

## POWER ALLOCATION RULES UNDER MULTICRITERIA SITUATION

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Received: September 2017 / Accepted: January 2018

**Abstract:** Under multicriteria situations, we define a power mensuration rule and its efficient extension by applying maximal-utilities among level (decision) vectors. We also adopt some axiomatic results to present the rationalities for these two rules. Based on the notions of reduced game and excess function respectively, we introduce different formulation and dynamic results for the efficient extension.

**Keywords:** Multicriteria Situation, Maximal-Utility, Reduced Game, Excess Function, Dynamic Process.

**MSC:** 91A, 91B.

### 1. INTRODUCTION

In the framework of transferable-utility (TU) games, the power indexes have been defined to measure the political power of each member of a voting system. A member in a voting system is, e.g., a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so its power will be different. Results of the power indexes may be found in, e.g., Dubey and Shapley [4], Haller [5], Lehrer [7], van den Brink and van der Laan [2] and so on.

Banzhaf [1] defined a power index in the framework of voting games that was essentially identical to that given by Coleman [3]. This index was later on extended to arbitrary games by Owen [13, 14]. In this paper, we focus on the Banzhaf-Owen index. Briefly speaking, the Banzhaf-Owen index is a rule that gathers each member's marginal contribution from all coalitions in which he/she/it has participated.

*Consistency* is an important property among the axiomatic formulations for allocation rules. Consistency states the independence of a value with respect to

fixing some members with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some members are settled on. This property has been investigated in various problems by applying *reduced games* always. In addition to axiomatizations for an allocation rule, *dynamic processes* can be described that lead the members to that allocation rule, starting from an arbitrary efficient payoff vector.

A *multi-choice TU game* could be regarded as a natural extension of a traditional TU game in which each member has various operational strategies (or decisions). By considering overall allocations for a given member on multi-choice TU games, Hwang and Liao [6], Liao [8, 9] and Nouweland et al. [12] proposed several extended allocations and related results for the core, the EANSC and the Shapley value [15], respectively. The above pre-existing results raise one motivation:

- whether the power indexes could be extended under multi-choice behavior and multicriteria situation simultaneously.

The paper is devoted to investigate the motivation. Different from the framework of multi-choice TU games, we consider the framework of *multicriteria multi-choice TU games* in Section 2. A power index and its efficient extension, the *multi-choice Banzhaf-Owen index* and the *multi-choice efficient Banzhaf-Owen index*, are further proposed by applying maximal-utilities among level (decision) vectors on multicriteria multi-choice TU games. By applying an extended reduction, we propose some axiomatic results to present the rationalities for these two indexes in Section 3. In order to establish the dynamic processes, we present alternative formulation for the multi-choice efficient Banzhaf-Owen index in terms of *excess functions*. In Section 4, we adopt reduction and excess function to show that the multi-choice efficient Banzhaf-Owen index can be reached by members who start from an arbitrary efficient payoff vector.

## 2. PRELIMINARIES

Let  $U$  be the universe of members. For  $i \in U$  and  $b_i \in \mathbb{N}$ ,  $B_i = \{0, 1, \dots, b_i\}$  could be treated as the level (decision) space of member  $i$  and  $B_i^+ = B_i \setminus \{0\}$ , where 0 denotes no participation. Let  $B^N = \prod_{i \in N} B_i$  be the product set of the level (decision) spaces of all members of  $N$ . For all  $T \subseteq N$ , we define  $\theta^T \in B^N$  is the vector with  $\theta_i^T = 1$  if  $i \in T$ , and  $\theta_i^T = 0$  if  $i \in N \setminus T$ . Denote  $0_N$  the zero vector in  $\mathbb{R}^N$ . For  $m \in \mathbb{N}$ , let  $0_m$  be the zero vector in  $\mathbb{R}^m$  and  $\mathbb{N}_m = \{1, \dots, m\}$ .

A **multi-choice TU game** is a triple  $(N, b, v)$ , where  $N$  is a non-empty and finite set of members,  $b = (b_i)_{i \in N}$  is the vector that presents the highest levels for each member, and  $v : B^N \rightarrow \mathbb{R}$  is a characteristic mapping with  $v(0_N) = 0$  which assigns to each  $\alpha = (\alpha_i)_{i \in N} \in B^N$  the worth that the members can gain when each member  $i$  participates at level  $\alpha_i$ . Given a multi-choice TU game  $(N, b, v)$  and  $\alpha \in B^N$ , we write  $A(\alpha) = \{i \in N \mid \alpha_i \neq 0\}$  and  $\alpha_T$  to be the restriction of  $\alpha$  at  $T$  for each  $T \subseteq N$ .

Further, we define  $v_*(T) = \max_{\alpha \in B^N} \{v(\alpha) | A(\alpha) = T\}$  is the **maximal-utility** among all action vector  $\alpha$  with  $A(\alpha) = T$ . A **multicriteria multi-choice TU game** is a triple  $(N, b, V^m)$ , where  $m \in \mathbb{N}$ ,  $V^m = (v^t)_{t \in \mathbb{N}_m}$  and  $(N, b, v^t)$  is a multi-choice TU game for all  $t \in \mathbb{N}_m$ .

Denote the collection of all multicriteria multi-choice TU games by  $\Gamma$ . Let  $(N, b, V^m) \in \Gamma$ . A **payoff vector** of  $(N, b, V^m)$  is a vector  $x = (x^t)_{t \in \mathbb{N}_m}$  and  $x^t = (x_i^t)_{i \in N} \in \mathbb{R}^N$ , where  $x_i^t$  denotes the payoff to member  $i$  in  $(N, b, v^t)$  for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ . A payoff vector  $x$  of  $(N, b, V^m)$  is **multicriteria efficient** if  $\sum_{i \in N} x_i^t = v_*^t(N)$  for all  $t \in \mathbb{N}_m$ . The collection of all multicriteria efficient vector of  $(N, b, V^m)$  is denoted by  $E(N, b, V^m)$ . A **solution** is a map  $\sigma$  assigning to each  $(N, b, V^m) \in \Gamma$  an element

$$\sigma(N, b, V^m) = (\sigma^t(N, b, V^m))_{t \in \mathbb{N}_m},$$

where  $\sigma^t(N, b, V^m) = (\sigma_i^t(N, b, V^m))_{i \in N} \in \mathbb{R}^N$  and  $\sigma_i^t(N, b, V^m)$  is the payoff of the member  $i$  assigned by  $\sigma$  in  $(N, b, v^t)$ .

