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ASYMPTOTIC RESULTS FOR THE FIRST AND SECOND MOMENTS AND NUMERICAL COMPUTATIONS IN DISCRETE-TIME BULK-RENEWAL PROCESS

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Abstract: This paper introduces a simplified solution to determine the asymptotic results for the renewal density. It also offers the asymptotic results for the first and second moments of the number of renewals for the discrete-time bulk-renewal process. The methodology adopted makes this study distinguishable compared to those previously published where the constant term in the second moment is generated. In similar studies published in the literature, the constant term is either missing or not clear how it was obtained. The problem was partially solved in the study by Chaudhry and Fisher where they provided a asymptotic results for the non-bulk renewal density and for both the first and second moments using the generating functions. The objective of this work is to extend their results to the bulk-renewal process in discrete-time, including some numerical results, give an elegant derivation of the asymptotic results and derive continuous-time results as a limit of the discrete-time results.

Keywords: Renewal Theory, Discrete-time, Bulk-renewal Process, Generating Function, Asymptotic Results.

MSC: 60K05, 62E20, 60K25.

1. INTRODUCTION

Renewal theory and its applications have a significant role in many different areas such as failure and replacement of equipment, risk-based asset management models and queues [8]. The asymptotic results for the first and second moments for the number of renewals in the non-bulk case are given in recent study by Van der Weide in [7]. This result provides a constant term in the second moment and states that it is not clear from Feller [4] as to how to obtain the constant term using generating functions. The same problem persists in [5] and [6]. Recently, Chaudhry and Fisher [1] have responded to this problem by providing the asymptotic results for the non-bulk renewal density as well as for both the first and second moments using generating functions. The purpose of this note is to extend their results to bulk-renewal process in discrete-time and give an elegant derivation of the asymptotic results. Some easy steps could have been avoided, but are included here for the sake of clarity. Numerical computations of both single-renewal and bulk-renewal processes are provided in order to demonstrate the accuracy of asymptotic results. This is done by comparing the analytic, numeric and asymptotic moments at various renewal times in order to provide readers with better understanding of our findings.

2. RENEWAL THEORY BASICS

A discrete-renewal process is a process $\{N_m,\ \ m \geq 1\}$ for which the state space belongs to a denumerable set $\{0, 1, 2, \ldots\}$. N_m can count the number of renewals within a time period $(0, m]$, and the time intervals between renewals are called renewal periods. Renewals occur at instants of time $s_1, s_2, s_3, \ldots,$ and renewal intervals $T_n = s_n - s_{n-1}$, $n \geq 1$, and $s_0 = 0$ are independent identically distributed random variables (i.i.d.r.v.s) distributed as T with probability mass function (p.m.f.)$f_k = P(T = k), k \geq 1, f_0 = 0$. This p.m.f. has a probability generating function (p.g.f) $f(z) = \sum_{k=1}^{\infty} f_k z^k,\ |z| < 1$ with $\mu \equiv \mu_1 = E[T] < \infty$, $\sigma^2 = E\left[T^2\right] - E^2[T] < \infty$, $a_n = \frac{d^n}{dz^n} f(z)|_{z=1},\ n \geq 1$ and $\mu_n = E[T^n],\ n \geq 1$. If W_n is the total waiting time until the n-th renewal occurs, then $W_n = \sum_{r=1}^{n} T_r$ with $W_0 = 0$. The renewal equation is defined as $\mathrm{m}_k = f_k + \sum_{j=1}^{k} m_{k-j} f_j$where $m_k = P(renewalattimek)$ with $m_1 = f_1$ and $m_0 = 0$ (implying no renewal at time 0). The left-hand side of the renewal equation is the probability of a renewal taking place at time k while the right-hand side is either a first renewal occurring at time k or a renewal occurring at time $j \geq 1$ with probability f_j and a subsequent renewal at time $(k-j)$ with probability m_{k-j}. The generating function (gf) for the renewal density is $m(z) = \sum_{k=1}^{\infty} m_k z^k = \frac{f(z)}{1-f(z)},\ (|z| < 1)$. The mean value of the discrete-time renewal process $\{N_m\}$ is referred to as the renewal function and is defined as $M_m \equiv E[N_m],\ (m \geq 1)$. A great portion of renewal theory is concerned with properties of the renewal function, and it is for this reason that its asymptotic results are of such great interest.

