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A PRIORITY BASED TIME MINIMIZATION TRANSPORTATION PROBLEM

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Abstract: This paper discusses a priority based time minimizing transporation problem in which destinations are prioritized so that the material is supplied, based upon the priorities of the destinations. All the destinations, which are at priority, are served first in Stage-I while the demands of the secondary destinations are met in Stage-II. It is assumed that secondary transportation can not take place until the primary transportation is done. The purpose is to transport in such a manner that the sum of the transportation time of primary and secondary destinations is minimum. To achieve this, two approaches are proposed. In the first approach, primary destinations are served optimally by giving weights while in the second approach, lexicographic optimization is used. From the generated pairs, the minimum sum of times corressponding to Stage-I and Stage-II times is picked up as the optimal solution. It is also shown, through Computational Details, that the lexicographic optimization converges to the optimal solution faster than the first approach as reported in Table 4.

Keywords: Time Minimizing Transportation Problem(TMTP), Non Linear Programming(NLP) , Lexicographic Solution.

MSC: 90B85, 90C26.

1. INTRODUCTION

Time minimizing transportation is an important class of transportation problem. In this, concave function is optimized over a convex polytope and for this reason it is included in the class of concave minimization problems[4]. Thus, the search is restricted to finding only the extreme points. In Literature, various authors have contributed in this field, where the initial contribution was due to Hammer([9],[10]). Later on, different authors have studied and proposed various methodologies to find the solution of this problem. Some important methods have been developed by Szwarc ([20],[21]), Garfinkel and Rao[8], Bhatia et al. [5], Ahuja[1], Prakash [15], Chandra et al. [7], Issermann[11], Arora and Puri[3]. Some methodologies are also developed to find lexicographic solution of bottleneck problem [6], [2]. In lexicographic optimization, our aim is to minimize the transportation cost not only on the routes of the longest duration but also on the routes of second longest, and third longest duration, and so on. Sherali [17], Mazzola and Neebee [13] developed methods for the computation of weights to find lexicographic optimal solutions. All the available techniques for time minimizing transportation problem(TMTP) involves cost minimizing transportation problem(CMTP) for which polynomial time algorithms already exist. Hence, TMTP is solvable in polynomial time. Sonia and Puri[18] discussed hierarchy in levels of TMTP, while an iterative procedure to solve this is developed by Anuj Sharma et al. [16].

Mathematically, TMTP can be defined as

$$\min[\max_{(i,j)\in IXJ} t_{ij}(x_{ij})]$$

Subject to

$$\begin{aligned} &\sum_{j=1}^{n} x_{ij} = a_i, \quad \forall i \in I = \{1,2,3,\ldots,m\} \\ &\sum_{i=1}^{m} x_{ij} = b_j, \quad \forall j \in J = \{1,2,3,\ldots,n\} \\ &\quad x_{ij} \geq 0, \quad \forall (i,j) \in IXJ \end{aligned} \tag{P1}$$

where,
I(Set of sources) = $\{1,2,\ldots,m\}$,
J(Set of destinations) = $\{1,2,\ldots,n\}$,
a_i : availability at each source; b_j : requirement at each destination,
x_{ij} : the quantity transported to destination j from source i,
t_{ij} : transportation time involved when the destination j is being supplied by the source i

$$t_{ij}(x_{ij}) = \begin{cases} t_{ij} & \text{, if } j\text{th destination is supplied by } i\text{th source} \\ 0 & \text{, otherwise} \end{cases}$$

A priority based assignment problem is studied by Prabhjot Kaur et al.[12], where an industrial project was discussed, which gives the optimal asssignment in a finite number of iterations. This is done by selecting m manufacturing units for the primary jobs, and the secondary jobs are performed(or assigned) optimally by selecting from the remaining $(n-m)$ units, where m and n are the number of units and the number of jobs, respectively. Two stage time minimizing assignment problem is also discussed by Sonia and M.C.Puri[19]. Ilija Nikolić[14] discussed total time minimizing transportation problem related to active routes, in which if more than one optimal solutions exist, it is conceivable to incorporate other criteria as a second level of the criteria. Again, if more than one solutions exist, then the third objective will be optimized in lexicographic order.

A priority based time minimizing transportaion problem is discussed in this paper, and to solve it, we used two approaches. The first approach is motivated by a priority based assignment problem in [12]: primary destinations are served optimally by giving weights as given by Mazzolla's [13] technique, and secondary destinations are served without giving any weights. Therefore, the serving of secondary destinations is dependent upon the serving of primary destinations. On the other hand, in the second approach, the problem is solved by using lexicographic optimization. Out of the lexicographic pairs so produced, the one with the minimum sum of time involved to serve primary and secondary destinations is the optimal solution. It is also shown, through a numerical example, and the computational results that by the second approach, the optimality is reached in lesser number of steps than by the first approach as reported in Table 4. The mathematical model of the problem is given in Section 2. In Section 3, some definitions and results are given, based on, the methodologies are developed. Section 4 shows the working procedure of the two approaches. A numerical illustration is shown in Section 5, and Computational Details for some random problems are shown in Section 6. Some conclusions are given in the Concluding Remarks.