Next, we provide the multi-choice Banzhaf-Owen index and the multi-choice efficient Banzhaf-Owen index under multicriteria situation.

**Definition 1.** *The multi-choice Banzhaf-Owen index (MBOI),  $\beta$ , is defined by*

$$\beta_i^t(N, b, V^m) = \sum_{\substack{S \subseteq N \\ i \in S}} [v_*^t(S) - v_*^t(S \setminus \{i\})]$$

for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ . Under the solution  $\beta$ , all members receive their **marginal contributions** related to maximal-utilities in each  $S \subseteq N$  respectively.

A solution  $\sigma$  satisfies **multicriteria efficiency (MEFF)** if for all  $(N, b, V^m) \in \Gamma$  and for all  $t \in \mathbb{N}_m$ ,  $\sum_{i \in N} \sigma_i^t(N, b, V^m) = v_*^t(N)$ . Property MEFF asserts that all members allocate all the utility completely. It is easy to check that the MBOI violates MEFF. Therefore, we consider an efficient normalization as follows.

**Definition 2.** *The multi-choice efficient Banzhaf-Owen index (MEBOI),  $\bar{\beta}$ , is defined by*

$$\bar{\beta}_i^t(N, b, V^m) = \beta_i^t(N, b, V^m) + \frac{1}{|N|} \cdot [v_*^t(N) - \sum_{k \in N} \beta_k^t(N, b, V^m)]$$

for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ .

**Lemma 2.1.** *The MEBOI satisfies MEFF on  $\Gamma$ .*

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From now on we consider bounded multi-choice TU games, defined as those games  $(N, b, v)$  such that, there exists  $K_v \in \mathbb{R}$  such that  $v(\alpha) \leq K_v$  for all  $\alpha \in B^N$ . We adopt it to ensure that  $v_*(T)$  is well-defined.

*Proof.* For all  $(N, b, V^m) \in \Gamma$  and for all  $t \in \mathbb{N}_m$ ,

$$\begin{aligned}
& \sum_{i \in N} \overline{\beta}_i^t(N, b, V^m) \\
&= \sum_{i \in N} \left[ \beta_i^t(N, b, V^m) + \frac{1}{|N|} \cdot \left[ v_*^t(N) - \sum_{k \in N} \beta_k^t(N, b, V^m) \right] \right] \\
&= \sum_{i \in N} \beta_i^t(N, b, V^m) + \frac{|N|}{|N|} \cdot \left[ v_*^t(N) - \sum_{k \in N} \beta_k^t(N, b, V^m) \right] \\
&= \sum_{i \in N} \beta_i^t(N, b, V^m) + v_*^t(N) - \sum_{k \in N} \beta_k^t(N, b, V^m) \\
&= v_*^t(N).
\end{aligned}$$

Thus, the MEBOI satisfies MEFF on  $\Gamma$ . ■

□

Here we provide a brief application of multicriteria multi-choice TU games in the setting of “management”. This kind of problem can be formulated as follows. Let  $N = \{1, 2, \dots, n\}$  be a set of all members of a grand management system  $(N, b, V^m)$ . The function  $v^t$  could be treated as an utility function which assigns to each level vector  $\alpha = (\alpha_i)_{i \in N} \in B^N$  the value that the members can obtain when each member  $i$  participates at operation strategy  $\alpha_i \in B_i$  in the sub-management system  $(N, b, v^t)$ . Modeled in this way, the grand management system  $(N, b, V^m)$  could be considered as a multicriteria multi-choice TU game, with  $v^t$  being each characteristic function and  $B_i$  being the set of all operation strategies of the member  $i$ . In the following sections, we show that the MBOI and the MEBOI could provide “*optimal allocation mechanisms*” among all members, in the sense that this organization can get payoff from each combination of operation strategies of all members under multi-choice behavior and multicriteria situation.

### 3. AXIOMATIZATIONS

In this section, we show that there exists corresponding reduced games that could be adopted to analyze the MBOI and the MEBOI.

Subsequently, we introduce a reduced game and the related consistency as follows. Let  $\psi$  be a solution,  $(N, b, V^m) \in \Gamma$  and  $S \subseteq N$ . The **1-reduced game**  $(S, b_S, V_{S,\psi}^{1,m})$  is defined by  $V_{S,\psi}^{1,m} = (v_{S,\psi}^{1,t})_{t \in \mathbb{N}_m}$  and

$$v_{S,\psi}^{1,t}(\alpha) = \begin{cases} \sum_{Q \subseteq N \setminus S} \left[ v_*^t(A(\alpha) \cup Q) - \sum_{i \in Q} \psi_i^t(N, b, V^m) \right] & \text{otherwise,} \\ 0 & \text{if } \alpha = 0_S. \end{cases}$$

$\psi$  satisfies **1-consistency (1CON)** if  $\psi_i^t(S, b_S, V_{S,\psi}^{1,m}) = \psi_i^t(N, b, V^m)$  for all  $(N, b, V^m) \in \Gamma$ , for all  $S \subseteq N$  with  $|S| = 2$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in S$ . Further,  $\psi$  satisfies **1-standard for games (1SG)** if  $\psi(N, b, V^m) = \beta(N, b, V^m)$  for all  $(N, b, V^m) \in \Gamma$  with  $|N| \leq 2$ . Next, we characterize the MBOI by applying the properties of 1CON and 1SG.