3. BULK RENEWAL PROCESS

Assume that the group of renewals occurs at time $s_1, s_2, \ldots$ with group size X_i, where X_i are independent and identically distributed random variables (i.i.d.r.v's) distributed as X with $P_X(z) = E\left[z^X\right] = \sum_{n=1}^{\infty} b_n z^n$, $\mu_X = P_X'(1)$ and $P_X''(1) = \frac{d^2}{dz^2} P_X(z)|_{z=1}$. If N_m is the number of groups arriving in the time interval $(0, m]$, then the total number of renewals is $Y_{N_m} = \sum_{i=1}^{N_m} X_i$ with pmf $B_n(m) = P(Y_{N_m} = n)$, $(n = 0, 1, 2, \ldots)$. Since Y_{N_m} is a random variable based on two parameters (n, m), we first take a gf with respect to n, such that $\mathrm{P}(z,m) = E\left[z^{Y_{N_m}}\right] = E\left[E\left[z^{\sum_{i=1}^{N_m} X_i}|N_m\right]\right] = \sum_{n=0}^{\infty} E\left[z^{\sum_{i=1}^{N_m} X_i}\middle|N_m = n\right] P_n(m)$
$= \sum_{n=0}^{\infty} (P_X(z))^n P_n(m)$, $(|z| < 1,\ m \geq 1)$ 1 Equation (1) reduces to a single-renewal process if $P_X(z) = z$, where Y_{N_m} becomes N_m. Given (1), the generating function (gf) of $P(z,m)$ with respect to m is given by $\mathrm{P}(z,v) = \sum_{m=1}^{\infty} P(z,m)v^m = \sum_{m=1}^{\infty} \{\sum_{n=0}^{\infty} (P_X(z))^n P_n(m)\} v^m$
$= \sum_{n=0}^{\infty} (P_X(z))^n \sum_{m=1}^{\infty} P_n(m) v^m$
$= \sum_{n=0}^{\infty} \frac{1-f(v)}{1-v} (P_X(z) f(v))^n$
$= \frac{1-f(v)}{(1-v)(1-P_X(z)f(v))}$, $(|z| < 1, |v| < 1)$ 2 where we have used $\sum_{m=1}^{\infty} P_n(m) v^m = \frac{f^n(v)(1-f(v))}{1-v}$ (see [5]) for further details).

4. FIRST MOMENT OF Y_{N_m}

If $M_m = E[Y_{N_m}]$, then the generating function (gf) of the first moment is given by
$\mathrm{M}(v) \equiv \sum_{m=1}^{\infty} E[Y_{N_m}] v^m = \sum_{m=1}^{\infty} M_m v^m = \frac{d}{dz} P(z,v)|_{z=1} = \frac{f(v)}{(1-v)(1-f(v))} \mu_X$, $(|v| < 1)$

Assuming that the renewal event is aperiodic recurrent with $\sigma < \infty$ and $\mu_X < \infty$, we now want to show that $\mathrm{M}_m = \left(\frac{m}{\mu}\right) \mu_x + \mu_x \left(\frac{\sigma^2 - \mu^2 + \mu}{2\mu^2}\right) + o(1)$is true, where $o(1) \to 0$ as $m \to \infty$.

Proof:

In the recurrent case $f(1) = 1$, following the procedure similar to the one used in [3] for the continuous-time, we have $\mathrm{M}(v) = \frac{C_{-2}}{(1-v)^2} + \frac{C_{-1}}{(1-v)} + O(1)$ 3 leading to $\mathrm{M}_m = (m+1) C_{-2} + C_{-1} + o(1)$ 4 with $O(1)$ indicating a function of v bounded as $v \to 1^-$ and $o(1)$ indicating a function of m tending to zero as $m \to \infty$. From equation (3), we get
$\mathrm{C}_{-2} = \lim_{v \to 1^-} (1-v)^2 M(v) = \lim_{v \to 1^-} (1-v)^2 \frac{f(v)}{(1-v)(1-f(v))}$
$P_X'(1) = \frac{P_X'(1)}{f'(1)} = \frac{\mu_X}{\mu}$and $\mathrm{C}_{-1} = \lim_{v \to 1^-} \left\{(1-v) \frac{f(v)P_X'(1)}{(1-v)(1-f(v))} - \frac{P_X'(1)}{\mu(1-v)}\right\} = P_X'(1) \lim_{v \to 1^-} \frac{\mu f(v)(1-v) - (1-f(v))}{\mu(1-v)(1-f(v))}$
$= P_X'(1) \left(\frac{f''(1)}{2\mu^2} - 1\right) = \mu_X \left(\frac{\sigma^2 + \mu^2 - \mu}{2\mu^2} - 1\right)$Substituting C_{-1} and C_{-2} into equation (4) gives $\mathrm{M}_m = \left(\frac{m}{\mu}\right) \mu_x + \mu_x \left(\frac{\sigma^2 - \mu^2 + \mu}{2\mu^2}\right) + o(1)$where $o(1) \to 0$ as $m \to \infty$.

