Yugoslav Journal of Operations Research
29 (2019), Number 1, 43–50
DOI: https://doi.org/10.2298/YJOR171115003G

FRACTIONAL PROGRAMMING APPROACH TO A COST MINIMIZATION PROBLEM IN ELECTRICITY MARKET

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Received: November 2017 / Accepted: February 2018

Abstract: This paper was motivated by a practical optimization problem that appeared in electricity market of Mongolia. We consider the total average cost minimization problem of power companies of the Ulaanbaatar city. By solving an identification problem, we developed a fractional model that quite adequately represents the real data. The obtained problem turned out to be a fractional minimization problem over a box constraint, and to solve it, we propose a method that employs the global search theory for d.c. minimization.

Keywords: Fractional Minimization, D.C. Programming, Local Search, Linearization, Average Cost, Electricity Market.

MSC: 90C26, 90C32, 90C90.

1. INTRODUCTION

The average cost minimization problem plays an important role not only in engineering and management sciences but also in optimization theory and its methods. As it has already been shown in [9], the average cost minimization problem for a company producing one product belongs to a class of global optimization. In particular, it was revealed that the average cost function is pseudoconvex [9]. Moreover, maximization of efficiency of the average productivity is formulated as the fractional programming [5]. In this paper, we consider the average cost minimization problem with multiple variables in the form of the following fractional program

$$(\mathcal{P}) \qquad f(x) := \frac{\sum\limits_{i=1}^{n} \psi_i(x_i)}{\langle e, x \rangle} \downarrow \min_x, \quad x \in \Pi,$$

where $e = (1, \dots, 1) \in I\!R^n$, Π is a box, f is the total average cost function, ψ_i is the cost function of the power company i, and x_i is the amount of electricity generated by the company i, $i = 1, \dots, n$.

Problem $(\mathcal{P})$ is obviously nonconvex (with numerous local extrema). However, the aim is to minimize only one ratio, which certainly facilitates the solution of the problem. The most common approach to solving the problem is the Dinkelbach method [8] with nonconvex subproblems. There are numerous methods in the literature for solving fractional program. They include variable transformation [24], direct nonlinear programming approach [1], duality approach [2], and parametric approach [18]. A detailed outline on the major areas of fractional programming applications is given in [24]. Another real-world application of fractional programming is described in [21]. Fractional terms composed of signomial functions are first decomposed into convex and concave terms by convexification strategies, and then converted into a convex program in [19]. Recently, an efficient global search method for fractional program, based on two different approaches, was proposed in [11, 12, 13, 15]. The first approach develops Dinkelbach's idea and uses a solution to an equation with the optimal value of an auxiliary d.c. optimization problem with a vector parameter. The second one deals with another auxiliary problem over nonconvex inequality constraints. Both auxiliary problems are d.c. optimization problems, which allows us to apply the Global Optimization Theory [26, 27, 28, 29], recently successfully applied to another practical problem arising in mineral processing industry of Mongolia [10, 16]. In this paper we develop a two-component algorithm based on the global optimality conditions for d.c. optimization problems [26, 28] to solve the average cost minimization problem for power companies of Mongolia.

2. THE D.C. PROGRAMMING APPROACH TO FRACTIONAL PROGRAM

Let us consider the following problem of the fractional optimization [4, 25]

$$(\mathcal{P}_f) \qquad f(x) := \sum_{i=1}^{m} \frac{\psi_i(x)}{\varphi_i(x)} \downarrow \min_x, \quad x \in S,$$

where $S \subset I\!R^n$ is a convex set, and $\psi_i : I\!R^n \to I\!R$, $\varphi_i : I\!R^n \to I\!R$,

$$(\mathcal{H}_0) \qquad \psi_i(x) > 0,\ \varphi_i(x) > 0\ \ \forall x \in S,\ \ i = 1, \ldots, m.$$

First, to solve Problem $(\mathcal{P}_f)$, using the d.c. minimization, we consider the following auxiliary optimization problem

$$(\mathcal{P}_\alpha) \qquad \Phi(x, \alpha) := \sum_{i=1}^{m} [\psi_i(x) - \alpha_i \varphi_i(x)] \downarrow \min_x,\ \ x \in S,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)^\top \in I\!R^m$ is a vector parameter.