2. PROBLEM DESCRIPTION

2.1. A Priority Based Time Minimizing Transportation Problem (PBTMTP)

A priority based transportation problem is a variant of the classical transportation problem. In this problem some destinatons are labelled with priorities i.e., out of the given destinations, some are to be served at priority in comparison to other. Such destinations are treated as primary destinations, and the rest are secondary destinations. From the application view point, such a situation in the real world, can be encountered in many situations, two of which are listed below:

1. During war time, the soldiers at some destinations require immediate supply due to deficiency of the material at that destination and hence, such destinations are treated as primary destinations.
2. Similarly, now a days, various e-retailers have launched priority services so that the customers receive their goods in one or two days, or sometimes

within few hours instead of the standard delivery time, depending upon the requirement of the product (e.g "Prime service of Amazon"). Such customers ("Prime members") are primary customers, and the rest are secondary.

It is assumed that $\min_{i \in I} a_i > \max_{j \in J} b_j$

Mathematically, PBTMTP can be stated as : (P2)

$$\min\{\min_{X \in S(X)} (\max_{IXJ_1} t_{ij}(x_{ij})) + \max_{\overline{x_{ij}} \in S(\overline{X})} (t_{ij}(\overline{x_{ij}}))\}$$

$$S(X) = \{x_{ij} \in R^{m \times n} \mid \sum_{j \in J_1} x_{ij} \leq a_i, \forall i \in I, \sum_{i \in I} x_{ij} = b_j,$$
$$\forall j \in J_1, x_{ij} \geq 0, \forall (i,j) \in IXJ_1\}$$

$$S\overline{(X)} = \{\overline{x_{ij}} \in R^{m \times (n-m)} \mid \sum_{j \in J_2} \overline{x_{ij}} \leq a_i^{'}, \forall i \in \overline{I}, \sum_{i \in \overline{I}} \overline{x_{ij}} = b_j,$$
$$\forall j \in J_2, \overline{x_{ij}} \geq 0, \forall (i,j) \in \overline{I}XJ_2\}$$

Here,
I (Set of sources) $= \{1, 2, \ldots, m\}$,
J (Set of destinations) $= \{1, 2, \ldots, n\}$,
a_i= availability at any source i,
b_j= requirement at any destination j,
t_{ij} : transportation time when ith source supply to jth destination,
$a_i{}' = a_i - \sum_{i \in I} x_{ij}, \forall j \in J_1$,
$i.e.$ updated availability at each source i when all primary destinations are served,
$\overline{I}$ = Set of sources which are available to supply secondary destinations,
$J_2 = J - J_1$: Set of secondary destinations.

3. THEORETICAL DEVELOPMENT

Some Definitions and Results

For a given PBTMTP, first partition the whole transportation routes $IXJ = \{(i,j)\}$, where $i = 1$ to m, $j = 1$ to n into disjoint sets $M_l, l = 1, 2, \ldots, p$, depending upon the corresponding time entries such that $t_1 > t_2 > t_3 > \ldots, \ldots, > t_p$.

Now, corresponding to each defined set M_l, positive weights are attached, say λ_{p-l+1} for $l = 1, 2 \ldots, \ldots, p$, where $\lambda_{l+1} >> \lambda_l, \forall\, l = 1, 2, \ldots, \ldots, p-1$. One can refer Mazzolla and Neebee [13], and Sherali [17] for computation of values of λ.

1. **Sherali's Technique**

    - The values of λ_{p-l+1} are given as
    $\lambda_{p-l+1} = B^{p-l}, l = 1, 2, \ldots, p.$
    $B = 1 + Max[UB(\sum \sum_{(i,j)\in M_l} x_{ij}, l = 1, 2, \ldots, p)]$, where UB denote the upper bound.

- The values of λ_{p-l+1} are found as
$\lambda_1 = 1$
$u_{p-l+1} = u_{p-l} + \lambda_{p-l+1} \sum_{M_l} x_{ij}, l = p-1, p-2, \ldots, \ldots, 2, 1.$
This gives
$u_p = \lambda_p \sum_{(i,j)\in M_1} x_{ij} + \lambda_{p-1} \sum_{(i,j)\in M_2} x_{ij} + \ldots \cdots + \lambda_1 \sum_{(i,j)\in M_p} x_{ij}$

2. **Mazzola's Technique**

    $\lambda_1 = 1$
    $\lambda_l = (m + n - 1)\lambda_{l-1} + 1, l = 2, 3, \ldots\ldots, p$