In the case of single-arrivals ($P_X(z) = z,\ \mu_X = 1$), the above result corresponds to Feller [5] and Hunter [6]. The above result leads to the well-known result, $\lim_{m\to\infty} \frac{M_m}{m} = \frac{\mu_X}{\mu}$, which gives the arrival rate for the discrete-time bulk-arrival renewal process. Further, it is interesting to see that the first asymptotic moment of continuous-time bulk-renewal process discussed in [2] can also be derived if we let $\mu = \frac{\hat{\mu}}{\Delta}$, $\sigma^2 = (\frac{\hat{\sigma}}{\Delta})^2$ and $m = \frac{t}{\Delta}$, and then take the limit of M_m as $\Delta \to 0$, where $\hat{\mu}$, $\hat{\sigma}$, and t are the parameters for the continuous time process. By doing so, M_m becomes $\mathrm{M(t)}=\left(\frac{t}{\hat{\mu}}\right)\mu_x + \mu_x\left(\frac{\hat{\sigma}^2-\hat{\mu}^2}{2\hat{\mu}^2}\right) + o(1)$where $o(1) \to 0$ as $t \to \infty$ and $t > 0$.

5. SECOND MOMENT OF Y_{N_m}

If $M_m^{(2)} = E[Y_{N_m}^2]$, its probability generating function (pgf) $M^{(2)}(v)$ can be expressed in terms of first and second derivatives of equation (2) at $z = 1$, in other words $\mathrm{M}^{(2)}(v) \equiv \sum_{m=1}^{\infty} M_m^{(2)} v^m = \frac{d^2}{dz^2}P(z,v)\,|_{z=1} + \frac{d}{dz}P(z,v)\,|_{z=1}$
$= \frac{f(v)}{(1-v)(1-f(v))}\left(\frac{2f(v)\mu_X^2+\mu_X-\mu_X f(v)}{1-f(v)} + P_X''(1)\right)$Assuming that the renewal event is aperiodic recurrent with $\mu_3 < \infty$ and $\frac{d^2}{dz^2}P_X(z)\,|_{z=1} < \infty$, we now want to show that $\mathrm{M}_m^{(2)} = m^2\left(\frac{\mu_x}{\mu}\right)^2 + m\left(\frac{\mu_x^2}{\mu^2} + \frac{\mu_x}{\mu} - \frac{2\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{2\mu_x^2\sigma^2}{\mu^3}\right)$
$+ 2\mu_x^2 - \mu_x - P_X''(1) + \frac{8\mu_x^2}{3\mu^2} - \frac{2\mu_x^2\mu_3}{3\mu^3} + \frac{\mu_x}{\mu} - \frac{4\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu}$
$+ \frac{P_X''(1)\sigma^2}{2\mu^2} - \frac{2(\sigma\mu_x)^2}{\mu^2} + \frac{\mu_x\sigma^2}{2\mu^2} + \frac{4(\sigma\mu_x)^2}{\mu^3} + \frac{3\mu_x^2\sigma^4}{\mu^4} + o(1)$where $o(1) \to 0$ as $m \to \infty$.

Proof:

Now we make similar assumptions as we did in the case of first moment in order to find the asymptotic result for the second moment. $\mathrm{M}^{(2)}(v) = \frac{C_{-3}}{(1-v)^3} + \frac{C_{-2}}{(1-v)^2} + \frac{C_{-1}}{(1-v)} + O(1)$ 5 leading to $\mathrm{M}_m^2 = E\left[Y_{N_m}^2\right] = \frac{(m+2)!}{2!m!}C_{-3} + (m+1)C_{-2} + C_{-1} + o(1)$ 6 All the constant terms can be found in a manner as we did for the first moment. $\mathrm{C}_{-3} = \lim_{v\to 1^-}(1-v)^3 M^{(2)}(v) = \frac{P_X'(1)}{\mu}\lim_{v\to 1^-}\frac{2(1-v)f(v)P_X'(1)+(1-v)(1-f(v))}{1-f(v)}$
$= \frac{P_X'(1)}{\mu}\left(\frac{2P_X'(1)}{f'(1)}\right) = 2\left(\frac{\mu_x}{\mu}\right)^2$ $\mathrm{C}_{-2} = \lim_{v\to 1^-}\left\{(1-v)^2 M^{(2)}(v) - \frac{C_{-3}}{(1-v)}\right\}$
$= 6\lim_{v\to 1^-}\left\{(1-v)^2 M^{(2)}(v) - \frac{2\left(P_X'(1)\right)^2}{\mu^2(1-v)}\right\}$
$= \lim_{v\to 1^-}\left\{\begin{array}{c}\frac{(1-v)f(v)(P_X'(1)-P_X'(1)f(v)+2f(v)\left(P_X'(1)\right)^2)}{(1-f(v))^2} \\ +\frac{(1-v)^2 f(v)P_X''(1)}{(1-v)(1-f(v))} - \frac{2\left(P_X'(1)\right)^2}{\mu^2(1-v)}\end{array}\right\}$
$= \lim_{v\to 1^-}\frac{f_1(x)}{(1-f(v))^2(1-v)(1-f(v))\mu^2(1-v)},$