**Lemma 3.1.**

1. The MBOI satisfies 1CON on  $\Gamma$ .
2. On  $\Gamma$ , the MBOI is the only solution satisfying 1SG and 1CON.

*Proof.* To prove result 1, let  $(N, b, V^m) \in \Gamma$  and  $S \subseteq N$ . If  $|N| = 1$ , then the proof is completed. Assume that  $|N| \geq 2$  and  $S = \{i, j\}$  for some  $i, j \in N$ . For all  $t \in \mathbb{N}_m$  and for all  $i \in S$ ,

$$\begin{aligned}
\beta_i^t(S, b_S, V_{S,\beta}^{1,m}) &= \sum_{\substack{T \subseteq S \\ i \in T}} \left[ (v_{S,\beta}^{1,t})_*(T) - (v_{S,\beta}^{1,t})_*(T \setminus \{i\}) \right] \\
&= \sum_{\substack{T \subseteq S \\ i \in T}} \sum_{Q \subseteq N \setminus S} \left[ v_*^t(T \cup Q) - v_*^t(T \setminus \{i\} \cup Q) \right] \\
&= \sum_{\substack{K \subseteq N \\ i \in K}} \left[ v(K) - v(K \setminus \{i\}) \right] \\
&= \beta_i^t(N, b, V^m).
\end{aligned}$$

Hence,  $\beta$  satisfies 1CON.

By result 1,  $\beta$  satisfies 1CON on  $\Gamma$ . Clearly,  $\beta$  satisfies 1SG on  $\Gamma$ . To prove uniqueness of result 2, suppose  $\psi$  satisfies 1SG and 1CON. Let  $(N, b, v) \in \Gamma$ . If  $|N| \leq 2$ , then  $\psi(N, b, v) = \beta(N, b, v)$  by 1SG. The case  $|N| > 2$ : Let  $i \in N$  and  $t \in \mathbb{N}_m$ , and let  $S \subset N$  with  $|S| = 2$  and  $i \in S$ . Then,

$$\begin{aligned}
&\psi_i^t(N, b, V^m) \\
&= \psi_i^t(S, b_S, V_{S,\psi}^{1,m}) && \text{(by 1CON)} \\
&= \beta_i^t(S, b_S, V_{S,\psi}^{1,m}) && \text{(by 1SG)} \\
&= \sum_{\substack{T \subseteq S \\ i \in T}} \left[ (v_{S,\psi}^{1,t})_*(T) - (v_{S,\psi}^{1,t})_*(T \setminus \{i\}) \right] \\
&= \sum_{\substack{T \subseteq S \\ i \in T}} \sum_{Q \subseteq N \setminus S} \left[ v_*^t(T \cup Q) - v_*^t((T \setminus \{i\}) \cup Q) \right] \\
&= \sum_{\substack{K \subseteq N \\ i \in K}} v_*^t(K) - v_*^t(K \setminus \{i\}) \\
&= \beta_i^t(N, b, V^m).
\end{aligned}$$

Thus,  $\psi(N, b, V^m) = \beta(N, b, V^m)$ . ■

□

The following examples are to show that each of the axioms adopted in Lemma 2 is logically independent of the remaining axioms.

**Example 3.2.** Define a solution  $\psi$  by for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ ,  $\psi_i^t(N, b, V^m) = 0$ . Clearly,  $\psi$  satisfies 1CON, but it violates 1SG.

**Example 3.3.** Define a solution  $\psi$  by for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ ,

$$\psi_i^t(N, b, V^m) = \begin{cases} \beta_i^t(N, b, V^m) & \text{if } |N| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

On  $\Gamma$ ,  $\psi$  satisfies 1SG, but it violates 1CON.

It is easy to check that the index  $\bar{\beta}$  violates 1CON. Therefore, we consider the 2-reduced game as follows. Let  $\psi$  be a solution,  $(N, b, V^m) \in \Gamma$  and  $S \subseteq N$ . The **2-reduced game**  $(S, b_S, V_{S,\psi}^{2,m})$  is defined by  $V_{S,\psi}^{2,m} = (v_{S,\psi}^{2,t})_{t \in \mathbb{N}_m}$  and

$$v_{S,\psi}^{2,t}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v_*^t(N) - \sum_{i \in N \setminus S} \psi_i^t(N, b, V^m) & \text{if } A(\alpha) = S, \\ \sum_{Q \subseteq N \setminus S} \left[ v_*^t(A(\alpha) \cup Q) - \sum_{i \in Q} \psi_i^t(N, b, V^m) \right] & \text{otherwise.} \end{cases}$$

$\psi$  satisfies **2-consistency (2CON)** if  $\psi_i^t(S, b_S, V_{S,\psi}^{2,m}) = \psi_i^t(N, b, V^m)$  for all  $(N, b, V^m) \in \Gamma$ , for all  $S \subseteq N$  with  $|S| = 2$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in S$ . Further,  $\psi$  satisfies **2-standard for games (2SG)** if  $\psi(N, b, V^m) = \bar{\beta}(N, b, V^m)$  for all  $(N, b, V^m) \in \Gamma$  with  $|N| \leq 2$ . Next, we characterize the MEBOI by applying the properties of 2CON and 2SG.

In order to establish consistency of the MEBOI, it will be useful to present alternative formulation for the MEBOI in terms of *excess*. Let  $(N, b, V^m) \in \Gamma$ ,  $S \subseteq N$  and  $x$  be a payoff vector in  $(N, b, V^m)$ . Define that  $x^t(S) = \sum_{i \in S} x_i^t$  for all  $t \in \mathbb{N}_m$ . The **excess** of a coalition  $S \subseteq N$  at  $x$  is the real number

$$e(S, V^m, x) = (e(S, v^t, x^t))_{t \in \mathbb{N}_m} \text{ and } e(S, v^t, x^t) = v_*^t(S) - x^t(S). \quad (1)$$

The value  $e(S, v^t, x^t)$  can be treated as the **complaint** of coalition  $S$  when all members receive their payoffs from  $x^t$  in  $(N, b, v^t)$ .