Here, we have followed Mazzola's technique [13] for computation of weights. Now, we find the optimal solution of Stage-I by the following cost minimizing transportation problem(CMTP)

$$\min Z(X) = \min_{X\in S(X)} \sum \sum_{(i,j)\in IXJ} c_{ij} x_{ij}$$

$$c_{ij} = \left\{ \begin{array}{ll} 0 & \forall (i,j) \in \overline{I} X J_2 \\ \lambda_{p-l+1} & \forall (i,j) \in (IXJ_1) \cap M_l, l = 1, 2.\ldots.\ldots p \end{array} \right\} \ldots\ldots \qquad \text{(CP)}$$

Let $T_1^o \,\&\, T_2^o$ be the time corresponding to OBFS (Optimal Basic Feasible Solution) of problem CP. The next pair of solutions can be obtained by the restricted version of (CP) i.e.(CP_k). Let T_1^{k-1}, T_2^{k-1} be the corresponding time of Stage I and Stage II, where T_1^{k-1}, $T_2^{k-1} \in \{t_1, t_2, \ldots, t_p\}$, then the next pair of solutions can be obtained as $T_1^k \,\&\, T_2^k$, respectively.

$$(\mathrm{CP}_k) \qquad \min_{x\in S(X)} \sum \sum_{(i,j)\in IXJ} c_{ij}x_{ij}$$

$$c_{ij} = \begin{cases} M & \text{if } t_{ij} \geq T_2^{k-1}, (i,j) \in \overline{I}XJ_2 \\ 0 & \text{if } t_{ij} < T_2^{k-1}, (i,j) \in \overline{I}XJ_2 \\ \lambda_{p-l+1}, & \forall (i,j) \in (\overline{I}XJ_1) \cap M_l, l = 1,2,\ldots,p \end{cases}$$

M-Feasible Solution

The problem (CP_k) is said to be M-feasible if there exists a feasible solution which satisfies the following condition, i.e.
$x_{ij} = 0\ \forall (i,j) \in IXJ$ for which $t_{ij} = M$.

Theorem 1. *Let* (T_1^k, T_2^k) *be the time corresponding to primary and secondary destinations at any* k^{th} *iteration. Then there does not exist a pair such that* $T_2 < T_2^{k-1}$ *and* $T_1 < T_1^k$. *(i.e.* T_1^k *is the minimum time of primary destinations corresponding to secondary destinations time).*

Proof. Let there exists a pair (T_1, T_2) for which the above conditions are satisfied, i.e. $T_2 < T_2^{k-1}$ and $T_1 < T_1^k$. Let $T_1 = t^u; T_1^k = t^v$ for some $u, v \in \{1,2,\ldots,p\}$.
Since $T_1 < T_1^k$ therefore, $t^u < t^v$ this implies $u > v$
$\Rightarrow -u < -v$
$\Rightarrow p-u < p-v$
$\Rightarrow p-u+1 < p-v+1$
Therefore

$$Z(X) = \sum \sum_{(i,j)\in IXJ} (c_{ij}x_{ij}) = \sum_{l=1}^{p} (\lambda_{p-l+1})(\sum_{(i,j)\in M_l} x_{ij}) = \sum_{l=u}^{p} (\lambda_{p-l+1})(\sum_{(i,j)\in M_l} x_{ij})$$

Also $Z(X^k) = \displaystyle\sum_{(i,j)\in(IXJ)} (c_{ij}x_{ij}^k) = \sum_{l=v}^{p} (\lambda_{p-l+1})(\sum_{(i,j)\in M_l} x_{ij}^k)$

Since $\lambda_i >> \lambda_j$ for $i > j$, where $(i,j) \in \{1,2,\ldots,p-1\}$

$$\sum_{l=u}^{p} \lambda_{p-l+1}(\sum_{(i,j)\in M_l} x_{ij}) < \sum_{l=v}^{p} \lambda_{(p-l+1)}(\sum_{(i,j)\in M_l} x_{ij}^k)$$

$Z(X) < Z(X^k)$
Hence, there is a contradiction. Therefore, $T_1^k \leq T_1$. □

Theorem 2. *The optimal solution corresponding to Stage-I is found using (CP).*

Proof. The proof follows on same lines of Theorem1. □

Remark 3. *With the formation of* (CP_k), *it is found that*
$T_2^o > T_2^1 > \ldots \cdots > T_2^t$

*$T_1^o \leq T_1^1 \leq \ldots \cdots \leq T_1^t$,
where T_1^t, be the Stage-I time coresponding to the optimal solution of (CP_t) & (CP_{t+1}), is not M-Feasible.
Proof:- Let $T_1^{k+1} < T_1^k$, for some k
Z_k & Z_{k+1} be the corresponding objective function value, we find $Z_{k+1} < Z_k$ as $T_1^{k+1} < T_1^k$.
X^{k+1} is the optimal solution, which is a contradiction as X_k is the optimal solution.*