Due to the theoretical foundation developed in [11, 15], we are able to avoid the consideration of fractional program, and address the parametrized problem $(\mathcal{P}_\alpha)$ with $\alpha \in I\!R^m_+$. Hence, we propose to combine solving Problem $(\mathcal{P}_\alpha)$ with a search with respect to the parameter $\alpha \in I\!R^m_+$ in order to find the vector $\alpha_0 \in I\!R^m_+$ such that $\mathcal{V}(\alpha_0) = 0$, where

$$\mathcal{V}(\alpha) := \inf_x \{\Phi(x, \alpha)\ \mid\ x \in S\} = \inf_x \left\{ \sum_{i=1}^{m} [\psi_i(x) - \alpha_i \varphi_i(x)]\ :\ x \in S \right\}.$$

Denote $\Phi_i(x) := \psi_i(x) - \alpha_i^k \varphi_i(x),\ i = 1, \ldots, m.$

Let $[0, \alpha_i^+]$ and $[v_i^k, u_i^k]$ be an initial segment and a k-segment for varying α_i, respectively $(0 \le v_i^k < u_i^k \le \alpha_i^+,\ \ i = 1, \ldots, m)$.

Let a solution $z(\alpha^k)$ to Problem $(\mathcal{P}_{\alpha^k})$ be given, and assume that we have computed $\mathcal{V}_k := \mathcal{V}(\alpha^k)$.

α_k-bisection algorithm

Step 1. If $\mathcal{V}_k > 0$, then set $v^{k+1} := \alpha^k$, $\alpha^{k+1} := \frac{1}{2}(u^k + \alpha^k)$.

Step 2. If $\mathcal{V}_k < 0$, then set $u^{k+1} := \alpha^k$, $\alpha^{k+1} := \frac{1}{2}(v^k + \alpha^k)$.

Step 3. If $\mathcal{V}_k = 0$ and $\min_i \Phi_i(z(\alpha^k)) < 0$, then set

$$\alpha_i^{k+1} := \begin{cases} \dfrac{\psi_i(z(\alpha^k))}{\varphi_i(z(\alpha^k))} & \forall i :\ \Phi_i(z(\alpha^k)) < 0, \\ \alpha_i^k & \forall i :\ \Phi_i(z(\alpha^k)) \ge 0. \end{cases}$$

In addition, set $v^{k+1} := 0$, $u^{k+1} := t_{k+1}\alpha^{k+1}$, where $t_{k+1} = \dfrac{\alpha^+}{\max_i \alpha_i^k}$.

Stop: we computed the values α^{k+1}, v^{k+1}, and u^{k+1}.

Therefore, in order to verify the equality $\mathcal{V}(\alpha_0) = 0$, we should be able to find a global solution to Problem $(\mathcal{P}_\alpha)$ for every $\alpha \in I\!R^m_+$. Since $\psi_i(\cdot)$, $\varphi_i(\cdot)$, $i = 1, \ldots, m$, are simply convex or generally d.c. functions, it can readily be observed that

Problem $(\mathcal{P}_\alpha)$ belongs to the class of d.c. minimization. As a consequence, in order to solve Problem $(\mathcal{P}_\alpha)$, we can apply the Global Search Theory [26, 28].

Let us emphasize the fact that this method for solving Problem $(\mathcal{P}_f)$ of fractional optimization consists of 3 basic stages: the (a) local and (b) global search methods for Problem $(\mathcal{P}_\alpha)$ with a fixed vector parameter α and (c) the method of finding the vector parameter α at which the optimal value of Problem $(\mathcal{P}_\alpha)$ is zero, i.e. $\mathcal{V}(\alpha) = 0$.