$\mathrm{f}_1(x) = (1-v)\,f(v)(P_X'(1) - P_X'(1)\,f(v)$
$+2\mathrm{f(v)P}_X'(1)^2)\,(1-v)\,(1-f(v))\,\mu^2\,(1-v)$

+(1-v)$^2 f(v)P''_X(1)(1-f(v))^2\mu^2(1-v)$
-2P$'_X(1)^2(1-f(v))^2(1-v)(1-f(v))$

by applying L'Hospital's rule, we have $C_2 = \frac{\mu^2\mu_x - 4\mu_x^2\mu^2 + 2\mu_x^2 a_2 + \mu^2 P''_X(1)}{\mu^3}$ and

$$C_{-1} = \lim_{v\to 1^-}\left\{(1-v)M^{(2)}(v) - \frac{C_{-3}}{(1-v)^2} - \frac{C_{-2}}{(1-v)}\right\}$$

$$= \lim_{v\to 1^-}\left\{\begin{array}{c}\frac{f(v)(P'_X(1)-P'_X(1)f(v)+2f(v)P'_X(1)^2)(1-v)}{(1-v)(1-f(v))^2} + \frac{P''_X(1)f(v)(1-f(v))^2}{(1-f(v))^3} \\ -\frac{2(P'_X(1)^2-f(v)P'_X(1)^2)(1-f(v))}{\mu^2(1-v)^2(1-f(v))^2} - \frac{\mu^2P'_X(1)-4P'_X(1)^2\mu^2+2P'_X(1)^2a_2+\mu^2P''_X(1)}{\mu^3(1-v)}\end{array}\right\}$$

$$= \lim_{v\to 1^-}\left\{\frac{f_2(x)}{(1-v)(1-f(v))^2(1-f(v))^3\mu^2(1-v)^2(1-f(v))^2\mu^3(1-v)}\right\},$$

$$\begin{aligned} f_2(x) = {} & f(v)(P'_X(1) - P'_X(1)f(v) + 2f(v)P'_X(1)^2)(1-v)^4(1-f(v))^5\mu^5 \\ & + P''_X(1)f(v)(1-f(v))^6\mu^2(1-v)^4\mu^3 \\ & -2(P'_X(1)^2 - f(v)P'_X(1)^2)(1-v)^2(1-f(v))^3\mu^3. \end{aligned}$$

By applying L'Hopital's rule, we have $C_{-1} = \frac{3\mu_x^2}{2} - \frac{\mu_x}{2} - \frac{P''_X(1)}{2} + \frac{\mu_x^2}{6\mu^2} - \frac{2\mu_x^2(a_3+3a_2+\mu)}{3\mu^3} + \frac{\mu_x}{2\mu} - \frac{\mu_x^2}{\mu}$
$+ \frac{P''_X(1)}{2\mu} + \frac{P''_X(1)(a_2+\mu-\mu^2)+2\mu_x^2(a_2+\mu-\mu^2)+\mu_x(a_2+\mu-\mu^2)}{2\mu^2}$
$+ \frac{\mu_x^2(a_2+\mu-\mu^2)}{\mu^3} + \frac{3\mu_x^2(a_2^2+2a_2\mu-2a_2\mu^2+\mu^2-2\mu^3+\mu^4)}{2\mu^4}$ Substituting C_{-1}, C_{-2} and C_{-3} into equation (6) with $a_3 \equiv E[(T-2)(T-1)T] = E[T^3] - 3E[T^2] + 2E[T] = \mu_3 - 3\mu_2 + 2\mu$ and $a_2 = E[(T-1)T] = E[T^2] - E[T] = \mu_2 - \mu$, we have

