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## DETERMINING FUZZY DISTANCE THROUGH NON-SELF FUZZY CONTRACTIONS

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**Abstract:** In the present work we solve the problem of finding the fuzzy distance between two subsets of a fuzzy metric space for which we use a non-self fuzzy contraction mapping from one set to the other. It is a fuzzy extension of the proximity point problem which is by its nature a problem of global optimization. The contraction is defined here by two control functions. We define a geometric property of the fuzzy metric space. The main result is illustrated with an example. Our result extends a fuzzy version of the Banach contraction mapping principle.

**Keywords:** Fuzzy Metric Spaces, Global Optimization, Proximity Point, Non-Self  $(\phi - \psi)$ - Proximal Contraction, Optimal Approximate Solution, Fuzzy P-property.

**MSC:** 47H10, 54H25.

## 1. INTRODUCTION

In this paper we establish a proximity point result in a fuzzy metric space so to find the fuzzy distance between two subsets. The problem originated from the work of Eldred et al [9] and has been well studied during the decade through works like [2, 5, 9, 15, 14, 16, 21, 22] . For our purpose we use a non-self contraction mapping which is defined by two control functions. The fuzzy metric space on which we deduce our results is as in George et al [10]. Due to its special features, it has become the platform of several extensions of metric related studies [1, 3, 4, 6, 7, 11, 12, 13, 19]. The problem sought to be considered here is essentially a global optimization problem which is solved by transforming it to a problem of finding the optimal approximate solution to a fixed point equation for a non-self contraction defined by use of two control functions. Control functions have been used in several fixed point problems in metric spaces [20]. Here, as the contraction function is non-self mapping, there is no exact solution of the fixed point equation. The following are two special features of the present work.

1. We define and use a non-self contraction with two control functions.
2. We define and use a geometric property in the fuzzy metric space.

## 2. MATHEMATICAL PRELIMINARIES

George and Veeramani in their paper [10] introduced the following definition of fuzzy metric space. Throughout this paper, we use this definition of fuzzy metric space.

**Definition 1.** [10] *The 3-tuple  $(X, M, *)$  is called a fuzzy metric space if  $X$  is an arbitrary non-empty set,  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions for each  $x, y, z \in X$  and  $t, s > 0$ :*

- (i)  $M(x, y, t) > 0$ ,
- (ii)  $M(x, y, t) = 1$  if and only if  $x = y$ ,
- (iii)  $M(x, y, t) = M(y, x, t)$ ,
- (iv)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$  and
- (v)  $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$  is continuous,

where  $*$  is a continuous  $t$ -norm, that is, a continuous function  $*$  :  $[0, 1]^2 \rightarrow [0, 1]$  such that

- (i)  $a * b = b * a$  for all  $a, b \in [0, 1]$ ,
- (ii)  $a * (b * c) = (a * b) * c$  for all  $a, b, c \in [0, 1]$ ,
- (iii)  $a * 1 = a$  for all  $a \in [0, 1]$ ,
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ , for each  $a, b, c, d \in [0, 1]$ .

Let  $(X, M, *)$  be a fuzzy metric space. For  $t > 0$  and  $r$  with  $0 < r < 1$ , the open ball  $B(x, t, r)$  with center  $x \in X$  is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset  $A \subset X$  is called open if for each  $x \in A$ , there exist  $t > 0$  and  $r$  with  $0 < r < 1$  such that  $B(x, t, r) \subset A$ . Let  $\tau$  denote the family of all open subsets of  $X$ . Then  $\tau$  is a topology and is called the topology on  $X$  induced by the fuzzy metric  $M$ . The topology  $\tau$  is a Hausdorff topology [10]. In fact, the definition 2.1 is a modification of the definition given in [17] for ensuring Hausdorff topology of the space.

**Definition 2.** [10] Let  $(X, M, *)$  be a fuzzy metric space. A sequence  $\{x_n\}$  in  $X$  is said to be convergent to a point  $x \in X$  if  $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$  for all  $t > 0$ .

**Definition 3.** [10] Let  $(X, M, *)$  be a fuzzy metric space. A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if for each  $\varepsilon$  with  $0 < \varepsilon < 1$  and  $t > 0$ , there exists a positive integer  $n_0$  such that  $M(x_n, x_m, t) > 1 - \varepsilon$  for each  $n, m \geq n_0$ .

A fuzzy metric space is said to be complete if every Cauchy sequence is convergent in it.

