i g_i(u)\geqq 0,\ i\in K, \tag{9}$$

$$\sum_{j=1}^{m}\mu_j(h_j(u)+u^Tw_j)\geqq 0,\ j\in M, \tag{10}$$

$$\lambda>0,\ \mu\geqq 0,\ \nu\geqq 0, \tag{11}$$

$$\lambda,\nu\in R^k,\ \mu\in R^m,\ u\in X, \tag{12}$$

$$z_i\in D_i\ \ (i\in k),\ w_j\in E_j\ (j\in M), \tag{13}$$

Theorem 4.1. *(Weak Duality) Let x and (u,λ,ν,μ,z,w) be feasible solutions of (**P**) and (**SD**) respectively . Let*

(i) $\rho_i^1(x,u) \geqq 0$ $(i \in K)$, $\rho^2(x,u) \geqq 0$ for the same $y \in R^n$,
(ii) $f_i(.) + (.)^T z_i$ be higher order $B-\left(b_i^1, \rho_i^1, \theta_i^1, \tilde{p}, \tilde{r}\right)$-invex at u with respect to $G_i(u,p)$, and $\nu_i g_i(.)$ be higher order $B-\left(b_i^1, \rho_i^1, \theta_i^1, \tilde{p}, \tilde{r}\right)$-invex at u with respect to $\nu_i G_i(u,p)$, $i \in K$,
(iii) $\sum_{j=1}^{m} \mu_j \left(h_j(.) + (.)^T w_j\right)$ be higher order $B-(b^2, \rho^2, \theta^2, \tilde{p}, \tilde{r})$-invex at u with respect to $-\sum_{i=1}^{k} \lambda_i(1-\nu_i)G_i(u,p)$,
(iv) $b_i^1(x,u) > 0$ for at least one $i \in K$.
Then $F(x) \nless \nu$.

Proof. Suppose, to the contrary, that $F(x) < \nu$, *i.e*, $\forall\ i \in K$,

$$\frac{f_i(x)+s(x/D_i)}{g_i(x)} < \nu_i.$$

Using hypothesis (*iv*) and $\lambda > 0$, we get

$$\sum_{i=1}^{k} \lambda_i b_i^1(x,u)\left[\frac{1}{\tilde{r}}\left((e^{\tilde{r}(f_i(x)+x^T z_i - f_i(u) - u^T z_i)} - 1) - \left(e^{\tilde{r}\nu_i(g_i(x)-g_i(u))} - 1\right)\right)\right] < 0. \tag{14}$$

It follows from hypothesis (*ii*) and for $i \in K$, that

$$b_i^1(x,u)\left(\frac{1}{\tilde{r}}(e^{\tilde{r}(f_i(x)+x^T z_i - f_i(u) - u^T z_i)} - 1)\right) \geqq \frac{1}{\tilde{p}}(\langle\ \nabla f_i(u) + z_i + \nabla_p G_i(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle)$$
$$+ G_i(u,p) - p^T \nabla p G_i(u,p) + \rho_i^1(x,u)\|\theta_i^1(x,u)\|^2 \tag{15}$$

And, we have

$$b_i^1(x,u)\left(\frac{1}{\tilde{r}}(e^{\tilde{r}\nu_i(g_i(x)-g_i(u))} - 1)\right) \geqq \frac{1}{\tilde{p}}(\langle\ \nu_i \nabla g_i(u) + \nu_i \nabla_p G_i(u,p), e^{\tilde{p}y} - \mathbf{1}\rangle)$$
$$+ \nu_i(G_i(u,p) - p^T \nabla p G_i(u,p)) + \rho_i^1(x,u)\|\theta_i^1(x,u)\|^2. \tag{16}$$