$M_m^{(2)} = m^2\left(\frac{\mu_x}{\mu}\right)^2 + m\left(\frac{\mu_x^2}{\mu^2} + \frac{\mu_x}{\mu} - \frac{2\mu_x^2}{\mu} + \frac{P''_X(1)}{\mu} + \frac{2\mu_x^2\sigma^2}{\mu^3}\right)$
$+ \Big(2\mu_x^2 - \mu_x - P''_X(1) + \frac{\mu_x^2}{6\mu^2} - \frac{2\mu_x^2\mu_3}{3\mu^3} + \frac{\mu_x}{2\mu} - \frac{2\mu_x^2}{\mu} + \frac{P''_X(1)}{2\mu}$
$+ \frac{P''_X(1)\mu_2}{2\mu^2} - \frac{2\mu_2\mu_x^2}{\mu^2} + \frac{\mu_2\mu_x}{2\mu^2} + \frac{\mu_2\mu_x^2}{\mu^3} + \frac{3(\mu_2\mu_x)^2}{2\mu^4}\Big) + o(1)$ and by substituting $\mu_2 = \sigma^2 + \mu$, the final expression is $M_m^{(2)} = m^2\left(\frac{\mu_x}{\mu}\right)^2 + m\left(\frac{\mu_x^2}{\mu^2} + \frac{\mu_x}{\mu} - \frac{2\mu_x^2}{\mu} + \frac{P''_X(1)}{\mu} + \frac{2\mu_x^2\sigma^2}{\mu^3}\right)$
$+ \Big(2\mu_x^2 - \mu_x - P''_X(1) + \frac{8\mu_x^2}{3\mu^2} - \frac{2\mu_x^2\mu_3}{3\mu^3} + \frac{\mu_x}{\mu} - \frac{4\mu_x^2}{\mu} + \frac{P''_X(1)}{\mu}$
$+ \frac{P''_X(1)\sigma^2}{2\mu^2} - \frac{2(\sigma\mu_x)^2}{\mu^2} + \frac{\mu_x\sigma^2}{2\mu^2} + \frac{4(\sigma\mu_x)^2}{\mu^3} + \frac{3\mu_x^2\sigma^4}{\mu^4}\Big) + o(1)$ The first two terms of the above expression correspond to Feller [5] and Hunter [6] when $P_X(z) = z$. However, this paper provides extra constant terms in addition to the first two terms. This result matches with that given in [1] if $P_X(z) = z$. Similar to the first moment, $\lim_{m\to\infty}\frac{M_m^{(2)}}{m^2} = \left(\frac{\mu_X}{\mu}\right)^2$ gives the 2nd moment of the number of renewals in discrete-time bulk-arrivals. Further, the second asymptotic moment of continuous-time bulk-renewal process discussed in [2] can be derived if we let $\mu = \frac{\hat{\mu}}{\Delta}$, $\sigma^2 = (\frac{\hat{\sigma}}{\Delta})$,

$\mu_3 = \frac{\hat{\mu}_3}{\Delta^3}$ and $m = \frac{t}{\Delta}$, and then take the limit of $M_m^{(2)}$ as $\Delta \to 0$. By doing so, $M_m^{(2)}$ becomes $\mathrm{M}^{(2)}(t) = t^2 \left(\frac{\mu_x}{\hat{\mu}}\right)^2 + t\left(\frac{P''_X(1) - 2\mu_x^2 + \mu_x}{\hat{\mu}} + \frac{2\hat{\sigma}^2\mu_x^2}{\hat{\mu}^3}\right)$ $+\left(\frac{\hat{\sigma}^2 P''_X(1)}{2\hat{\mu}^2} + \frac{\hat{\sigma}^2\mu_x}{2\hat{\mu}^2} - \frac{\mu_x}{2} - \frac{2\hat{\mu}_3\mu_x^2}{3\hat{\mu}^3} + \frac{3\hat{\sigma}^4\mu_x^2}{2\hat{\mu}^4} + \frac{\hat{\sigma}^2\mu_x^2}{\hat{\mu}^2} + \frac{3\mu_x^2}{2} - \frac{P''_X(1)}{2}\right) + o(1)$where $o(1) \to 0$ as $t \to \infty$ and $t > 0$.

6. NUMERICAL COMPUTATIONS

In Section 6.1, we first compute the distribution of a single-renewal process $(P_n(m))$, which are used in Section 6.2 to compute the distribution of a bulk-renewal process $(B_n(m))$.

6.1. Numerical computations in discrete-time single-renewal process

$P_n(m)$ is the probability mass function (pmf) of the number of renewals (N_m) that occur over the time period $(0, m]$. In computing $P_n(m)$, we consider various inter-renewal times such as geometric, negative binomial, and Poisson distributions. All computations were done using MAPLE software, calibrated to compute up to ninth decimal place. In presenting our numerical work, all numerical results were rounded to four decimal places in the tables below

The inter-renewal time (k) has a probability mass function (pmf), f_k, which follows a geometric distribution such that $f_k = pq^{k-1}$, $(k \geq 1)$ with probability generating function (pgf) $f(v) = \frac{pv}{(1-qv)}$, $|v| < 1$ and $p = 0.3$, $q = 0.7$. $P_n(m)$ is computed at $m = 1,\ 5,\ 10,\ 15,\ 20$ and $0 \leq n \leq 6$.