**Lemma 3.4.** Let  $(N, b, V^m) \in \Gamma$  and  $x \in E(N, b, V^m)$ . Then, for all  $i, j \in N$  and  $t \in \mathbb{N}_m$

$$\begin{aligned} \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{i\}, v^t, \frac{x^t}{2^{|\mathbb{N}|-1}}) &= \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{j\}, v^t, \frac{x^t}{2^{|\mathbb{N}|-1}}) \\ \iff x &= \bar{\beta}(N, b, V^m). \end{aligned}$$

*Proof.* Let  $(N, b, V^m) \in \Gamma$  and  $x \in E(N, b, V^m)$ . For all  $t \in \mathbb{N}_m$  and for all  $i, j \in N$ ,

$$\begin{aligned} \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{i\}, v^t, \frac{x^t}{2^{|\mathbb{N}|-1}}) &= \sum_{S \subseteq N \setminus \{i,j\}} e(S \cup \{j\}, v^t, \frac{x^t}{2^{|\mathbb{N}|-1}}) \\ \iff \sum_{S \subseteq N \setminus \{i,j\}} \left[ v_*^t(S \cup \{i\}) - \frac{x^t(S \cup \{i\})}{2^{|\mathbb{N}|-1}} \right] &= \sum_{S \subseteq N \setminus \{i,j\}} \left[ v_*^t(S \cup \{j\}) - \frac{x^t(S \cup \{j\})}{2^{|\mathbb{N}|-1}} \right] \\ \iff \left[ \sum_{S \subseteq N \setminus \{i,j\}} v_*^t(S \cup \{i\}) \right] - \frac{x_i^t}{2} &= \left[ \sum_{S \subseteq N \setminus \{i,j\}} v_*^t(S \cup \{j\}) \right] - \frac{x_j^t}{2} \\ \iff x_i^t - x_j^t &= 2 \cdot \sum_{S \subseteq N \setminus \{i,j\}} \left[ v_*^t(S \cup \{i\}) - v_*^t(S \cup \{j\}) \right]. \end{aligned} \quad (2)$$

By definition of  $\bar{\beta}$ ,

$$\bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m) = 2 \cdot \sum_{S \subseteq N \setminus \{i,j\}} \left[ v_*^t(S \cup \{i\}) - v_*^t(S \cup \{j\}) \right] \quad (3)$$

By equations (2) and (3), for all  $i, j \in N$ ,

$$x_i^t - x_j^t = \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m).$$

Hence,

$$\sum_{j \in N} [x_i^t - x_j^t] = \sum_{j \in N} [\bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m)].$$

That is,  $|N| \cdot x_i^t - \sum_{j \in N} x_j^t = |N| \cdot \bar{\beta}_i^t(N, b, V^m) - \sum_{j \in N} \bar{\beta}_j^t(N, b, V^m)$ . Since  $x \in E(N, b, V^m)$  and  $\bar{\beta}$  satisfies MEFF,  $|N| \cdot x_i^t - v_*^t(N) = |N| \cdot \bar{\beta}_i^t(N, b, V^m) - v_*^t(N)$ . Therefore,  $x_i^t = \bar{\beta}_i^t(N, b, V^m)$  for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ , i.e.,  $x = \bar{\beta}(N, b, V^m)$ . ■ □

**Remark 1.** It is easy to check that

$$\sum_{S \subseteq N \setminus \{i, j\}} e(S \setminus \{i\}, V^m, \beta(N, b, V^m)) = \sum_{S \subseteq N \setminus \{i, j\}} e(S \setminus \{j\}, V^m, \beta(N, b, V^m))$$

for all  $(N, b, V^m) \in \Gamma$  and for all  $i, j \in N$ .

**Theorem 1.**

1. The MEBOI satisfies 2CON on  $\Gamma$ .
2. If  $\psi$  satisfies 2SG and 2CON, then it also satisfies MEFF.
3. On  $\Gamma$ , the MEBOI is the only solution satisfying 2SG and 2CON.

*Proof.* To verify result 1, let  $(N, b, V^m) \in \Gamma$  and  $S \subseteq N$ . If  $|N| = 1$ , then the proof is completed. Assume that  $|N| \geq 2$ ,  $x = \bar{\beta}(N, b, V^m)$  and  $S = \{i, j\}$  for some  $i, j \in N$ . For all  $t \in \mathbb{N}_m$  and for all  $l \in S$ ,

$$\begin{aligned} & \sum_{T \subseteq S \setminus \{i, j\}} e(T \cup \{l\}, v_{S, x}^{2, t}, \frac{x_S^t}{2^{|S|-1}}) \\ &= e(\{l\}, v_{S, x}^{2, t}, \frac{x_S^t}{2}) \\ &= v_{S, x}^{2, t}(\{l\}) - \frac{x_l^t}{2} \\ &= \left( \sum_{Q \subseteq N \setminus S} [v_*^t(\{l\} \cup Q) - \sum_{k \in Q} x_k^t] \right) - \frac{x_l^t}{2} \\ &= \sum_{Q \subseteq N \setminus S} \left[ v_*^t(\{l\} \cup Q) - \sum_{k \in Q} x_k^t - \frac{x_l^t}{2 \cdot 2^{|N \setminus S|}} \right] \\ &= \sum_{Q \subseteq N \setminus S} \left[ v_*^t(\{l\} \cup Q) - \sum_{k \in Q} x_k^t - \frac{x_l^t}{2^{|N|-1}} \right] \tag{4} \\ &= \sum_{Q \subseteq N \setminus S} \left[ v_*^t(\{l\} \cup Q) - \sum_{k \in Q} x_k^t + \sum_{k \in Q} \frac{x_k^t}{2^{|N|-1}} - \sum_{k \in Q} \frac{x_k^t}{2^{|N|-1}} - \frac{x_l^t}{2^{|N|-1}} \right] \\ &= \sum_{Q \subseteq N \setminus S} \left[ v_*^t(\{l\} \cup Q) - \sum_{k \in Q} \left( 1 - \frac{1}{2^{|N|-1}} \right) x_k^t - \sum_{k \in Q \cup \{l\}} \frac{x_k^t}{2^{|N|-1}} \right] \\ &= \sum_{Q \subseteq N \setminus S} \left[ e(\{l\} \cup Q, v^t, \frac{x}{2^{|N|-1}}) - \sum_{k \in Q} \left( 1 - \frac{1}{2^{|N|-1}} \right) x_k^t \right] \\ &= \sum_{Q \subseteq N \setminus S} e(\{l\} \cup Q, v^t, \frac{x}{2^{|N|-1}}) - \sum_{Q \subseteq N \setminus S} \sum_{k \in Q} \left( 1 - \frac{1}{2^{|N|-1}} \right) x_k^t. \end{aligned}$$