In addition, to solve the fractional program as the d.c. constraints problem, we consider the following optimization problem with a nonconvex feasible set

$$(\mathcal{P}_c) \qquad \begin{cases} f_0 := \sum\limits_{i=1}^{m} \alpha_i \downarrow \min\limits_{(x,\alpha)}, \quad x \in S, \\ f_i := \psi_i(x) - \alpha_i \varphi_i(x) \leq 0, \quad i = 1, \ldots, m. \end{cases}$$

In [12] it was proved that for any solution $(x_*, \alpha_*) \in I\!R^n \times I\!R^m$ to Problem $(\mathcal{P}_c)$, the point x_* will be a solution to Problem $(\mathcal{P}_f)$. Therefore, we can solve Problem $(\mathcal{P}_c)$ using the exact penalization approach for d.c. optimization developed in [28] as well as the global search theory for solving the d.c. constraints problem [26].

It should be noted that, unlike solving Problem $(\mathcal{P}_f)$ by d.c. minimization, α_i will be found simultaneously with the solution vector x, because α_i are variables, although auxiliary ones, of Problem $(\mathcal{P}_c)$.

The results [14] of computational testing of the two approaches to a fractional programming problem suggest that we should combine them to solve fraction programs. For example, we can use the solution to Problem $(\mathcal{P}_c)$ to search for the parameter α that reduces the optimal value function of Problem $(\mathcal{P}_\alpha)$ to zero. This idea could be implemented by the following algorithm.

Two-component algorithm

Step 0. (Initialization) $k := 0$, $v^k := 0$, $u^k := \alpha^+$.

Step 1. Starting from feasible point (x^k, α_k), implement the local search method from [12] to find a critical point $(z^k, \hat{\alpha}_k)$ in d.c. constraints problem $(\mathcal{P}_c)$.

Step 2. (Stopping criterion) If $\mathcal{V}_k := \mathcal{V}(\hat{\alpha}^k) = 0$ and $\min\limits_i \Phi_i(z^k) \geq 0$, then STOP: $z^k \in Sol(\mathcal{P}_f)$.

Step 3. Starting from the critical point z^k, find a solution $z(\hat{\alpha}^k)$ to Problem $(\mathcal{P}_{\hat{\alpha}^k})$, using the global search strategy for d.c. minimization problem [26].

Step 4. (Stopping criterion) If $\mathcal{V}_k := \mathcal{V}(\hat{\alpha}^k) = 0$ and $\min\limits_i \Phi_i(z(\hat{\alpha}^k)) \geq 0$, then STOP: $z(\hat{\alpha}^k) \in Sol(\mathcal{P}_f)$.

Step 5. Set $x^{k+1} := z(\hat{\alpha}^k)$. Implement α_k-bisection algorithm to find new parameters α^{k+1}, v^{k+1} and u^{k+1}; $k := k + 1$ and go to Step 1.

Further, we describe in detail how we use the two-component algorithm to solve the average cost minimization problem provided by power companies of Mongolia.

3. IMPLEMENTATION ISSUES AND NUMERICAL RESULTS

The Mongolian electric energy system consists of five subsystems such as Central, Western, Altai, Dornod, and South. These subsystems provide almost entirely electric energy consumption of the country while the Central electric energy system produces 90% of the whole electric energy. In the Central energy system, Thermal power station 2 (TPS2), Darkhan power station, Thermal power station 3 (TPS3), Thermal power station 4 (TPS4), Erdenet power station, and Salkhit wind renewal energy station operate. Therefore, it is important to minimize average cost of the Central energy system. For this purpose, we consider the above six stations or power companies ($i = 1, \ldots, 6$).

In practice, the most popular forms of cost functions are cubic functions

$$\psi_i(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i, \tag{1}$$

where $a_i,\ b_i,\ c_i,\ d_i$ are parameters of functions $\psi_i(x_i),\ \ i = 1, \ldots, n$.