Table 1: Geometric arrival pattern

m	$P_0(m)$	$P_1(m)$	$P_2(m)$	$P_3(m)$	$P_4(m)$	$P_5(m)$	$P_6(m)$	...	$E[N_m]$	$E[N_m^2]$
1	0.7000	0.3000	0.0000	0.0000	0.0000	0.0000	0.0000	...	0.3000	0.3000
5	0.1681	0.3602	0.3087	0.1323	0.0284	0.0024	0.0000	...	1.5000	3.3000
10	0.0283	0.1211	0.2335	0.2668	0.2001	0.1029	0.0368	...	3.0000	11.1000
15	0.0048	0.0305	0.0916	0.1700	0.2186	0.2061	0.1472	...	4.5000	23.4000
20	0.0008	0.0068	0.0279	0.0716	0.1304	0.1789	0.1916	...	6.0000	40.2000

The inter-renewal time (k) has a probability mass function (pmf), f_k, which follows a negative binomial distribution such that $f_k = (\,k + r - 2k - 1)\,p^r q^{k-1}$, $(k \geq 1)$ with probability generating function (pgf) $f(v) = v\left(\frac{p}{1-qv}\right)^r$, $|v| < 1$ and $p = 0.75$, $q = 0.25$ and $r = 13$. $P_n(m)$ is computed at $m = 1,\ 10,\ 20,\ 30$ and $0 \leq n \leq 6$.

Table 2: Negative binomial arrival pattern

m	$P_0(m)$	$P_1(m)$	$P_2(m)$	$P_3(m)$	$P_4(m)$	$P_5(m)$	$P_6(m)$	...	$E[N_m]$	$E[N_m^2]$
1	0.9762	0.0238	0.0000	0.0000	0.0000	0.0000	0.0000	...	0.0238	0.0238
10	0.0295	0.4571	0.4317	0.0772	0.0045	0.0001	$1.4277x10^{-6}$	...	1.5703	2.9526
20	$7.9845x10^{-6}$	0.0064	0.1343	0.4058	0.3328	0.1041	0.0154	...	3.4453	12.7406
30	$7.4214x10^{-11}$	$6.0263x10^{-6}$	0.0015	0.0354	0.1929	0.3535	0.2779	...	5.3203	29.5570

The inter-renewal time (k) has a probability mass function (pmf), f_k, which follows a Poisson distribution such that $f_k = \frac{\alpha^{k-1}}{(k-1)!}e^{-\alpha},\ (k \geq 1)$ with probability generating function (pgf) $f(v) = ve^{-\alpha(1-v)},\ |v| < 1$, where $\alpha = 2$. $P_n(m)$ is computed at $m = 1,\ 5,\ 10,\ 15$ and $n = 0,\ 1,\ 2,\ 3,\ 4$.

Table 3: Poisson arrival pattern

m	$P_0(m)$	$P_1(m)$	$P_2(m)$	$P_3(m)$	$P_4(m)$	...	$E[N_m]$	$E[N_m^2]$
1	0.8647	0.1353	0.0000	0.0000	0.0000	...	0.1353	0.1353
5	0.0527	0.5139	0.3715	0.0590	0.0030	...	1.4459	2.5791
10	$4.6498x10^{-5}$	0.02	13 0.2347	0.4306	0.2463	...	3.1111	10.5432
15	$4.2000x10^{-9}$	$7.6325x10^{-5}$	0.0088	0.1031	0.3050	...	4.7778	24.0617

6.2. Numerical computations in discrete-time bulk-renewal process

$B_n(m)$ is the probability mass function (pmf) of the total number of renewals (Y_{N_m}) that occur over the time period $(0, m]$. In computing $B_n(m)$, we consider the same inter-renewal time distributions used in Section 6.1., while incorporating different batch size distributions such as binomial and 1-3-6-9 distributions (renewals occur in a group of size 1, 3, 6, and 9 with corresponding probabilities). Moreover, we find the first and second moments of Y_{N_m} using three different approaches; analytically, asymptotically, and numerically. These are then compared at various values of m. Analytic moments $\left(M_{analytic} \text{ and } M^{(2)}_{analytic}\right)$ are determined directly from the inversion of the equations $\mathrm{M}(v) = \frac{f(v)}{(1-v)(1-f(v))}\mu_X$ $M^{(2)}(v) = \frac{f(v)}{(1-v)(1-f(v))}\left(\frac{2f(v)\mu_X^2 + \mu_X - \mu_X f(v)}{1-f(v)} + P''_X(1)\right)$The asymptotic moments $\left(M_{asymptotic}\ and\ M^{(2)}_{asymptotic}\right)$ are computed using the derived results in Sections 4 and 5. The numeric moments $\left(M_{numeric}\ and\ M^{(2)}_{numeric}\right)$ are computed from $B_n(m)$ found from the coefficients of Taylor's series expansion of the expression $\mathrm{E}\left[z^{Y_m}\right] = \sum_{n=0}^{\infty} B_n(m)\, z^n = \sum_{n=0}^{\infty} (P_X(z))^n\, P_n(m)$where $P_n(m)$ are provided in Tables 1, 2, and 3 of Section 6.1.