Since  $\bar{\beta}$  satisfies MEFF,  $x_S^t \in E(S, b_S, v_{S,x}^{2,t})$  by definition of 2-reduced game. In addition, by equation (4) and Lemma 3,

$$\begin{aligned}
& e(\{i\}, v_{S,x}^{2,t}, x_S^t) \\
&= \sum_{Q \subseteq N \setminus S} e(\{i\} \cup Q, v^t, \frac{x}{2^{|N|-1}}) - \sum_{Q \subseteq N \setminus S} \sum_{k \in Q} \left(1 - \frac{1}{2^{|N|-1}}\right) x_k^t \quad (\text{by equation (4)}) \\
&= \sum_{Q \subseteq N \setminus S} e(\{j\} \cup Q, v^t, \frac{x}{2^{|N|-1}}) - \sum_{Q \subseteq N \setminus S} \sum_{k \in Q} \left(1 - \frac{1}{2^{|N|-1}}\right) x_k^t \quad (\text{by Lemma 3}) \\
&= e(\{j\}, v_{S,x}^{2,t}, x_S^t). \quad (\text{by equation (4)})
\end{aligned}$$

So,  $x_S = \bar{\beta}(S, b_S, V_{S,\bar{\beta}}^{2,m})$ . That is,  $\bar{\beta}$  satisfies 2CON.

To prove result 2, suppose  $\psi$  satisfies 2SG and 2CON. Let  $(N, b, V^m) \in \Gamma$  and  $t \in \mathbb{N}_m$ . If  $|N| \leq 2$ , it is trivial that  $\psi$  satisfies MEFF by 2SG. The case  $|N| > 2$ : Assume, on the contrary, that there exists  $(N, b, V^m) \in \Gamma$  such that  $\sum_{i \in N} \psi_i^t(N, b, V^m) \neq v_*^t(N)$ . This means that there exist  $i \in N$  and  $j \in N$  such that  $[v_*^t(N) - \sum_{k \in N \setminus \{i,j\}} \psi_k^t(N, b, V^m)] \neq [\psi_i^t(N, b, V^m) + \psi_j^t(N, b, V^m)]$ . By 2CON and  $\psi$  satisfies MEFF for two-person games, this contradicts with

$$\begin{aligned}
& \psi_i^t(N, b, V^m) + \psi_j^t(N, b, V^m) \\
&= \psi_i^t(\{i, j\}, b_{\{i,j\}}, v_{\{i,j\},\psi}^{2,t}) + \psi_j^t(\{i, j\}, b_{\{i,j\}}, v_{\{i,j\},\psi}^{2,t}) \\
&= v_*^t(N) - \sum_{k \in N \setminus \{i,j\}} \psi_k^t(N, b, V^m).
\end{aligned}$$

Hence,  $\psi$  satisfies MEFF.

To prove result 3,  $\bar{\beta}$  satisfies 2CON by result 1. Clearly,  $\bar{\beta}$  satisfies 2SG. To prove uniqueness, suppose  $\psi$  satisfies 2SG and 2CON, hence by result 2,  $\psi$  also satisfies MEFF. Let  $(N, b, V^m) \in \Gamma$ . If  $|N| \leq 2$ , it is trivial that  $\psi(N, b, V^m) = \bar{\beta}(N, b, V^m)$  by 2SG. The case  $|N| > 2$ : Let  $i \in N$ ,  $t \in \mathbb{N}_m$  and  $S = \{i, j\}$  for some  $j \in N \setminus \{i\}$ . Then

$$\begin{aligned}
& \psi_i^t(N, b, V^m) - \bar{\beta}_i^t(N, b, V^m) \\
&= \psi_i^t(S, b_S, V_{S,\psi}^{2,m}) - \bar{\beta}_i^t(S, b_S, V_{S,\bar{\beta}}^{2,m}) \quad (\text{by 2CON of } \psi, \bar{\beta}) \\
&= \bar{\beta}_i^t(S, b_S, V_{S,\psi}^{2,m}) - \bar{\beta}_i^t(S, b_S, V_{S,\bar{\beta}}^{2,m}) \quad (\text{by 2SG of } \psi, \bar{\beta}) \quad (5) \\
&= \frac{1}{2} \cdot \left[ (v_{S,\psi}^{2,t})_*(S) + (v_{S,\psi}^{2,t})_*(\{i\}) - (v_{S,\psi}^{2,t})_*(\{j\}) \right] \\
&\quad - \frac{1}{2} \cdot \left[ (v_{S,\bar{\beta}}^{2,t})_*(S) + (v_{S,\bar{\beta}}^{2,t})_*(\{i\}) - (v_{S,\bar{\beta}}^{2,t})_*(\{j\}) \right].
\end{aligned}$$

By definitions of  $v_{S,\psi}^{2,t}$  and  $v_{S,\bar{\beta}}^{2,t}$ ,

$$\begin{aligned}
(v_{S,\psi}^{2,t})_*(\{i\}) - (v_{S,\psi}^{2,t})_*(\{j\}) &= \frac{1}{2^{|N \setminus S|}} \cdot \sum_{Q \subseteq N \setminus S} \left[ v_*^t(\{i\} \cup Q) - v_*^t(\{j\} \cup Q) \right] \\
&= (v_{S,\bar{\beta}}^{2,t})_*(\{i\}) - (v_{S,\bar{\beta}}^{2,t})_*(\{j\}). \quad (6)
\end{aligned}$$