Subtracting equations (15) and (16) and using $\lambda > 0$, we get

$$\sum_{i=1}^{k} \lambda_i b_i^1(x,u)\left[\frac{1}{\tilde{r}}\left((e^{\tilde{r}(f_i(x)+x^Tz_i-f_i(u)-u^Tz_i)}-1)-\left(e^{\tilde{r}\nu_i(g_i(x)-g_i(u))}-1\right)\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\sum_{i=1}^{k}\lambda_i(\langle \nabla f_i(u)+z_i-\nu_i\nabla g_i(u), e^{\tilde{p}y}-\mathbf{1}\rangle)$$
$$+\frac{1}{\tilde{p}}\sum_{i=1}^{k}\lambda_i(1-\nu_i)(\langle \nabla_p G_i(u,p), e^{\tilde{p}y}-\mathbf{1}\rangle)$$
$$+\sum_{i=1}^{k}\lambda_i(1-\nu_i)(G_i(u,p)-p^T\nabla_p G_i(u,p)). \tag{17}$$

It follows from hypothesis (*iii*),

$$b^2(x,u)\left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\sum_{j=1}^{m}\mu_j(h_j(x)+x^Tw_j-h_j(u)-u^Tw_j)}-1\right)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle \sum_{j=1}^{m}\mu_j(\nabla h_j(u)+w_j)-\sum_{i=1}^{k}\lambda_i(1-\nu_i)\nabla_p G_i(u,p), e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$-\sum_{i=1}^{k}\lambda_i(1-\nu_i)(G_i(u,p)-p^T\nabla_p G_i(u,p))+\rho^2(x,u)||\theta^2(x,u)||^2. \tag{18}$$

Adding equations (17) and (18), we get

$$\sum_{i=1}^{k} \lambda_i b_i^1(x,u)\left[\frac{1}{\tilde{r}}\left((e^{\tilde{r}(f_i(x)+x^Tz_i-f_i(u)-u^Tz_i)}-1)-\left(e^{\tilde{r}\nu_i(g_i(x)-g_i(u))}-1\right)\right)\right]$$
$$+b^2(x,u)\left[\frac{1}{\tilde{r}}(e^{\tilde{r}\sum_{j=1}^{m}\mu_j(h_j(x)+x^Tw_j-h_j(u)-u^Tw_j)}-1)\right]$$
$$\geqq \frac{1}{\tilde{p}}\left(\langle \nabla\left\{\sum_{i=1}^{k}\lambda_i(f_i(u)+u^Tz_i-\nu_i g_i(u))+\sum_{j=1}^{m}\mu_j(h_j(u)+u^Tw_j)\right\}, e^{\tilde{p}y}-\mathbf{1}\rangle\right)$$
$$+\rho^2(x,u)||\theta^2(x,u)||^2.$$

Using the primal constraint (2), dual constraints (8), (10), $\mu \geqq 0$, $b^2(x,u) \geq 0$, $x^Tw_j \leqq s(x/E_j)$ $j \in M$, and the fact that $\rho^2(x,u) \geqq 0$, we get

$$\sum_{i=1}^{k} \lambda_i b_i^1(x,u)\left[\frac{1}{\tilde{r}}\left((e^{\tilde{r}(f_i(x)+x^Tz_i-f_i(u)-u^Tz_i)}-1)-\left(e^{\tilde{r}\nu_i(g_i(x)-g_i(u))}-1\right)\right)\right] \geqq 0.$$

Which contradicts equation (14). Hence, we have the result. □

Theorem 4.2. *(Strong Duality) Let $\bar{x}$ be an efficient solution for* **(P)** *and let the Slater's constraint qualification be satisfied. Then there exist $\bar{v}, \bar{\eta} \in R^k, \bar{\mu} \in R^m, \bar{z}_i \in R^n (i \in k)$ and $\bar{w}_j \in R^m (j \in M)$, such that $(\bar{x}, \bar{\eta}, \bar{v}, \bar{\mu}, \bar{z}, \bar{w})$ is feasible for* **(SD)** *and the two objectives are equal.*
Also, if weak duality holds for all feasible solutions of the problems **(P)** *and* **(SD)***, then $(\bar{x}, \bar{\eta}, \bar{v}, \bar{\mu}, \bar{z}, \bar{w})$ is an efficient solution for* **(SD)***.*