Remark 4. *If the problem $(CP_{t+1})$is not M-feasible, then optimal time of Stage-II is T_2^t.*

Theorem 5. *The optimal solution of PBTMTP is given by $\min\limits_{k=1,2...,t}[T_1^k + T_2^k]$, where ($T_1^k$ and $T_2^k, k \geq 0$) are the generated pairs corresponding to Stage-I and Stage-II, respectively.*

Proof. Let us consider, $\exists$ a pair $X' = (T_1, T_2)$ such that $T_1+T_2 < \min\limits_{k=1,2,...t}[T_1^k+T_2^k]$. We know that $T_2^o > T_2^1 > \ldots \cdots > T_2^t$ & $T_1^o \leq T_1^1 \leq \ldots \cdots \leq T_1^t$. Hence, the following cases arise

1. **Case I:** Let $T_2 > T_2^o \quad \ldots(1)$
   By consruction of (CP), T_1^o be the optimal time therefore, $T_1^o \leq T_1 \quad \ldots(2)$
   From (1) and (2) we get
   $T_1 + T_2 > T_1^o + T_2^o$
   $\Rightarrow T_1 + T_2 > min_{k=0,1,...,t}[T_1^k + T_2^k]$

2. **Case II:** Let $T_2 < T_2^t$
   $\Rightarrow$ that $X^{'}$ be the optimal solution of problem (CP_{t+1}), which is not possible as (CP_{t+1}) is not M-feasible.

3. **Case III:** Let $T_2 \in (T_2^o, T_2^t)$
   This implies either $T_2 = T_2^k$ for some $k = \{0, 1, \ldots, t\}$, or $T_2 \in (T_2^{k-1}, T_2^k)$

   - If $T_2 = T_2^k$ for some $k = \{0, 1, \ldots, t\}$, then by construction of (CP_k), we get X' be the M-feasible solution of (CP_k).
     As, $T_1 \geq T_1^k$
     we get, $T_1 + T_2 \geq T_1^k + T_2^k$
     $\Rightarrow T_1 + T_2 \geq \min\limits_{k=0,1,...,t}[T_1^k + T_2^k]$

   - If $T_2 \in (T_2^{k-1}, T_2^k)$
     i.e. if $T_2^k < T_2 < T_2^{k-1}$
     we get, $X^{'}$ is the M-feasible solution of problem (CP_k)

but $T_1 \geq T_1^k$, as T_1^k is the minimum time of (CP_k) corresponding to Stage-I and $T_2 > T_2^k$.
Thus, we have $T_1 + T_2 > T_1^k + T_2^k$
$\Rightarrow \mathrm{T}_1 + T_2 > \min\limits_{k=0,1,\ldots,t} [T_1^k + T_2^k]$

Hence, there exists no feasible solution $X^{'}$ which would yield a minimum value less than $\min\limits_{k=0,1,\ldots,t} [T_1^k + T_2^k]$. Thus, optimal solution of PBTMTP is given by $\min\limits_{k=0,1,\ldots,t} [T_1^k + T_2^k]$. $\square$

4. WORKING STEPS

Here, we give the working steps to find the optimal solution to problem (P2), using the above mentioned approches.

4.1. First Approach

Initial Step Find the optimal solution of problem (CP) and label the time corresponding to Stage-I and Stage-II as time T_1^o and T_2^o, respectively.

General Step If (T_1^{k-1}, T_2^{k-1}) are the time corresponding to Stage-I and Stage-II for $k \geq 1$, then solve (CP_k) for the next pair of solutions, i.e. (T_1^k, T_2^k).

Terminal Step If OBFS obtained by solving (CP_k) is not M-feasible, then stop, and the optimal value is given by $\min\limits_{k=0,1,\ldots,t-1} [T_1^k + T_2^k]$.

4.2. Second Approach

Now, we find lexicographic optimal solution to problem (P2)

Lexicographic Optimal Solution(LOS)

Let $F : S \rightarrow R^q$ be the q-dimensional function, where $S \subset R^q$ and f_k be the kth component of F. Let $X \in S$ be the lexicographic feasible solution, in addition to minimizing the function f_1, one is also interested to minimize f_2 and if f_1 is as small as possible and if $f_1$and f_2 are as small as possible then to minimize f_3 and so on. Hence, lexicographic optimal solution (LOS) of any time minimization transportation (P1) is given by $Lexmin_{X \in S} F(X)$.
A feasible solution $\overline{X} \in S$ is lexicographically better than $X \in S$ for lexicographic general optimization problem (LGOP) if there exists an index $k \in \{1, 2, \ldots, \ldots, p-1\}$, such that $f_r(\overline{X}) = f_r(X), r = 1, 2, \ldots\ldots, k$ & $f_{r+1}(\overline{X}) < f_{r+1}(X)$. It is represented by $F(\overline{X}) < F(X)$, or we can say, $\overline{X}$ is LOS of LGOP if there does not exist $X \in S$ for which $F(X) < F(\overline{X})$, where (LGOP) is $Lexmin_{X \in S} F(X)$. The Lexicographic time minimization transportation problem can be defined as