By equation (6), equation (5) becomes

$$\begin{aligned} & \psi_i^t(N, b, V^m) - \bar{\beta}_i^t(N, b, V^m) \\ &= \frac{1}{2} \cdot \left( (v_{S, \psi}^{2,t})_*(S) - (v_{S, \bar{\beta}^t}^{2,t})_*(S) \right) \\ &= \frac{1}{2} \cdot \left( \psi_i^t(N, b, V^m) + \psi_j^t(N, b, V^m) - \bar{\beta}_i^t(N, b, V^m) + \bar{\beta}_j^t(N, b, V^m) \right). \end{aligned}$$

That is,

$$\psi_i^t(N, b, V^m) - \psi_j^t(N, b, V^m) = \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m)$$

for all  $i, j \in N$ . By MEFF of  $\psi$  and  $\bar{\beta}$ ,

$$|N| \cdot \psi_i^t(N, b, V^m) - v_i^t(N) = |N| \cdot \bar{\beta}_i^t(N, b, V^m) - v_i^t(N).$$

Thus,  $\psi_i^t(N, b, V^m) = \bar{\beta}_i^t(N, b, V^m)$  for all  $i \in N$ , i.e.,  $\psi(N, b, V^m) = \bar{\beta}(N, b, V^m)$ . ■

□

The following examples are to show that each of the axioms adopted in Theorem 1 is logically independent of the remaining axioms.

**Example 3.5.** Define a solution  $\psi$  by for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ ,  $\psi_i^t(N, b, V^m) = 0$ . Clearly,  $\psi$  satisfies 2CON, but it violates 2SG.

**Example 3.6.** Define a solution  $\psi$  by for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ ,

$$\psi_i^t(N, b, V^m) = \begin{cases} \bar{\beta}_i^t(N, b, V^m) & \text{if } |N| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

On  $\Gamma$ ,  $\psi$  satisfies 2SG, but it violates 2CON.

#### 4. DYNAMIC PROCESSES

In this section, we adopt excess function and reduction to propose dynamic processes for the MEBOI.

In order to establish the dynamic processes of the MEBOI, we firstly define correction function by means of excess functions. The correction function is based on the idea that each member shortens the complaint relating to his own and others' nonparticipation, and adopts these regulations to correct the original payoff.

**Definition 3.** Let  $(N, b, V^m) \in \Gamma$  and  $i \in N$ . The **correction function** is defined to be  $f = (f^t)_{t \in \mathbb{N}_m}$ , where  $f^t = (f_i^t)_{i \in N}$  and  $f_i^t : E(N, b, V^m) \rightarrow \mathbb{R}$  is define by

$$f_i^t(x) = x_i^t + w \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i, j\}} \left[ e(Q \setminus \{j\}, v^t, \frac{x^t}{2^{|N|-1}}) - e(Q \setminus \{i\}, v^t, \frac{x^t}{2^{|N|-1}}) \right],$$

where  $w \in \mathbb{R}$  is a fixed positive number, which reflects the assumption that member  $i$  does not ask for full correction (when  $w = 1$ ) but only (usually) a fraction of it. Define  $[x]^0 = x$ ,  $[x]^1 = f([x]^0)$ ,  $\dots$ ,  $[x]^q = f([x]^{q-1})$  for all  $q \in \mathbb{N}$ .

**Lemma 4.1.**  $f(x) \in E(N, b, V^m)$  for all  $(N, b, V^m) \in \Gamma$  and for all  $x \in E(N, b, V^m)$ .

*Proof.* Let  $(N, b, V^m) \in \Gamma$ ,  $t \in \mathbb{N}_m$ ,  $i, j \in N$  and  $x \in E(N, b, V^m)$ . Similar to equation (2),

$$\begin{aligned} & \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i, j\}} \left[ e(Q \setminus \{j\}, v^t, \frac{x^t}{2^{|N|-1}}) - e(Q \setminus \{i\}, v^t, \frac{x^t}{2^{|N|-1}}) \right] \\ &= \sum_{j \in N \setminus \{i\}} \left[ \left( \sum_{Q \subseteq N \setminus \{i, j\}} \left[ v^t(Q \setminus \{j\}) - v^t(Q \setminus \{i\}) \right] \right) - \frac{x_i^t}{2} + \frac{x_j^t}{2} \right]. \end{aligned} \quad (7)$$

By definition of  $\bar{\beta}$ ,

$$\bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m) = 2 \sum_{Q \subseteq N \setminus \{i, j\}} \left[ v^t(Q \setminus \{j\}) - v^t(Q \setminus \{i\}) \right]. \quad (8)$$

By equations (7) and (8),

$$\begin{aligned} & \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i, j\}} \left[ e(Q \setminus \{j\}, v^t, \frac{x^t}{2^{|N|-1}}) - e(Q \setminus \{i\}, v^t, \frac{x^t}{2^{|N|-1}}) \right] \\ &= \frac{1}{2} \cdot \sum_{j \in N \setminus \{i\}} \left( \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m) - x_i^t + x_j^t \right) \\ &= \frac{1}{2} \cdot \left( (|N| - 1) \left( \bar{\beta}_i^t(N, b, V^m) - x_i^t \right) - \sum_{j \in N \setminus \{i\}} \bar{\beta}_j^t(N, b, V^m) + \sum_{j \in N \setminus \{i\}} x_j^t \right) \\ &= \frac{1}{2} \cdot \left( |N| \left( \bar{\beta}_i^t(N, b, V^m) - x_i^t \right) - v_*^t(N) + v_*^t(N) \right) \\ & \quad \text{(by MEFF of } \bar{\beta}, x \in E(N, b, V^m)) \\ &= \frac{|N|}{2} \cdot \left( \bar{\beta}_i^t(N, b, V^m) - x_i^t \right). \end{aligned} \quad (9)$$