6.2.1. Binomial group size distribution

The probability mass function (pmf) of the group size (X) follows a binomial distribution
$b_n = (rn - 1)\, p^n q^{r-n+1},\ (1 \leq n \leq 4)$ with probability generating function (pgf) $P_X(z) = z\,(q + pz)^r$ where $p = 0.45$, $q = 0.55$ and $r = 3$.

6.2.2. 1-3-6-9 group size distribution

The probability mass function (pmf) of the group size (X) is $b_1 = 0.1,\ b_3 = 0.25,\ b_6 = 0.45,\ b_9 = 0.2$ with probability generating function (pgf) $P_X(z) = 0.1z + 0.25z^3 + 0.45z^6 + 0.2z^9$.

Table 4: Geometric arrival pattern

m	$B_0(m)$	$B_1(m)$	$B_2(m)$	$B_3(m)$	$B_4(m)$	$B_5(m)$	$B_6(m)$	...
1	0.7000	0.0499	0.1225	0.1002	0.0273	0.0000	0.0000	...
5	0.1681	0.0599	0.1556	0.1629	0.1231	0.1085	0.0864	...
10	0.0283	0.0201	0.0559	0.0734	0.0851	0.1019	0.1069	...
15	0.0048	0.0051	0.0150	0.0234	0.0342	0.0483	0.0608	...
20	0.0008	0.0011	0.0036	0.0064	0.0109	0.0174	0.0251	...

m	M_{analytic}	$M_{\text{asymptotic}}$	M_{numeric}	$M^{(2)}_{\text{analytic}}$	$M^{(2)}_{\text{asymptotic}}$	$M^{(2)}_{\text{numeric}}$
1	0.7050	0.7050	0.7050	1.8795	1.8795	1.8796
5	3.5250	3.5250	3.5250	19.3380	19.3380	19.3376
10	7.0500	7.0500	7.0500	63.5273	63.5273	63.5273
15	10.5750	10.5750	10.5750	132.5678	132.5678	132.5678
20	14.1000	14.1000	14.1000	226.4595	226.4595	226.4595

Table 5: Negative binomial arrival pattern

m	$B_0(m)$	$B_1(m)$	$B_2(m)$	$B_3(m)$	$B_4(m)$	$B_5(m)$	$B_6(m)$	...
1	0.9762	0.0040	0.0097	0.0079	0.0022	0.0000	0.0000	...
2	0.8990	0.0167	0.0410	0.0336	0.0093	0.0002	0.0001	...
3	0.7639	0.0386	0.0948	0.0780	0.0223	0.0013	0.0008	...
4	0.5950	0.0646	0.1591	0.1320	0.0400	0.0050	0.0031	...
5	0.4261	0.0879	0.2170	0.1826	0.0606	0.0136	0.0085	...

m	M_{analytic}	$M_{\text{asymptotic}}$	M_{numeric}	$M^{(2)}_{\text{analytic}}$	$M^{(2)}_{\text{asymptotic}}$	$M^{(2)}_{\text{numeric}}$
1	0.0558	N/A	0.0558	0.1488	0.7995	0.1488
2	0.2386	0.1652	0.2386	0.6423	1.1005	0.6423
3	0.5648	0.6059	0.5648	1.5528	1.7898	1.5528
4	0.9911	1.0465	0.9911	2.8290	2.8674	2.8290
5	1.4575	1.4871	1.4575	4.4059	4.3333	4.4059

Table 6: Poisson arrival pattern

m	$B_0(m)$	$B_1(m)$	$B_2(m)$	$B_3(m)$	$B_4(m)$	$B_5(m)$	$B_6(m)$	...
1	0.8647	0.0225	0.0553	0.0452	0.0123	0.0000	0.0000	...
5	0.0527	0.0855	0.2201	0.2225	0.1521	0.1192	0.0817	...
10	$4.6498x10^{-5}$	0.0036	0.0152	0.0410	0.0820	0.1208	0.1433	...
15	$4.1957x10^{-9}$	$1.2699x10^{-5}$	0.0003	0.0017	0.0062	0.0164	0.0343	...

m	M_{analytic}	$M_{\text{asymptotic}}$	M_{numeric}	$M^{(2)}_{\text{analytic}}$	$M^{(2)}_{\text{asymptotic}}$	$M^{(2)}_{\text{numeric}}$
1	0.3180	0.2611	0.3181	0.8479	1.2415	0.8481
5	3.3978	3.3945	3.3979	15.3169	15.3219	15.3170
10	7.3111	7.3111	7.3111	60.5347	60.5349	60.5347
15	11.2278	11.2278	11.2278	136.4284	136.4284	136.4284