The parameters $a_i,\ b_i,\ c_i,\ d_i$ for modeling cost functions of the stations of Mongolian Central energy system are found by solving identification problems for all $i = 1, \ldots, 6$:

$$F_i(a, b, c, d) = \sum_{j=1}^{M} \left[a_i (x_i^j)^3 + b_i (x_i^j)^2 + c_i x_i^j + d_i - \psi_i^j \right]^2 \downarrow \min_{a_i, b_i, c_i, d_i}, \tag{2}$$

based on real industrial data employing the MATLAB, where M is the number of observations. As a result, the parameters are:

$$\begin{array}{l} a = (2.988595; 0.295274; 0.004967; -0.002076; -0.1775425; -0.315017), \\ b = (-6.711518; -1.654832; -0.107325; 0.145404; 0.4377455; 1.251723), \\ c = (5.536400; 3.014577; 1.951971; -1.94024; 1.124094; 0.158459), \\ d = (-0.599326; -0.289433; -0.834204; 23.081606; 0.0189135; 0.426788). \end{array} \tag{3}$$

The technological requirements for the variables are given by the box constraints Π:

$$\begin{array}{rcccrrcccr} 0.35 & \leq & x_1 & \leq & 1.31, & 20.56 & \leq & x_4 & \leq & 33.56, \\ 0.85 & \leq & x_2 & \leq & 3.17, & 0.54 & \leq & x_5 & \leq & 1.64, \\ 2.12 & \leq & x_3 & \leq & 9.23, & 0.61 & \leq & x_6 & \leq & 1.72. \end{array} \tag{4}$$

In the particular case of the applied Problem $(\mathcal{P})$, we should consider Problem $(\mathcal{P}_f)$ with one ratio, where $\sum\limits_{i=1}^{n} \psi_i(x_i)$ is the numerator and $\langle e, x \rangle = \sum\limits_{i=1}^{6} x_i$ is the denominator of the fraction. Thus, auxiliary Problem $(\mathcal{P}_\alpha)$ is formulated as follows

$$\sum_{i=1}^{6} \psi_i(x_i) - \alpha \sum_{i=1}^{6} x_i \downarrow \min_x, \quad x \in \Pi, \tag{5}$$

where $\alpha \in I\!R_+$ is the parameter.

In addition, Problem $(\mathcal{P}_c)$ is formulated as follows

$$\alpha \downarrow \min_{(x,\alpha)}, \quad x \in \Pi, \quad \sum_{i=1}^{6} \psi_i(x_i) - \alpha \sum_{i=1}^{6} x_i \le 0, \tag{6}$$

where $\alpha \in I\!R_+$ is the auxiliary variable.

Further, in order to solve problems (5) and (6), we need an explicit d.c. representation of the nonconvex functions (1) and the nonconvex term $\alpha \sum_{i=1}^{6} x_i = \langle \alpha e, x \rangle$ from (6), where $e = (1, \ldots, 1) \in I\!R^6$. We use the following clear formulas for the d.c. representation of nonconvex terms x_i^3, $-x_i^3$ and $\langle \alpha e, x \rangle$:

$$\begin{array}{c} x_i^3 = 0.25[(x_i+1)^4 - 4x_i - 1] - 0.25(x_i^4 + 6x_i^2) = p^{(1)}(x_i) - q^{(1)}(x_i); \\ -x_i^3 = 0.25(x_i^4 + 6x_i^2 + 4x_i + 1) - 0.25(x_i+1)^4 = p^{(2)}(x_i) - q^{(2)}(x_i); \\ \langle \alpha e, x \rangle = 0.25 \parallel \alpha e - x \parallel^2 - 0.25 \parallel \alpha e + x \parallel^2 = p^{(3)}(x) - q^{(3)}(x). \end{array}$$

Next, we can construct the d.c. functions as follows

$$\psi_i(x_i) = g_i(x_i) - h_i(x_i), \;\; i = 1, \ldots, 6,$$

where

$$\begin{array}{lll} g_i(x_i) = a_i p^{(1)}(x_i) + c_i x_i + d_i, & h_i(x_i) = a_i q^{(1)}(x_i) - b_i x_i^2, & i = 1,2,3; \\ g_i(x_i) = -a_i p^{(2)}(x_i) + b_i x_i^2 + c_i x_i + d_i, & h_i(x_i) = -a_i q^{(2)}(x_i), & i = 4,5,6. \end{array}$$