Moreover,

$$\begin{aligned} & \sum_{i \in N} \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i, j\}} \left[ e(Q \setminus \{j\}, v^t, x^t) - e(Q \setminus \{i\}, v^t, x^t) \right] \\ &= \sum_{i \in N} \frac{|N|}{2} \cdot \left( \bar{\beta}_i^t(N, b, V^m) - x_i^t \right) \\ &= \frac{|N|}{2} \cdot \left( \sum_{i \in N} \bar{\beta}_i^t(N, b, V^m) - \sum_{i \in N} x_i^t \right) \\ &= \frac{|N|}{2} \cdot \left( v_*^t(N) - v_*^t(N) \right) \text{ (by MEFF of } \bar{\beta}, x \in E(N, b, V^m)) \\ &= 0. \end{aligned} \quad (10)$$

So we have that

$$\begin{aligned}
& \sum_{i \in N} f_i^t(x) \\
&= \sum_{i \in N} \left[ x_i^t + w \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i,j\}} \left[ e(Q \setminus \{j\}, v^t, x^t) - e(Q \setminus \{i\}, v^t, x^t) \right] \right] \\
&= \sum_{i \in N} x_i^t + w \sum_{i \in N} \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i,j\}} \left[ e(Q \setminus \{j\}, v^t, x^t) - e(Q \setminus \{i\}, v^t, x^t) \right] \\
&= v_*^t(N) + 0 \quad (\text{by equation (10) and } x \in E(N, b, V^m)) \\
&= v_*^t(N).
\end{aligned}$$

Hence,  $f(x) \in E(N, b, V^m)$  if  $x \in E(N, b, V^m)$ . ■ □

**Theorem 2.** Let  $(N, b, V^m) \in \Gamma$ . If  $0 < t < \frac{2}{|N|}$ , then  $\{[x]^q\}_{q=1}^\infty$  converges geometrically to  $\bar{\beta}(N, b, V^m)$  for each  $x \in E(N, b, V^m)$ .

*Proof.* Let  $(N, b, V^m) \in \Gamma$ ,  $t \in \mathbb{N}_m$ ,  $i \in N$  and  $x \in E(N, b, V^m)$ . By equation (9) and definition of  $f$ ,

$$\begin{aligned}
f_i^t(x) - x_i^t &= w \sum_{j \in N \setminus \{i\}} \sum_{Q \subseteq N \setminus \{i,j\}} \left[ e(Q \setminus \{j\}, v^t, x^t) - e(Q \setminus \{i\}, v^t, x^t) \right] \\
&= w \cdot \frac{|N|}{2} \cdot (\beta_i^t(N, b, V^m) - x_i^t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\bar{\beta}_i^t(N, b, V^m) - f_i^t(x) &= \bar{\beta}_i^t(N, b, V^m) - x_i^t + x_i^t - f_i^t(x) \\
&= \bar{\beta}_i^t(N, b, V^m) - x_i^t - w \cdot \frac{|N|}{2} \cdot (\beta_i^t(N, b, V^m) - x_i^t) \\
&= \left(1 - w \cdot \frac{|N|}{2}\right) [\bar{\beta}_i^t(N, b, V^m) - x_i^t].
\end{aligned}$$

So, for all  $q \in \mathbb{N}$ ,

$$\bar{\beta}(N, b, V^m) - [x]^q = \left(1 - w \cdot \frac{|N|}{2}\right)^q [\bar{\beta}(N, b, V^m) - x].$$

If  $0 < w < \frac{4}{|N|}$ , then  $-1 < \left(1 - w \cdot \frac{|N|}{2}\right) < 1$  and  $\{[x]^q\}_{q=1}^\infty$  converges geometrically to  $\bar{\beta}(N, b, V^m)$ . ■

□

Inspired by Maschler and Owen [10], we find a dynamic process under reductions.

**Definition 4.** Let  $\psi$  be a solution,  $(N, b, V^m) \in \Gamma$ ,  $S \subseteq N$  and  $x \in E(N, b, V^m)$ . The  $(x, \psi)$ -reduced game  $(S, b_S, V_{\psi, S, x}^{r, m})$  is given by  $V_{\psi, S, x}^{r, m} = (v_{\psi, S, x}^{r, t})_{t \in \mathbb{N}_m}$  and for all  $T \subseteq S$ ,

$$v_{\psi, S, x}^{r, t}(\alpha) = \begin{cases} v_*^t(N) - \sum_{i \in N \setminus S} x_i^t & A(\alpha) = S, \\ v_{S, \psi}^{2, t}(\alpha) & \text{otherwise.} \end{cases}$$

Inspired by Maschler and Owen [10], we also define different correction function as follow. The **R-correction function** to be  $g = (g^t)_{t \in \mathbb{N}_m}$ , where  $g^t = (g_i^t)_{i \in N}$  and  $g_i^t : E(N, b, V^m) \rightarrow \mathbb{R}$  is define by

$$g_i^t(x) = x_i^t + w \sum_{k \in N \setminus \{i\}} (\overline{\beta}_i^t(\{i, k\}, v_{\overline{\beta}, \{i, k\}, x}^t) - x_i^t).$$

Define  $[\eta]^0 = x$ ,  $[\eta]^1 = g([\eta]^0), \dots$ ,  $[\eta]^q = g([\eta]^{q-1})$  for all  $q \in \mathbb{N}$ .

