Yugoslav Journal of Operations Research
27 (2017), Number 3, 385–390
DOI: 10.2298/YJOR151025011B

A NOTE ON ENTROPY OF LOGIC

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Received: October 2015 / Accepted: April 2016

Abstract: We propose an entropy based classification of propositional calculi. Our method can be applied to finite–valued propositional logics and then, extended asymptotically to infinite–valued logics. In this paper we consider a classification depending on the number of truth values of a logic and not on the number of its designated values. Furthermore, we believe that almost the same approach can be useful in classification of finite algebras.

Keywords: Many–valued Propositional Logics, Lindenbaum–Tarski Algebra, Partition, Entropy.

MSC: 03B50, 03B05, 94A17, 37A35.

1. INTRODUCTION

We present one way of logical systems classification based on their entropies (see [2] and [3]). The concept of generalized Shannons entropy, entropy of a partition and the logical system represented by its Linednbaum–Tarski algebra, make it possible to define the entropy of a many–valued propositional logic, and then to extend it asymptotically to infinite–valued logics. Our finite measure of uncertainty H of a finite–valued logic monotonically increases with the growth of truth values number. This measure is sensitive to both the number of truth values of a finite–valued logic and the number of its designated (true) values (see [2] and [3]). In this paper we deal with a classification depending only on the number of truth values.

2. LINDEBAUM–TARSKI ALGEBRA

Let us keep in mind the following two well–known facts. The first is related to the number 2^{2^n} of mutually non–equivalent formulae over the finite set of propositional letters $\{p_1, \ldots, p_n\}$ in the classical two–valued logic. The second one is that Nishimura has shown that in case of Heyting's propositional logic, an infinite–valued logic, even in case when the set of propositional formulae is built up over the set of a single propositional letter, there exist countably many mutually nonequivalent formulae (see [11]). These examples show the essential difference between finite–valued and infinite–valued logics from the stand point of our intentions. Namely, our aim is to consider possibilities for defining probabilistic measure over partitions of propositional formulae set, denoted by For, defined by the corresponding Linednbaum–Tarski algebra.

By a *partition* of a nonempty set X we mean any finite or denumerable collection (A_i) of nonempty subsets of X such that $(\forall i, j)(i \neq j \rightarrow A_i \cap A_j = \emptyset)$ and $\cup_i A_i = X$. Partition of the set For of propositional formulae is defined on the basis of an equivalence relation $\equiv_{\mathbf{L}}$, related to a propositional logic $\mathbf{L}$, by the following condition $A \equiv_{\mathbf{L}} B$ iff both consequences $A \vdash B$ and $B \vdash A$ are derivable in $\mathbf{L}$, for any $A, B \in$ For. This equivalence relation $\equiv_{\mathbf{L}}$ divides the set For on non–empty mutually disjoint sets and forms a quotient algebra $\mathrm{For}/_{\equiv_{\mathbf{L}}}$, usually called the Lindenbaum–Tarski algebra of $\mathbf{L}$. If by For_n we denote a subset of For built up over a finite set of propositional letters $\{p_1, \ldots, p_n\}$ and a usual list of propositional connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$, then, in case of an m–valued propositional logic $\mathbf{L}$, the corresponding quotient algebra $\mathrm{For}_n/_{\equiv_{\mathbf{L}}}$ will consist of at most m^{m^n} elements.

3. ENTROPY OF PARTITIONS OF For

A natural generalization of Shannon's entropy, appearing in Measure theory, is defined over a measurable partition $\alpha = \{A_i | i \in I\}$ of a space X, equipped with a measure μ, such that $(\forall i)(i \in I \rightarrow \mu(A_i) \geq 0)$, $(\forall i, j)(i \in I \wedge j \in I \wedge i \neq j \rightarrow A_i \cap A_j = \emptyset)$ and $\mu(X \setminus \cup_{i \in I} A_i) = 0$. In this context, the entropy is defined as follows:

$$H(\alpha) = -\sum_{i \in I} \mu(A_i) \log_2 \mu(A_i)$$

with the usual convention that $\mu(A_i) \log_2 \mu(A_i) = 0$, for $\mu(A_i) = 0$, by definition, having in mind that $\lim_{x \to 0+} x \log_2 x = 0$.