$$(LTMTP) : Lexmin_{X \in S} F(X)$$

Any rth component of $F(X)$ is given by
$f_r(X) = \sum \sum_{(i,j) \in M_r} c_{ij} x_{ij}, r = 1, 2, \ldots\ldots, p$. Now, to find the optimal solution we find $Lexmin[f_1(X), f_2(X), \ldots\ldots, f_p(X)]$,
i.e. $Lexmin[\sum \sum_{(i,j) \in M_1} c_{ij} x_{ij}, \sum \sum_{(i,j) \in M_2} c_{ij} x_{ij}, \ldots, \ldots, \sum \sum_{(i,j) \in M_p} c_{ij} x_{ij}]$

Theorem 6. *The necessary and the sufficient condition for $\overline{X}$ to be LOS of (LGOP) is that $\overline{X}$ is the optimal solution of $\min_{X \in S} \sum_{r=1}^{p} \lambda_r f_r(X)$, where F is a non-constant p-dimentional vector function, and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are positive real numbers such that $\sum_{r=1}^{p} \lambda_r f_r(X)$ has the sign of $\lambda_t f_t(X)$ where $t = \min_{r=1,2,\ldots,p} (r : f_r(X) \neq 0)$.*

Proof. One can refer to Theorem 1 of [2], or to Appendix A. □

Working Steps

Step 1 Partition the cells $(i, j) \in IXJ$ into disjoint sets $M_l, l = 1, 2, \ldots, p$ according to the time entries such that $t_1 > t_2 > t_3, \ldots, \ldots, > t_p$.

Step 2 Attach weights, say λ_{p-l+1} to each of the above set $M_l, l = 1, 2, \ldots, \ldots, p$ such that $\lambda_{l+1} > \lambda_l, \forall l = 1, 2, \ldots, \ldots, p-1$. These weights can be calculated using [13].

Step 3 While (true)

Find the optimal solution of $\sum_{l=1}^{p} \lambda_l (\sum \sum_{M_l} c_{ij} x_{ij})$ using UV method.

Step 4 Find $T^r = Max(t_{ij})$

- If T^r is M-feasible
  Start If
  For this solution T^r find $[T_r, T_{r'}]$, i.e. the time corresponding to primary and secondary destinations, respectively.
  End If
- Else
  Start Else
  T^r is non M-feasible.
  Break
  End Else

Step 5 Find f_r =[$\mathrm{T}_r + T_{r'}$].

Step 6 Set $t_{ij} = M$ $\forall \mathrm{t}_{ij} > T^r$.
End While Loop

Step 7 Stop. For each pair of lexicographic solution, we find
$Minf_r = Min[T_r + T_{r'}] = T, r = 1, 2, \ldots, p$
Hence, $T = (T_r, T_{r'})$ is the lexicographical optimal solution.

5. NUMERICAL ILLUSTRATION

Consider the following $6X8$ priority based time minimization transportation problem as shown in Table 1. Each cell represents the time of transportation between every source destination pair. Entries which are marked bold show primary destinations, and others show secondary destination.
$I = \{1, 2, 3, 4, 5, 6\}$=Number of given sources
$J = \{1, 2, 3, 4, 5, 6, 7, 8\}$=Number of given destinations
$J_1 = \{1, 3, 4, 6, 8\}$=Primary destinations
$J_2 = \{2, 5, 7\}$ = Secondary destinations

Now, partition various time entries given as $t_1(= 13) > t_2(= 12) > t_3(= 10) > t_4(= 9) > t_5(= 8) > t_6(= 7) > t_7(= 6) > t_8(= 5) > t_9(= 4) > t_{10}(= 3) > t_{11}(= 2) > t_{12}(= 1)$. Here, $t_p = t_{12}$, so $p = 12$
Let $M_l = \{(i, j) : t_{ij} = t^l\}, l = \{1, 2, \ldots, p\}$ and λ_{p-l+1} be the weights attached to the set M_l shown in Table 2,
$M_1 = \{(2, 1), (3, 2), (4, 8), (4, 7), (5, 5)\}$
$M_2 = \{(2, 4), (2, 5), (5, 6), (5, 8)\}$
$M_3 = \{(1, 7), (2, 6), (6, 2)\}$
$M_4 = \{(1, 4), (2, 7), (3, 4), (3, 7), (4, 5), (6, 1)\}$
$M_5 = \{(3, 1), (3, 6), (6, 4)\}$
$M_6 = \{(1, 3), (6, 5), (6, 8)\}$
$M_7 = \{(1, 8), (2, 3), (3, 8), (4, 6), (5, 2), (5, 4), (6, 6)\}$
$M_8 = \{(1, 1), (1, 5), (5, 7), (6, 5), (6, 8)\}$
$M_9 = \{(2, 2), (4, 1), (4, 3), (4, 4), (6, 3), (6, 7)\}$
$M_{10} = \{(1, 2), (2, 8), (3, 5)\}$
$M_{11} = \{(3, 3), (5, 1), (5, 3)\}$
$M_{12} = \{(1, 6), (4, 2)\}$