Proof. Since $\bar{x}$ is an efficient solution for **(P)** and the Slater's constraint qualification is satisfied, then there exist $0 \leq \bar{\gamma} \in R^k,\ \bar{\mu} \in R^m,\ \bar{z}_i \in R^n (i \in K)$ and $\bar{w}_j \in R^m (j \in M)$, such that

$$\sum_{i=1}^{k} \bar{\gamma}_i \nabla \left(\frac{f_i(\bar{x}) + \bar{x}^T \bar{z}_i}{g_i(\bar{x})} \right) + \sum_{j=1}^{m} \bar{\mu}_j (\nabla h_j(\bar{x}) + \bar{w}_j) = 0, \tag{19}$$

$$\sum_{j=1}^{m} \bar{\mu}_j (h_j(\bar{x}) + \bar{x}^T \bar{w}_j) = 0,$$

$$\bar{x}^T \bar{z}_i = s(\bar{x}/D_i),\ i \in K,$$

$$\bar{x}^T \bar{w}_j = s(\bar{x}/E_j),\ j \in M,$$

$$\bar{\mu} \geqq 0,\ \bar{z}_i \in D_i\ (i \in K),\ \bar{w}_j \in E_j\ (j \in M).$$

We can write Equation (19), as follows

$$\sum_{i=1}^{k} \frac{\bar{\gamma}_i}{g_i(\bar{x})} \left(\nabla (f_i(\bar{x}) + \bar{x}^T \bar{z}_i) - \left(\frac{f_i(\bar{x}) + \bar{x}^T \bar{z}_i}{g_i(\bar{x})} \right) \nabla g_i(\bar{x}) \right) + \sum_{j=1}^{m} \bar{\mu}_j (\nabla h_j(\bar{x}) + \bar{w}_j) = 0.$$

Putting $\bar{\eta}_i = \frac{\bar{\gamma}_i}{g_i(\bar{x})}$ and $\bar{v}_i = \frac{f_i(\bar{x}) + \bar{x}^T \bar{z}_i}{g_i(\bar{x})}, i \in K$, we get

$$\nabla \left(\sum_{i=1}^{k} \bar{\eta}_i (f_i(\bar{x}) + \bar{x}^T \bar{z}_i - \bar{v}_i g_i(\bar{x})) \right) + \sum_{j=1}^{m} \bar{\mu}_j (\nabla h_j(\bar{x}) + \bar{w}_j) = 0, \tag{20}$$

$$f_i(\bar{x}) + \bar{x}^T \bar{z}_i - \bar{v}_i g_i(\bar{x}) = 0, \tag{21}$$

$$\sum_{j=1}^{m} \bar{\mu}_j (h_j(\bar{x}) + \bar{x}^T \bar{w}_j) = 0, \tag{22}$$

$$\bar{x}^T \bar{z}_i = s(\bar{x}/D_i),\ i \in K, \tag{23}$$

$$\bar{x}^T \bar{w}_j = s(\bar{x}/E_j),\ j \in M, \tag{24}$$

$$\bar{\eta} \geq 0,\ \bar{\mu} \geqq 0,\ \bar{v} \geqq 0,\ \bar{z}_i \in D_i\ (i \in K),\ \bar{w}_j \in E_j\ (j \in M). \tag{25}$$

Thus $(\bar{x}, \bar{\eta}, \bar{v}, \bar{\mu}, \bar{z}, \bar{w})$ is feasible for **(SD)**. Also, from (21), the two objectives are equal and hence, $(\bar{x}, \bar{\eta}, \bar{v}, \bar{\mu}, \bar{z}, \bar{w})$ is an efficient solution for **(SD)**. □

Theorem 4.3. *(Strict Converse Duality) Let $\bar{x}$ and $(\bar{u}, \bar{\lambda}, \bar{v}, \bar{\mu}, \bar{z}, \bar{w})$ be efficient solutions for **(P)** and **(SD)**, respectively, such that $F(\bar{x}) = \bar{v}$ and let for the same $y \in R^n$, we have*