**Lemma 4.2.**  $g(x) \in E(N, b, V^m)$  for all  $(N, b, V^m) \in \Gamma$  and for all  $x \in E(N, b, V^m)$ .

*Proof.* Let  $(N, b, V^m) \in \Gamma$ ,  $t \in \mathbb{N}_m$ ,  $i, k \in N$  and  $x \in E(N, b, V^m)$ . Let  $S = \{i, k\}$ , by MEFF of  $\overline{\beta}$  and Definition 4,

$$\overline{\beta}_i^t(S, b_S, V_{\overline{\beta}, S, x}^{r, m}) + \overline{\beta}_k^t(S, b_S, V_{\overline{\beta}, S, x}^{r, m}) = x_i^t + x_k^t.$$

By 2CON and 2SG of  $\overline{\beta}$ ,

$$\begin{aligned} \overline{\beta}_i^t(S, b_S, V_{\overline{\beta}, S, x}^{r, m}) - \overline{\beta}_k^t(S, b_S, V_{\overline{\beta}, S, x}^{r, m}) &= (v_{\overline{\beta}, S, x}^{r, t})_*(\{i\}) - (v_{\overline{\beta}, S, x}^{r, t})_*(\{k\}) \\ &= (v_{S, \overline{\beta}}^{2, t})_*(\{i\}) - (v_{S, \overline{\beta}}^{2, t})_*(\{k\}) \\ &= \overline{\beta}_i^t(S, b_S, V_{S, \overline{\beta}}^{2, m}) - \overline{\beta}_k^t(S, b_S, V_{S, \overline{\beta}}^{2, m}) \\ &= \overline{\beta}_i^t(N, b, V^m) - \overline{\beta}_k^t(N, b, V^m). \end{aligned}$$

Therefore,

$$2 \cdot [\overline{\beta}_i^t(S, b_S, V_{\overline{\beta}, S, x}^{r, m}) - x_i^t] = \overline{\beta}_i^t(N, b, V^m) - \overline{\beta}_k^t(N, b, V^m) - x_i^t + x_k^t. \quad (11)$$

By definition of  $g$  and equation (11),

$$\begin{aligned} g_i^t(x) &= x_i^t + \frac{w}{2} \cdot \left[ \sum_{k \in N \setminus \{i\}} \overline{\beta}_i^t(N, b, V^m) - \sum_{k \in N \setminus \{i\}} x_i^t \right. \\ &\quad \left. - \sum_{k \in N \setminus \{i\}} \overline{\beta}_k^t(N, b, V^m) + \sum_{k \in N \setminus \{i\}} x_k^t \right] \\ &= x_i^t + \frac{w}{2} \cdot \left[ \sum_{k \in N \setminus \{i\}} \overline{\beta}_i^t(N, b, V^m) - (|N| - 1)x_i^t \right. \\ &\quad \left. - \sum_{k \in N \setminus \{i\}} \overline{\beta}_k^t(N, b, v) + (v_*^t(N) - x_i^t) \right] \\ &= x_i^t + \frac{w}{2} \cdot \left[ (|N| - 1)\overline{\beta}_i^t(N, b, V^m) - (|N| - 1)x_i^t \right. \\ &\quad \left. - (v_*^t(N) - \overline{\beta}_i^t(N, b, V^m)) + (v_*^t(N) - x_i^t) \right] \\ &= x_i^t + \frac{|N| \cdot w}{2} \cdot [\overline{\beta}_i^t(N, b, V^m) - x_i^t]. \end{aligned} \quad (12)$$

So we have

$$\begin{aligned} \sum_{i \in N} g_i^t(x) &= \sum_{i \in N} \left[ x_i^t + \frac{|N| \cdot w}{2} \cdot [\overline{\beta}_i^t(N, b, V^m) - x_i^t] \right] \\ &= \sum_{i \in N} x_i^t + \frac{|N| \cdot w}{2} \cdot \left[ \sum_{i \in N} \overline{\beta}_i^t(N, b, V^m) - \sum_{i \in N} x_i^t \right] \\ &= v_*^t(N) + \frac{|N| \cdot w}{2} \cdot [v_*^t(N) - v_*^t(N)] \\ &= v_*^t(N). \end{aligned}$$

Thus,  $g(x) \in E(N, b, V^m)$  for all  $x \in E(N, b, V^m)$ . ■ □

**Theorem 3.** Let  $(N, b, V^m) \in \Gamma$ . If  $0 < \alpha < \frac{4}{|N|}$ , then  $\{[\eta]^q\}_{q=1}^\infty$  converges to  $\bar{\beta}(N, b, V^m)$  for each  $x \in E(N, b, V^m)$ .

*Proof.* Let  $(N, b, V^m) \in \Gamma$ ,  $t \in \mathbb{N}_m$  and  $x \in E(N, b, V^m)$ . By equation (12),  $g_i^t(x) = x_i^t + \frac{|N| \cdot w}{2} \cdot [\beta_i^t(N, b, V^m) - x_i^t]$  for all  $i \in N$ . Therefore,

$$\left(1 - \frac{|N| \cdot w}{2}\right) \cdot [\beta_i^t(N, b, V^m) - x_i^t] = [\beta_i^t(N, b, V^m) - g_i^t(x)]$$

So, for all  $q \in \mathbb{N}$ ,

$$\bar{\beta}(N, b, V^m) - [\eta]^q = \left(1 - \frac{|N| \cdot w}{2}\right)^q [\bar{\beta}(N, b, V^m) - x].$$

If  $0 < w < \frac{4}{|N|}$ , then  $-1 < \left(1 - \frac{|N| \cdot w}{2}\right) < 1$  and  $\{[\eta]^q\}_{q=1}^\infty$  converges to  $\bar{\beta}(N, b, v)$  for all  $(N, b, V^m) \in \Gamma$ , for all  $t \in \mathbb{N}_m$  and for all  $i \in N$ . ■ □

## 5. CONCLUSIONS

In this paper, we investigate an extended Banzhaf-Owen index and its efficient extension in the framework of multicriteria multi-choice TU games. Based on reduced game, two axiomatizations for these two indexes are proposed. By applying reduction and excess function respectively, we also introduce different formulations and related dynamic processes for the multi-choice efficient Banzhaf-Owen index. One should compare our results with related pre-existing results:

- The multi-choice Banzhaf-Owen index and the multi-choice efficient Banzhaf-Owen index are introduced initially in the framework of multicriteria multi-choice TU games.
- The idea of our correction functions in Definitions 3, 4 and related dynamic processes are based on that of Maschler and Owen's [10] dynamic results for the Shapley value [15]. The major difference is that our correction functions in Definition 4 are based on "excess function", and Maschler and Owen's [10] correction function is based on "reduced games".

The above mentioned raise one question:

- Whether there exist other normalizations and the related results for some more solutions in the framework of multicriteria multi-choice TU games.

To our knowledge, these issues are still open questions.

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