Then define the corresponding cost minimizing transportation problem(CP)(Ref. Table 2).

Iteration 1 An OBFS of (CP) gives Stage-I time as $T_1^o = 4$ and Stage-II time as $T_2^o = 13$ with the corresponding solution given as

Table 1: Entries in each cell shows trasportation time between sources and destinations, the rightmost entry shows the availability at each source, entries at the bottom show the demand at each destination.

5	3	**7**	**9**	5	**1**	10	**6**	9
13	4	**6**	**12**	12	**10**	9	**3**	8
8	13	**2**	**9**	3	**8**	9	**6**	8
4	1	**4**	**4**	9	**6**	13	**13**	10
2	6	**2**	**6**	13	**12**	5	**5**	9
9	10	**4**	**8**	7	**6**	4	**4**	8
5	8	6	2	6	3	2	3	

Stage-I: $x_{16} = 3, x_{28} = 3, x_{33} = 2, x_{53} = 4, x_{44} = 2, , x_{51} = 5, x_{53} = 4$, and otherwise $x_{ij} = 0$
Stage-II: $\overline{x_{45}} = 6, \overline{x_{47}} = 2, \overline{x_{62}} = 8$ and otherwise $\overline{x_{ij}} = 0$
Hence, the first generated pair is (4,13).

Iteration 2 An OBFS of (CP) gives Stage-I time as $T_1^o = 4$ and Stage-II time as $T_2^o = 10$ with the corresponding solution given as
Stage-I: $x_{16} = 3, x_{28} = 3, x_{33} = 2, x_{53} = 4, x_{44} = 2, , x_{51} = 5, x_{53} = 4$, and otherwise, $x_{ij} = 0$
Stage-II: $\overline{x_{45}} = 6, \overline{x_{37}} = 2, \overline{x_{62}} = 8$, otherwise, $\overline{x_{ij}} = 0$
Hence, the first generated pair is (4,10).

Iteration 3 An OBFS of (CP) gives Stage-I time as $T_1^o = 4$ and Stage-II time as $T_2^o = 7$ with the corresponding solution given as
Stage-I: $x_{16} = 3, x_{28} = 3, x_{33} = 2, x_{53} = 4, x_{44} = 2, , x_{51} = 5, x_{53} = 4$, otherwise, $x_{ij} = 0$
Stage-II: $\overline{x_{42}} = 8, \overline{x_{65}} = 6, \overline{x_{67}} = 2$, otherwise, $\overline{x_{ij}} = 0$
Hence, the first generated pair is (4,7).

Iteration 4 An OBFS of (CP) gives Stage-I time as $T_1^o = 4$ and Stage-II time as $T_2^o = 3$ with the corresponding solution given as

Table 2: Cost minimizing problem when weights are assigned using Mazzolla [13]

λ_5	0	λ_7	λ_9	0	λ_1	0	λ_6
λ_{12}	0	λ_6	λ_{11}	0	λ_{10}	0	λ_3
λ_8	0	λ_2	λ_9	0	λ_8	0	λ_6
λ_4	0	λ_4	λ_4	0	λ_6	0	λ_{12}
λ_2	0	λ_2	λ_6	0	λ_{11}	0	λ_{11}
λ_9	0	λ_4	λ_8	0	λ_6	0	λ_7

Stage-I: $x_{16} = 3, x_{28} = 3, x_{33} = 2, x_{53} = 4, x_{44} = 2, , x_{51} = 5, x_{53} = 4$, otherwise, $x_{ij} = 0$
Stage-II: $\overline{x_{42}} = 8, \overline{x_{35}} = 6, \overline{x_{67}} = 2$, otherwise, $\overline{x_{ij}} = 0$
Hence, the first generated pair is (4,3).