Table 7: Geometric arrival pattern

m	$B_0(m)$	$B_1(m)$	$B_2(m)$	$B_3(m)$	$B_4(m)$	$B_5(m)$	$B_6(m)$	...
1	0.7000	0.0300	0.0000	0.0750	0.0000	0.0000	0.1350	...
5	0.1681	0.0360	0.0031	0.0902	0.0154	0.0010	0.1814	...
10	0.0282	0.0121	0.0023	0.0305	0.0117	0.0020	0.0693	...
15	0.0048	0.0031	0.0009	0.0078	0.0046	0.0013	0.0197	...
20	0.0008	0.0007	0.0003	0.0018	0.0014	0.0005	0.0050	...

m	M_{analytic}	$M_{\text{asymptotic}}$	M_{numeric}	$M^{(2)}_{\text{analytic}}$	$M^{(2)}_{\text{asymptotic}}$	$M^{(2)}_{\text{numeric}}$
1	1.6050	1.6050	1.6050	10.4250	10.4250	10.4250
5	8.0250	8.0250	8.0250	103.6455	103.6455	103.6455
10	16.0500	16.0500	16.0500	336.0923	336.0923	336.0923
15	24.0750	24.0750	24.0750	697.3403	697.3403	697.3403
20	32.1000	32.1000	32.1000	1187.3895	1187.3895	1187.3895

Table 8: Negative binomial arrival pattern

m	$B_0(m)$	$B_1(m)$	$B_2(m)$	$B_3(m)$	$B_4(m)$	$B_5(m)$	$B_6(m)$	...
1	0.9762	0.0024	0.0000	0.0059	0.0000	0.0000	0.0107	...
2	0.8990	0.0100	$5.6441x10^{-6}$	0.0251	$2.8220x10^{-5}$	0.0000	0.0452	...
3	0.7639	0.0232	$4.2197x10^{-2}$	0.0580	0.0002	$1.0057x10^{-7}$	0.1046	...
4	0.5950	0.0388	0.0002	0.0971	0.0008	$1.0787x10^{-6}$	0.1758	...
5	0.4261	0.0528	0.0005	0.1321	0.0022	$5.9502x10^{-6}$	0.2406	...

m	M_{analytic}	$M_{\text{asymptotic}}$	M_{numeric}	$M^{(2)}_{\text{analytic}}$	$M^{(2)}_{\text{asymptotic}}$	$M^{(2)}_{\text{numeric}}$
1	0.1271	N/A	0.1271	0.8256	3.8765	0.8255
2	0.5432	0.3762	0.5432	3.5606	5.8639	3.5606
3	1.2858	1.3793	1.2858	8.5955	9.8639	8.5955
4	2.2563	2.3824	2.2563	15.6234	15.8764	15.6234
5	3.3180	3.3856	3.3180	24.2488	23.9014	24.2488

Table 9: Poisson arrival pattern

m	$B_0(m)$	$B_1(m)$	$B_2(m)$	$B_3(m)$	$B_4(m)$	$B_5(m)$	$B_6(m)$	...
1	0.8647	0.0134	0.0000	0.0338	0.0000	0.0000	0.0609	...
5	0.0527	0.0514	0.0037	0.1285	0.0186	0.0004	0.2545	...
10	0.0005	0.0021	0.0024	0.0058	0.0118	0.0032	0.0245	...
15	$4.1957x10^{-9}$	$7.6325x10^{-6}$	$8.7512x10^{-5}$	0.0001	0.0005	0.0008	0.0009	...

m	M_{analytic}	$M_{\text{asymptotic}}$	M_{numeric}	$M^{(2)}_{\text{analytic}}$	$M^{(2)}_{\text{asymptotic}}$	$M^{(2)}_{\text{numeric}}$
1	0.7240	0.5944	0.7240	4.7029	6.6880	4.7029
5	7.7353	7.7278	7.7352	82.6810	82.7040	82.6808
10	16.6445	16.6444	16.6445	320.8354	320.8364	320.8354
15	25.5611	25.5611	25.5611	717.9826	717.9827	717.9826

7. CONCLUSION

The method of generating function (gf) as illustrated in this paper, provides a shorter and simpler alternative to the usually used method for determining the asymptotic results of the discrete bulk-arrival renewal process. The generating function (gf) of first and second moments are first described as $M(v)$ and $M^{(2)}(v)$ respectively, and then the desired asymptotic results are easily derived. If the first renewal period (T_1) has a different distribution than the other renewal periods, then the first and second moments can be derived along similar lines. Moreover, higher order moments and their corresponding asymptotic results can be found similarly. Numerical examples of various cases have also been presented for the sake of completeness.

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