(i) $f_i(.)+(.)^T\bar{z}_i-\bar{v}_ig_i(.)$ be strictly higher order $B-\left(b_i^1, \rho_i^1, \theta_i^1, \tilde{p}, \tilde{r}\right)$-invex at $\bar{u}$ with respect to $G_i(\bar{u}, \bar{p}), i \in K$ and $\bar{\lambda} > 0$, $f_r(.)+(.)^T\bar{z}_r-\bar{v}_rg_r(.)$ be strictly higher order $B-\left(b_r^1, \rho_r^1, \theta_r^1, \tilde{p}, \tilde{r}\right)$-invex at $\bar{u}$ with respect to $G_r(\bar{u}, \bar{p})$ for at least one $r \in k$, $f_i(.) + (.)^T\bar{z}_i - \bar{v}_ig_i(.)$ be higher order $B - \left(b_i^1, \rho_i^1, \theta_i^1, \tilde{p}, \tilde{r}\right)$-invex at $\bar{u}$ with respect to $G_i(\bar{u}, \bar{p}), i \in K_r$,

(ii) $\sum\limits_{j=1}^{m} \bar{\mu}_j \left(h_j(.) + (.)^T\bar{w}_j\right)$ is higher order $B - (b^2, \rho^2, \theta^2, \tilde{p}, \tilde{r})$-invex at $\bar{u}$ with respect to $-\sum\limits_{i=1}^{k} \bar{\lambda}_iG_i(\bar{u}, \bar{p})$, and

(iii) $\rho_i^1(x, u) \geqq 0$ $(i \in K)$, $\rho^2(x, u) \geqq 0$ for the same $y \in R^n$.

*Then $\bar{x} = \bar{u}$, i.e., $\bar{u}$ is an efficient solution of **(P)**.*

Proof. Suppose $\bar{x} \neq \bar{u}$. From (9), (10), $\bar{\lambda} > 0$, $b_i^1(\bar{x}, \bar{u}) \geq 0$ $(i \in K)$, and $b^2(\bar{x}, \bar{u}) \geq 0$, it yields

$$\sum_{i=1}^{k} \bar{\lambda}_ib_i^1(\bar{x}, \bar{u}) \left[\frac{1}{\tilde{r}}(e^{-\tilde{r}(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{v}_ig_i(\bar{u}))} - 1)\right]$$
$$+b^2(\bar{x}, \bar{u}) \left[\frac{1}{\tilde{r}}(e^{-\tilde{r}\sum\limits_{j=1}^{m}\bar{\mu}_j(h_j(\bar{u})+\bar{u}^T\bar{w}_j)} - 1)\right] \leqq 0. \tag{26}$$

Using hypothesis (i), we get

$$b_r^1(\bar{x}, \bar{u}) \left[\frac{1}{\tilde{r}}(e^{\tilde{r}(f_r(\bar{x})+\bar{x}^T\bar{z}_r-\bar{v}_rg_r(\bar{x})-f_r(\bar{u})-\bar{u}^T\bar{z}_r+\bar{v}_rg_r(\bar{u}))} - 1)\right]$$
$$> \frac{1}{\tilde{p}}(\langle \nabla f_r(\bar{u}) + \bar{z}_r - \bar{v}_r\nabla g_r(\bar{u}) + \nabla_pG_r(\bar{u}, \bar{p}), e^{\tilde{p}y} - \mathbf{1}\rangle)$$
$$+G_r(\bar{u}, \bar{p}) - \bar{p}^T\nabla_pG_r(\bar{u}, \bar{p}) + \rho_r^1(\bar{x}, \bar{u})||\theta_r^1(\bar{x}, \bar{u})||^2,$$

for at least one $r \in K$, we have

$$b_i^1(\bar{x}, \bar{u}) \left[\frac{1}{\tilde{r}}(e^{\tilde{r}(f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{v}_ig_i(\bar{x})-f_i(\bar{u})-\bar{u}^T\bar{z}_i+\bar{v}_ig_i(\bar{u}))} - 1)\right]$$
$$\geqq \frac{1}{\tilde{p}}(\langle \nabla f_i(\bar{u}) + \bar{z}_i - \bar{v}_i\nabla g_i(\bar{u}) + \nabla_pG_i(\bar{u}, \bar{p}), e^{\tilde{p}y} - \mathbf{1}\rangle)$$
$$+G_i(\bar{u}, \bar{p}) - \bar{p}^T\nabla_pG_i(\bar{u}, \bar{p}) + \rho_i^1(\bar{x}, \bar{u})||\theta_i^1(\bar{x}, \bar{u})||^2, i \in K_r.$$