Second Approach
Step 1 Partition various time entries given as
$t_1(= 13) > t_2(= 12) > t_3(= 10) > t_4(= 9) > t_5(= 8) > t_6(= 7) > t_7(= 6) > t_8(= 5) > t_9(= 4) > t_{10}(= 3) > t_{11}(= 2) > t_{12}(= 1)$. Here, $t_p = t_{12}$, so $p = 12$. Let $M_l = \{(i, j) : t_{ij} = t^l\}$ and λ_{p-l+1} be the weights attached to the set M_l shown in Table 3.
$M_1 = \{(2, 1), (3, 2), (4, 8), (4, 7), (5, 5)\}$
$M_2 = \{(2, 4), (2, 5), (5, 6), (5, 8)\}$
$M_3 = \{(1, 7), (2, 6), (6, 2)\}$
$M_4 = \{(1, 4), (2, 7), (3, 4), (3, 7), (4, 5), (6, 1)\}$
$M_5 = \{(3, 1), (3, 6), (6, 4)\}$
$M_6 = \{(1, 3), (6, 5), (6, 8)\}$
$M_7 = \{(1, 8), (2, 3), (3, 8), (4, 6), (5, 2), (5, 4), (6, 6)\}$
$M_8 = \{(1, 1), (1, 5), (5, 7), (6, 5), (6, 8)\}$
$M_9 = \{(2, 2), (4, 1), (4, 3), (4, 4), (6, 3), (6, 7)\}$
$M_{10} = \{(1, 2), (2, 8), (3, 5)\}$
$M_{11} = \{(3, 3), (5, 1), (5, 3)\}$
$M_{12} = \{(1, 6), (4, 2)\}$

Step 2 Attach weights according to Mazzolla technique[13] as shown in Table 3.

Table 3: Entries in each cell represents the weights assigned using Mazzolla's technique[13] corresponding to the given time entries, Bold entries in brackets show the allocations after the first iteration.

λ_5	λ_3**(0)**	λ_7	λ_9	λ_5	λ_1**(3)**	λ_{10}	λ_6
λ_{12}	λ_4	λ_6	λ_{11}	λ_{11}	λ_{10}	λ_9	λ_3**(3)**
λ_8	λ_{12}	λ_2**(2)**	λ_9	λ_3**(6)**	λ_8	λ_9	λ_6
λ_4	λ_1**(8)**	λ_4	λ_4**(2)**	λ_9	λ_6	λ_{12}	λ_{12}
λ_2**(5)**	λ_6	λ_2**(4)**	λ_6	λ_{12}	λ_{11}	λ_5	λ_{11}
λ_9	λ_{10}	λ_4	λ_8	λ_7	λ_6	λ_4**(2)**	λ_7

Step 3 Find the optimal solution to $\sum_{l=1}^{12} \lambda_l (\sum \sum_{M_l} c_{ij} x_{ij})$ using UV method.

Step 4 Find $T^1 = Max(t_{ij}) = T^1 = 4$ (Ref. Table 3). It is M-feasible solution and $(T_1, T_{1'}) = (4, 3)$.

Step 5 Find $f_1 = [T_1 + T_{1'}] = (4 + 3) = 7$.

Step 6 For all $t_{ij} > 4$, set $t_{ij} = M$.

Step 3 Find the optimal solution of $\sum_{r=1}^{12} \lambda_r (\sum \sum_{M_r} c_{ij} x_{ij})$ using UV method.

Step 4 Find $Max(t_{ij}) = T^1 = M$, it is Non M-feasible solution.

Step 7 Stop. The LOS is obtained and is equal to $min(f_1) = 7$.

6. COMPUTATIONAL DETAILS

The algorithm has been coded in MATLAB and successfully verified for random generated PBTMTP of different sizes. Implementation is done on Intel Processor i5 with 2.40 gigahertz, 4 gigabyte RAM on 64 - bit window operating system. Table 5 shows the computational behaviour of the algorithm for some classes of a different size.

Table 4: Average run time (taken over 1000 instances) of PBTMTP for randomly generated problem of different sizes using MATLAB.

Source(m)	Destination(n)	Run Time(sec)(Approach-I)	Run Time(sec)(Approach-II)
10	10	0.024485	0.008345
10	20	0.120363	0.027441
20	20	0.228380	0.054738
20	30	0.531934	0.117426
30	30	1.044533	0.200807
30	40	1.538710	0.304291
40	40	1.432821	0.442290
40	50	2.637870	0.624100
50	50	3.958245	0.846890
50	60	4.405050	1.449748
60	60	7.660967	1.691123
60	70	8.296886	2.367269
70	70	11.020997	2.980602
70	80	13.840961	3.761623
80	80	15.435807	4.309930
80	90	17.392803	4.908302
90	90	22.181180	6.597987

CONCLUDING REMARKS

(a) An exact method to find the solution of PBTMTP is proposed using two approaches.

(b) To the best of authors' knowledge, PBTMTP which is discussed in this paper has not been studied so far. Though, the first proposed approach is motivated by [12], we were unable to solve PBTMTP by first serving secondary destinations, and therefore the approach was modified by first serving the primary destinations. In the absence of any other approach, we are unable to provide any comparative study.