The following lemma was proved by Grabiec [11] for fuzzy metric spaces defined by Kramosil et al [17]. The proof is also applicable to the fuzzy metric space given in definition 2.1.

**Lemma 4.** [11] Let  $(X, M, *)$  be a fuzzy metric space. Then  $M(x, y, \cdot)$  is non-decreasing for all  $x, y \in X$ .

**Lemma 5.** [18]  $M$  is a continuous function on  $X^2 \times (0, \infty)$ .

We will require for use in our results the following two functions.

**Definition 6. ( $\Psi$ -function)**[23] A function  $\psi : [0, \infty) \rightarrow [0, \infty)$  is a  $\Psi$ -function if

- (i)  $\psi$  is nondecreasing and continuous,
- (ii)  $\sum_{n=1}^{\infty} \psi^n(t) < \infty$  for all  $t > 0$ , where  $\psi^{n+1}(t) = \psi(\psi^n(t))$ ,  $n \geq 1$ .

It is clear that  $\psi(t) < t$  for all  $t > 0$  whenever  $\psi$  is a  $\Psi$ -function.

The following function is an example of a  $\psi$ -function:

$$\psi(t) = \begin{cases} t - \frac{t^2}{2}, & \text{if } t \in [0, 1], \\ \frac{1}{2}, & t > 1. \end{cases}$$

**Definition 7.** [20] A function  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a  $\Phi$ -function if

- (i)  $\phi$  is nondecreasing and continuous,  
(ii)  $\phi(0) = 0$ .

**Lemma 8.** [23] If  $*$  is a continuous  $t$ -norm, and  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences such that  $\alpha_n \rightarrow \alpha$ ,  $\gamma_n \rightarrow \gamma$  as  $n \rightarrow \infty$ , then  $\lim_{k \rightarrow \infty} (\alpha_k * \beta_k * \gamma_k) =$

$$\alpha * \overline{\lim}_{k \rightarrow \infty} \beta_k * \gamma \text{ and}$$

$$\underline{\lim}_{k \rightarrow \infty} (\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim}_{k \rightarrow \infty} \beta_k * \gamma.$$

**Lemma 9.** [23] Let  $\{f(k, \cdot) : [0, \infty) \rightarrow [0, 1], k = 0, 1, 2, \dots\}$  be a sequence of functions such that  $f(k, \cdot)$  is continuous and monotone increasing for each  $k \geq 0$ . Then  $\lim_{k \rightarrow \infty} f(k, t)$  is a left continuous function in  $t$  and  $\underline{\lim}_{k \rightarrow \infty} f(k, t)$  is a right continuous function in  $t$ .

### 3. MAIN RESULTS

**Definition 10.** [24] Let  $(X, M, *)$  be a fuzzy metric space. The fuzzy distance of a point  $x \in X$  from a nonempty subset  $A$  of  $X$  is

$$M(x, A, t) = \sup_{a \in A} M(x, a, t) \text{ for all } t > 0$$

and the fuzzy distance between two nonempty subsets  $A$  and  $B$  of  $X$  is

$$M(A, B, t) = \sup\{M(a, b, t) : a \in A, b \in B\} \text{ for all } t > 0.$$

Let  $A$  and  $B$  be two nonempty disjoint subsets of a fuzzy metric space  $(X, M, *)$ .

We write

$$A_0 = \{x \in A : \exists y \in B \text{ such that } M(x, y, t) = M(A, B, t) \text{ for all } t > 0\},$$

$$B_0 = \{y \in B : \exists x \in A \text{ such that } M(x, y, t) = M(A, B, t) \text{ for all } t > 0\}.$$

**Definition 11.** Let  $(X, M, *)$  be a fuzzy metric space and  $A, B$  are two non-empty subsets of  $X$ . An element  $x^* \in A$  is defined as a fuzzy best proximity point of the mapping  $f : A \rightarrow B$  if it satisfies the condition that for all  $t > 0$

$$M(x^*, fx^*, t) = M(A, B, t).$$

In the following we define a property of a pair of subsets in a fuzzy metric space. It is essentially a geometric property.