Our central problem is how to define a measure over a finite family of sets consequently extendable to a denumerable family, in order to get a finite philosophically well founded and logically justified entropy of partition. Let us describe the basic idea and the construction. More accurately, the problem is to define a measure μ over the set $\mathrm{For}_n/_{\equiv_{\mathbf{L}}}$ and to extend it into $\mathrm{For}/_{\equiv_{\mathbf{L}}}$, obtaining the finite entropy for $\mathrm{For}/_{\equiv_{\mathbf{L}}}$. As we stated in M. Boričić (2013, 2014), the measures distributed uniformly or binomially do not give satisfiable results. Namely, even in the case of classical two–valued propositional logic, neither uniform, nor binomial probability distribution do not give a finite entropy. If we suppose that the measure $\mu(A_i)$ of the class A_i is uniformly distributed, meaning that $\mu(A_i) = \frac{1}{2^{2^n}}$, then, by (1), the corresponding entropies $H(\mathbf{L}_m^n)$ and $H(\mathbf{L}_m)$ over partitions of sets $\mathrm{For_n}$ and For, respectively, are: $H(\mathbf{L}_2^n) = 2^n \ln 2$ and $H(\mathbf{L}_2) = \lim_{n \to \infty} H(\mathbf{L}_2^n) = +\infty$, where $H(\mathbf{L}_2^n)$ and $H(\mathbf{L}_2)$ denote entropies of two–valued logic $\mathbf{L}_2$ over the sets

$\mathrm{For_n}$ and For, respectively. Alternatively, if we suppose that these measures are binomially distributed, meaning that

$$p(A_i) = \binom{2^n}{i} \frac{1}{2^{2^n}}$$

then, by (1) and the known asymptotic relation:

$$H(\mathbf{L}_2^n) \sim \frac{1}{2} \ln \left(e\pi 2^{n-1} \right)$$

as $n \to \infty$ (see [13]), we also conclude that $H(\mathbf{L}_2) = \lim_{n\to\infty} H(\mathbf{L}_2^n) = +\infty$.

Here we will present, according to [2], a definition enabling a good possibility for classification of finite–valued propositional logics on the basis of a finite entropy of a countable partition of For.

In order to give a simple and clear definition, we will consider here the case when an m–valued logic has only one designated value, meaning that only one value of m designates the truth. The case of m–valued logic with $k = 1, 2, \ldots, m-1$ designated values is considered in [2] and [3].

Let $\mathbf{L}^m$ be an m–valued logic with one designated value, and $\mathbf{L}_m^n$ its part built up over a set consisting of n propositional letters only. By $H(\mathbf{L}_m^n)$ and $H(\mathbf{L}_m)$ we denote entropies of $\mathbf{L}_m^n$ and $\mathbf{L}_m$, respectively. Let

$$\mu(A_i) = \frac{1}{m} \left(1 - \frac{1}{m} \right)^{i-1}$$

for $i = 1, 2, \ldots, m^{m^n} - 1$ and

$$\mu(A_i) = \left(1 - \frac{1}{m} \right)^{i}$$

for $i = m^{m^n}$.

Lemma 1. $H(\mathbf{L}_m) = m \log_2 m - (m-1) \log_2 (m-1)$

Proof. Using the formula for a geometric series, following essentially from [5], i.e. from the fact that

$$\sum_{k=1}^{n} kz^k = z \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2}$$

which is provable, for example, by mathematical induction on n, for $M = m^{m^n}$, and using (1), we calculate:

$$
\begin{aligned}
H(\mathbf{L}_m^n) &= -\sum_{i=1}^{M} \mu(A_i)\log_2 \mu(A_i)\\
&= -\frac{1}{m}\log_2\frac{1}{m} - \frac{1}{m}\left(1-\frac{1}{m}\right)\log_2\left(\frac{1}{m}\left(1-\frac{1}{m}\right)\right) - \cdots -\\
&\quad -\frac{1}{m}\left(1-\frac{1}{m}\right)^{M-2}\log_2\left(\frac{1}{m}\left(1-\frac{1}{m}\right)^{M-2}\right) - \left(1-\frac{1}{m}\right)^{M}\log_2\left(1-\frac{1}{m}\right)^{M}\\
&= -\left(1-\left(1-\frac{1}{m}\right)^{M-1}\right)\log_2\frac{1}{m} -\\
&\quad -\left(1-\frac{1}{m}\right)\log_2\left(1-\frac{1}{m}\right)\frac{1-(M-1)\left(1-\frac{1}{m}\right)^{M-2}+(M-2)\left(1-\frac{1}{m}\right)^{M-1}}{\frac{1}{m}} -\\
&\quad -\left(1-\frac{1}{m}\right)^{M}\log_2\left(1-\frac{1}{m}\right)^{M}
\end{aligned}
$$

and finally, we find:

$$H(\mathbf{L}_m) = \lim_{n\to\infty} H(\mathbf{L}_m^n) = m\log_2 m - (m-1)\log_2(m-1)7pt$$

Using this Lemma we justified the definition of entropy of m–valued logic $\mathbf{L}_m$. Consequently, we find (see [2]) that:

m	$H(\mathbf{L}_m)$
2	2.0000
3	2.7549
4	3.2451
5	3.6096
6	3.9001

Simple monotonicity analysis of the function $f(x) = x\log_2 x - (x-1)\log_2(x-1)$ leads us to the following conclusion:

Lemma 2. *For any two m–valued and n–valued logics $\mathbf{L}_m$ and $\mathbf{L}_n$, if $m \leq n$, then $H(\mathbf{L}_m) \leq H(\mathbf{L}_n)$.*

4. ENTROPY OF SOME KNOWN LOGICS

Here we mention some features of the well known finite–valued logics, give their entropies and consider entropies of infinite–valued propositional logics.

First of all, we note that the classical propositional logic has the entropy less than or equal to 2, and that both Lukasiewicz's (see [8], [9], [14] and [12]) and Kleene's (see [6], [7] and [12]) three–valued logics, with one designated value, have the entropies less than or equal to 2.7549. Belnap's four–valued logic (see [1]), with one designated value, has the entropy less than or equal to 3.2451.

Let us consider the sequence $\mathbf{H} + \mathbf{E}_m$ of finite–valued extensions of Heyting's propositional logic $\mathbf{H}$ by axiom–schemata $\mathbf{E}_m$:

$$\bigvee_{1 \leq i < j \leq m} (A_i \leftrightarrow A_j)$$

for $m \geq 3$, where $A \leftrightarrow B$ is an abbreviation for $(A \to B) \wedge (B \to A)$, introduced by McKay (see [10]), presents a strictly descending sequence $\mathbf{H} + \mathbf{E}_m$ of $(m-1)$–valued logics, with one designated value, intermediate between $\mathbf{H}$ and classical two–valued logic $\mathbf{L}_2$ (see [4])), i.e.

$$\mathbf{H} \subset \ldots \subset \mathbf{H} + \mathbf{E}_{m+1} \subset \mathbf{H} + \mathbf{E}_m \subset \ldots \subset \mathbf{H} + \mathbf{E}_4 \subset \mathbf{H} + \mathbf{E}_3 = \mathbf{L}_2$$

having the following property:

$$\lim_{m \to +\infty} (\mathbf{H} + \mathbf{E}_m) = \bigcap_{m \geq 3} (\mathbf{H} + \mathbf{E}_m) = \mathbf{H}$$

(see [10]), gives us the reason to consider an asymptotic approximation of the entropy of Heyting propositional logic, as well. For the entropy of $\mathbf{H} + \mathbf{E}_m$, we have:

$$H(\mathbf{H} + \mathbf{E}_m) \leq m \log_2 m - (m-1) \log_2 (m-1)$$

Acknowledgement: The author thanks the anonymous referees of YUJOR and the members of the Seminar for Probabilistic Logic of Mathematical Institute of SANU, Belgrade, particularly her teachers N. Ikodinović, Z. Marković, Z. Ognjanović, A. Perović and M. Rašković, for making very valuable suggestions and helpful comments regarding earlier versions of this paper. This research was supported in part by the Ministry of Science and Technology of Serbia, grant number 174026.

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