(c) It is shown that lexicographic optimization converges to the optimal solution faster than the first approach, as listed in Table 4, and shown in the Numerical Illustrations.

(d) The problem may be extended to the case when there are three or more prior destinations.

(e) The proposed problem has been coded in MATLAB and tested successfully for randomly generated problems of different sizes.

Appendix A

Proof of Theorem 6 **(Ref. Theorem1 [2])**

Proof. Let $\overline{X}$ be a LOS of (LGOP), hence there does not exist $X \in S$, for which $k \in \{1, 2, \ldots, p-1\}$ such that $f_r(X) = f_r(\overline{X}), r = 1, 2, \ldots, k$ and $f_{r+1}(X) < f_{r+1}(\overline{X})$. It means for no $X \in S$, we have:

$$\sum_{r=1}^{p} \lambda_r (f_r(X) - f_r(\overline{X}) = \lambda_{k+1}(f_{k+1}(X) - f_{k+1}(\overline{X})) + \sum_{j=k+2}^{p} \lambda_j (f_j(X) - f_j(\overline{X}) < 0 (\text{ by the nature of } \lambda_r' s)$$

Therefore, for no such $X \in S$

$$\sum_{r=1}^{p} \lambda_r f_r(X) < \sum_{r=1}^{p} \lambda_r f_r(\overline{X})$$

Thus, $\overline{X}$ is an optimal solution to $\min\limits_{X \in S} \sum_{r=1}^{p} \lambda_r f_r(X)$.

Converse: Let $\overline{X}$ be the optimal solution of $\min\limits_{X \in S} \sum_{r=1}^{p} \lambda_r f_r(X)$

This means

$$\sum_{r=1}^{p} \lambda_r f_r(\overline{X}) \leq \sum_{r=1}^{p} \lambda_r f_r(X), \forall X \in S$$
$$\text{i.e. } \sum_{r=1}^{p} \lambda_r (f_r(\overline{X}) - f_r(X)) \leq 0, \forall X \in S$$

Now, the following two cases arise

(a) Suppose there exist $X \in S$ such that

$$\textstyle\sum_{r=1}^{p} \lambda_r (f_r(\overline{X}) - f_r(X)) = 0$$

Let there be an index $k \in \{1, 2, \ldots\ldots, p-1\}$ such that $f_r(\overline{X}) = f_r(X), r = 1, 2, \ldots, k$ & $f_{r+1}(\overline{X}) \neq f_{r+1}(X)$

This implies that $\sum_{r=1}^{p} \lambda_r (f_r(\overline{X}) - f_r(X)) = 0$ has the sign of $\lambda_{r+1}(f_{r+1}(\overline{X}) - f_{r+1}(X))$, and $\sum_{r=1}^{p} \lambda_r (f_r(\overline{X}) - f_r(X)) \neq 0$, which is a contradiction.

Hence, for any $X \in S$ for which $\sum_{r=1}^{p} \lambda_r (f_r(\overline{X}) - f_r(X)) = 0$, we get $f_r(X) = f_r(\overline{X}), \forall r = 1, 2, \ldots, p$.

(b) Let $\sum_{r=1}^{p} \lambda_r(f_r(\overline{X}) - f_r(X)) = 0, \forall X \in S$, this means $f_r(X) = f_r(\overline{X}), \forall r = 1, 2, \ldots, p$. Hence, F becomes a constant p-dimentional real valued function, which is a contradiction to the assumption.
Hence, there exists an $X \in S$ for which either
$\sum_{r=1}^{p} \lambda_r(f_r(\overline{X}) - f_r(X)) = 0$ or $\sum_{r=1}^{p} \lambda_r(f_r(\overline{X}) - f_r(X)) < 0$
In earlier case, $f_r(\overline{X}) = f_r(X), \forall r = 1, 2, \ldots, p$.
Consider the latter case when $X \in S$ for which $\sum_{k=1}^{p} \lambda_r(f_r(\overline{X}) - f_r(X)) < 0$.
Now, as from the construction of $\lambda_r' s$, it follows that for $X \in S$ there exists an index $k \in \{1, 2, \ldots, p-1\}$ s.t $f_r(\overline{X}) = f_r(X), r = 1, 2, \ldots, k$ and $f_{k+1}(\overline{X}) < f_{k+1}(X)$. Hence, such $X \in S$ is not lexicographic better than $\overline{X}$ for LGOP. Thus, in all situations there does not exist an $X \in S$ such that $F(X) < F(\overline{X})$. Hence, $\overline{X}$ is the LOS of LGOP.

□

Theorem 7. *An optimal solution of* $\min \sum_{r=1}^{p} \lambda_r(\sum \sum_{(i,j)\in M_r} c_{ij}x_{ij})$ *is LOS of time minimization transportation and conversely.*

Proof. Proof of this theorem follows from the above theorem. □

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