**Definition 12.** Let  $(A, B)$  be a pair of nonempty disjoint subsets of a fuzzy metric space  $(X, M, *)$ . Then the pair  $(A, B)$  is said to satisfy the fuzzy  $P$ -property if for all  $t > 0$  and  $x_1, x_2 \in A$ ,  $y_1, y_2 \in B$ ,

$$M(x_1, y_1, t) = M(A, B, t) \text{ and } M(x_2, y_2, t) = M(A, B, t)$$

jointly implies that

$$M(x_1, x_2, t) = M(y_1, y_2, t).$$

The *P*-property is a geometric property which is automatically valid in Hilbert spaces for non- empty closed and convex pairs of sets [21], but does not hold in arbitrary Banach spaces. In metric spaces such property for pairs of subsets is separately assumed for specific purposes. The above definition is a fuzzy extension of that.

**Definition 13.** Let  $(X, M, *)$  be a fuzzy metric space and  $f : A \rightarrow B$  be a mapping. The mapping  $f$  is non-self  $(\phi - \psi)$ - contraction mapping if there exist  $\Psi$ -function (definition 6)  $\psi$ , a  $\Phi$ -function (definition 7)  $\phi$  and  $0 < c < 1$  such that for all  $t > 0$  and  $x, y \in A$  we have

$$\left(\frac{1}{M(fx, fy, \phi(ct))} - 1\right) \leq \psi\left(\frac{1}{M(x, y, \phi(t))} - 1\right). \tag{3.1}$$

**Note.** The above contraction condition with some variations in the condition on  $\psi$  has already appeared in the context of fixed point studies in probabilistic metric spaces [8].

**Theorem 14.** Let  $(X, M, *)$  be a complete fuzzy metric space. Let  $A$  and  $B$  be two closed subsets of  $X$  and  $f : A \rightarrow B$  be an  $(\phi - \psi)$ - contractive mapping such that the following conditions are satisfied.

- (i)  $(A, B)$  satisfies the fuzzy *P*-property,
- (ii)  $f(A_0) \subseteq B_0$ ,
- (iii)  $A_0$  is nonempty,

Then there exists an element  $x^* \in A$  which is a fuzzy best proximity point of  $f$ .

**Proof.** By assumption (iii),  $A_0$  is nonempty. Let  $x_0 \in A_0$ . Since  $f(A_0) \subseteq B_0$ , there exists  $x_1 \in A_0$  such that

$$M(x_1, fx_0, t) = M(A, B, t) \text{ for all } t > 0.$$

Again since  $f(A_0) \subseteq B_0$ , there exists  $x_2 \in A_0$  such that

$$M(x_2, fx_1, t) = M(A, B, t) \text{ for all } t > 0.$$

Continuing this process, we construct a sequence  $\{x_n\}$  in  $A_0$  such that for all  $n \geq 1$ , for all  $t > 0$ ,

$$M(x_n, fx_{n-1}, t) = M(A, B, t). \tag{3.2}$$

Also, we can write the above as

$$M(x_{n+1}, fx_n, t) = M(A, B, t) \text{ for all } n \geq 1, \text{ for all } t > 0. \tag{3.3}$$

Since  $(A, B)$  satisfies the fuzzy *P*-property, we get from (3.2) and (3.3), for all  $t > 0$

$$M(x_n, x_{n+1}, t) = M(fx_{n-1}, fx_n, t) \text{ for all } n > 1. \tag{3.4}$$

From the property of  $\phi$  it is clear that for each  $t > 0$  there exists  $t_0 > 0$  such that  $\phi(t_0) = t$ .

Since  $f$  is  $(\phi - \psi)$ -contraction and from the property of  $\phi$ , we have for all  $n \geq 1$ , for all  $t > 0$  there exist  $t_0 > 0$  such that

$$\begin{aligned} \left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1\right) &= \left(\frac{1}{M(fx_{n-1}, fx_n, \phi(t_0))} - 1\right) \\ &\leq \left(\frac{1}{M(fx_{n-1}, fx_n, \phi(ct_0))} - 1\right) \\ &\leq \psi\left(\frac{1}{M(x_{n-1}, x_n, \phi(t_0))} - 1\right) \\ &= \psi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right) \end{aligned}$$

Therefore, we have for all  $n \geq 1$ , for all  $t > 0$ ,

$$\left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1\right) \leq \psi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right). \quad (3.5)$$

Combining (3.4) and (3.5), we have for all  $n \geq 1$ , for all  $t > 0$ ,

$$\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq \psi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right). \quad (3.6)$$

If for some  $k > 0$ ,  $x_k = x_{k+1}$ , then  $x_k$  is a best proximity point of  $f$ .