As a result, the goal function for Problem (5) has the form

$$\sum_{i=1}^{6} \psi_i(x_i) - \alpha \sum_{i=1}^{6} x_i = G_\alpha(x) - H_\alpha(x),$$

where $G(x) = \sum_{i=1}^{6} g_i(x_i) - \alpha \sum_{i=1}^{6} x_i$, $\;H(x) = \sum_{i=1}^{6} h_i(x_i)$, and the constraint for Problem (6) has the form

$$\sum_{i=1}^{6} \psi_i(x_i) - \alpha \sum_{i=1}^{6} x_i = G_c(x,\alpha) - H_c(x,\alpha) \le 0,$$

where $G_c(x,\alpha) = \sum_{i=1}^{6} g_i(x_i) + q^{(3)}(x)$, $\;H_c(x,\alpha) = \sum_{i=1}^{6} h_i(x_i) + p^{(3)}(x)$.

Further, according to the two-component algorithm from Sect. 2, we should implement the local search method from [12] to find a critical point in the d.c. constraints Problem $(\mathcal{P}_c)$. As it is known, it consists of applying a classical idea of linearization with respect to the basic nonconvexity of Problem $(\mathcal{P}_c)$ (i.e. linearization of (6) with respect to $H_c(x,\alpha)$) at the point (x^s, α_s). Thus, we obtain the following linearized problem:

$$\left.\begin{array}{c} \alpha \downarrow \min\limits_{(x,\alpha)}, \quad x \in \Pi, \quad \alpha \in I\!R_+, \\ G_c(x,\alpha) - \langle \nabla H_c(x^s,\alpha_s), (x,\alpha) - (x^s,\alpha_s) \rangle - H_c(x^s,\alpha_s) \le 0. \end{array}\right\} \tag{7}$$

It is easy to see that the local search algorithm constructed in this way provides critical points by employing only tools of the convex analysis.

Next, in Step 3 of the two-component algorithm for solving the fraction program $(\mathcal{P})$, we should find a solution $z(\alpha^k)$ to Problem $(\mathcal{P}_{\alpha^k})$ using the global search strategy for d.c. minimization [26], i.e. we should implement a procedure of escaping from the critical point found by the local search.

If the stopping criterion is satisfied, then the algorithm has found the solution to Problem $(\mathcal{P})$. Otherwise, we should implement the α_k-bisection algorithm to find the new parameter α^{k+1}.

The solutions to Problem $(\mathcal{P})$ representing optimal quantity of electricity found by the two-component algorithm are $x_1^* = 1.031, x_2^* = 3.082, x_3^* = 2.120, x_4^* = 20.560, x_5^* = 0.540, x_6^* = 0.610$, respectively.

The cost functions $\psi_1(x_1^*) = 1.249860003, \psi_2(x_2^*) = 1.926867122, \psi_3(x_3^*) = 2.868939252, \psi_4(x_4^*) = 26.6120213, \psi_5(x_5^*) = 0.725614296, \psi_6(x_6^*) = 0.917711245$ correspond to the power companies TPS2, Darkhan, TPS3, TPS4, Erdenet, and Salkhit, resptectively. The minimum total average cost is $f(x^*) = 1.227535$.

The optimal solutions show that in order to minimize the total average cost, the power companies do not necessarily have to produce their products at the maximum level but it is sufficient to produce at levels $x_1^*, \ldots, x_6^*$.

4. CONCLUSIONS

We formulated the average cost minimization problem for power companies of Mongolia as a fractional program. Based on the global optimality conditions for d.c. programming, we proposed a two-component algorithm for solving this problem. The algorithm can be applied to solving a general fractional program.

Acknowledgement: This work has been fully supported by the Russian Science Foundation, Project N 15-11-20015, except of collecting and processing the real industrial data, which were carried out by Tungalag N. (supported by the research grant P2017-2455 of the National University of Mongolia).

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