As $\bar{\lambda} > 0$, we get

$$\sum_{i=1}^{k} \bar{\lambda}_i b_i^1(\bar{x},\bar{u}) \left[\frac{1}{\tilde{r}}(e^{\tilde{r}(f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{v}_i g_i(\bar{x})-f_i(\bar{u})-\bar{u}^T\bar{z}_i+\bar{v}_i g_i(\bar{u}))} - 1)\right]$$

$$> \frac{1}{\tilde{p}} \sum_{i=1}^{k} \bar{\lambda}_i \left(\langle \nabla f_i(\bar{u}) + \bar{z}_i - \bar{v}_i \nabla g_i(\bar{u}) + \nabla_p G_i(\bar{u},\bar{p}), e^{\tilde{p}y} - \mathbf{1}\rangle\right)$$

$$+ \sum_{i=1}^{k} \bar{\lambda}_i (G_i(\bar{u},\bar{p}) - \bar{p}^T \nabla_p G_i(\bar{u},\bar{p})) + \sum_{i=1}^{k} \bar{\lambda}_i \rho_i^1(\bar{x},\bar{u}) \|\theta_i^1(\bar{x},\bar{u})\|^2. \tag{27}$$

Using hypothesis (*ii*),

$$b^2(\bar{x},\bar{u}) \left[\frac{1}{\tilde{r}}(e^{\tilde{r}\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{x})+\bar{x}^T\bar{w}_j-h_j(\bar{u})-\bar{u}^T\bar{w}_j)} - 1)\right]$$

$$\geqq \frac{1}{\tilde{p}} \left(\langle \sum_{j=1}^{m} \bar{\mu}_j(\nabla h_j(\bar{u}) + \bar{w}_j) - \sum_{i=1}^{k} \bar{\lambda}_i \nabla_p G_i(\bar{u},\bar{p}), e^{\tilde{p}y} - \mathbf{1}\rangle\right)$$

$$- \sum_{i=1}^{k} \bar{\lambda}_i (G_i(\bar{u},\bar{p}) - \bar{p}^T \nabla_p G_i(\bar{u},\bar{p})) + \rho^2(\bar{x},\bar{u}) \|\theta^2(\bar{x},\bar{u})\|^2. \tag{28}$$

Adding equations (27), (28) and using (8), $\bar{\mu} \geqq 0$ and hypothesis (*iii*), we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i b_i^1(\bar{x},\bar{u}) \left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}(f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{v}_i g_i(\bar{x})-f_i(\bar{u})-\bar{u}^T\bar{z}_i+\bar{v}_i g_i(\bar{u}))} - 1\right)\right]$$

$$+ b^2(\bar{x},\bar{u}) \left[\frac{1}{\tilde{r}}\left(e^{\tilde{r}\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{x})+\bar{x}^T\bar{w}_j-h_j(\bar{u})-\bar{u}^T\bar{w}_j)} - 1\right)\right] > 0.$$

It follows from (2), $F(\bar{x}) = \bar{v}$, and $\bar{x}^T\bar{w}_j \leqq s(\bar{x}/E_j)$, $j \in M$, we get

$$\sum_{i=1}^{k} \bar{\lambda}_i b_i^1(\bar{x},\bar{u}) \left[\frac{1}{\tilde{r}}(e^{-\tilde{r}(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{v}_i g_i(\bar{u}))} - 1)\right]$$

$$+ b^2(\bar{x},\bar{u}) \left[\frac{1}{\tilde{r}}(e^{-\tilde{r}\sum_{j=1}^{m} \bar{\mu}_j(h_j(\bar{u})+\bar{u}^T\bar{w}_j)} - 1)\right] > 0.$$

Which contradicts (26). Hence, we have $\bar{x} = \bar{u}$. □

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