Assuming  $x_{n-1} \neq x_n$  for all  $n \geq 1$ , and making repeated applications of (3.6), we have for all  $n \geq 1$ , for all  $t > 0$ ,

$$\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq \psi^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right). \quad (3.7)$$

Taking  $n \rightarrow \infty$  in the above inequality (3.7), for all  $t > 0$ , we obtain

$\lim_{n \rightarrow \infty} \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq \lim_{n \rightarrow \infty} \psi^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right) \rightarrow 0$  as  $n \rightarrow \infty$ , (by a property of  $\psi$ ).

that is,  $\lim_{n \rightarrow \infty} \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) = 0$ , which implies that for all  $t > 0$ ,

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+1}, t) = 1. \quad (3.8)$$

Next, we show that  $\{x_n\}$  is a Cauchy sequence in  $A$ . We suppose, if possible, that  $\{x_n\}$  is not a Cauchy sequence in  $A$ . Then definition 3 is not satisfied by the sequence  $\{x_n\}$  and, therefore, there exist some  $\epsilon > 0$  and some  $\lambda$  with  $0 < \lambda < 1$ , for which we can find two subsequences  $\{x_{m(k)}\}$  and  $\{x_{n(k)}\}$  of  $\{x_n\}$  with

$$n(k) > m(k) > k \quad \text{such that}$$

$$M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1 - \lambda), \quad (3.9)$$

for all positive integer  $k$ .

We may choose the  $n(k)$  as the smallest integer exceeding  $m(k)$  for which (3.9) holds. Then, for all positive integer  $k$ ,

$$M(x_{m(k)}, x_{n(k)-1}, \epsilon) > (1 - \lambda) \quad (3.10)$$

Then, for all  $k \geq 1, 0 < s < \frac{\epsilon}{2}$ , we obtain,

$$\begin{aligned} (1 - \lambda) &\geq M(x_{m(k)}, x_{n(k)}, \epsilon) \\ &\geq M(x_{m(k)}, x_{m(k)-1}, s) * M(x_{m(k)-1}, x_{n(k)-1}, \epsilon - 2s) \\ &\quad * M(x_{n(k)-1}, x_{n(k)}, s). \end{aligned} \tag{3.11}$$

For all  $t > 0$ , we denote

$$h_1(t) = \overline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, t). \tag{3.12}$$

Taking limit supremum on both sides of (3.11), using (3.8), the properties of  $M$  and  $*$ , by lemma (8), we obtain

$$(1 - \lambda) \geq 1 * \overline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon - 2s) * 1 = h_1(\epsilon - 2s) \tag{3.13}$$

Since  $M$  is bounded within the range in  $[0,1]$ , continuous and, by lemma 4, monotone increasing in the third variable  $t$ , it follows by an application of lemma 9 that  $h_1$ , as given in (3.12) is continuous from the left. From the above, letting  $s \rightarrow 0$  in (3.13), it then follows that

$$\overline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \leq (1 - \lambda). \tag{3.14}$$

Let,

$$h_2(t) = \underline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, t), t > 0. \tag{3.15}$$

Again, for all  $k \geq 1, s > 0$ ,

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon + s) &\geq M(x_{m(k)-1}, x_{m(k)}, s) * M(x_{m(k)}, x_{n(k)-1}, \epsilon) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, s) * (1 - \lambda), \text{ (by (3.14))} \end{aligned} \tag{3.16}$$

Taking limit infimum as  $k \rightarrow \infty$  in (3.16), by virtue of (3.8), we obtain

$$\begin{aligned} h_2(\epsilon + s) = \underline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon + s) &\geq \underline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{m(k)}, s) * (1 - \lambda) \\ &= 1 * (1 - \lambda) = (1 - \lambda). \end{aligned} \tag{3.17}$$

Since  $M$  is bounded within the range in  $[0,1]$ , continuous and by lemma 4, it is monotone increasing in the third variable  $t$ , it follows by an application of lemma 9 that  $h_2$ , as given in (3.15) is continuous from the right.

From the above, letting  $s \rightarrow 0$  in (3.17), it then follows that

$$\underline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \geq (1 - \lambda). \tag{3.18}$$

The inequalities (3.14) and (3.18) jointly imply that

$$\lim_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) = (1 - \lambda). \tag{3.19}$$

Again by (3.9),

$$\overline{\lim}_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1 - \lambda) \quad (3.20)$$

Also for all  $k \geq 1, s > 0$ , we obtain

$$M(x_{m(k)}, x_{n(k)}, \epsilon + 2s) \geq M(x_{m(k)}, x_{m(k)-1}, s) * M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) * M(x_{n(k)-1}, x_{n(k)}, s)$$

Taking limit infimum as  $k \rightarrow \infty$  in the above inequality, using (3.8), (3.19) and the properties of  $M$  and  $*$ , by lemma 8, we obtain

$$\underline{\lim}_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}, \epsilon + 2s) \geq 1 * \underline{\lim}_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) * 1 = 1 - \lambda.$$

Since  $M$  is bounded within the range in  $[0, 1]$ , is continuous and, by lemma 4, monotone increasing in the third variable  $t$ , it follows by an application of lemma 9 that  $\underline{\lim}_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}, t)$  is continuous function of  $t$  from the right.

Taking  $s \rightarrow 0$  in the above inequality, and using lemma 9, we obtain

$$\underline{\lim}_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}, \epsilon) \geq (1 - \lambda), \quad (3.21)$$

Combining (3.20) and (3.21), we obtain

$$\lim_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}, \epsilon) = (1 - \lambda) \quad (3.22)$$

From (3.3), we have

$$M(x_{m(k)}, fx_{m(k)-1}, t) = M(A, B, t) \quad (3.23)$$

$$M(x_{n(k)}, fx_{n(k)-1}, t) = M(A, B, t) \quad (3.24)$$

Since  $(A, B)$  satisfies the fuzzy  $P$ -property, we get from (3.23) and (3.24), for all  $t > 0$ ,

$$M(x_{m(k)}, x_{n(k)}, t) = M(fx_{m(k)-1}, fx_{n(k)-1}, t). \quad (3.25)$$

Now by the property of  $\phi$ , there exists  $\epsilon_0 > 0$  such that  $\phi(\epsilon_0) = \epsilon$ .

Therefore, from the above and by (3.25),

$$\begin{aligned} \left( \frac{1}{M(x_{m(k)}, x_{n(k)}, \epsilon)} - 1 \right) &= \left( \frac{1}{M(fx_{m(k)-1}, fx_{n(k)-1}, \epsilon)} - 1 \right) \\ &= \left( \frac{1}{M(fx_{m(k)-1}, fx_{n(k)-1}, \phi(\epsilon_0))} - 1 \right) \\ &\leq \left( \frac{1}{M(fx_{m(k)-1}, fx_{n(k)-1}, \phi(c\epsilon_0))} - 1 \right) \\ &\leq \psi \left( \frac{1}{M(x_{m(k)-1}, x_{n(k)-1}, \phi(\epsilon_0))} - 1 \right) \\ &= \psi \left( \frac{1}{M(x_{m(k)-1}, x_{n(k)-1}, \epsilon)} - 1 \right) \end{aligned}$$



Taking  $k \rightarrow \infty$  in the above inequality, we have

$$\left(\lim_{k \rightarrow \infty} \frac{1}{M(x_{m(k)}, x_{n(k)}, \epsilon)} - 1\right) \leq \psi\left(\lim_{k \rightarrow \infty} \frac{1}{M(x_{m(k)-1}, x_{n(k)-1}, \epsilon)} - 1\right). \text{ (since } \psi \text{ is continuous)}$$

Using (3.19) and (3.22), we have

$$\left(\frac{1}{1-\lambda} - 1\right) \leq \psi\left(\frac{1}{1-\lambda} - 1\right) < \left(\frac{1}{1-\lambda} - 1\right),$$

which is a contradiction.

Thus, it is established that  $\{x_n\}$  is a Cauchy sequence. Since  $(X, M, *)$  is complete, there exists  $x^* \in A$  such that

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Since  $f$  is  $(\phi - \psi)$ - proximal contractive mapping, by using (3.1), we have for all  $n \geq 0, t > 0$

$$\begin{aligned} \left(\frac{1}{M(fx_n, fx^*, t)} - 1\right) &= \left(\frac{1}{M(fx_n, fx^*, \phi(t_0))} - 1\right) \\ &\leq \left(\frac{1}{M(fx_n, fx^*, \phi(ct_0))} - 1\right) \\ &\leq \psi\left(\frac{1}{M(x_n, x^*, \phi(t_0))} - 1\right) \\ &\leq \psi\left(\frac{1}{M(x_n, x^*, t)} - 1\right) \end{aligned}$$

Taking limit  $n \rightarrow \infty$  on both sides of the above inequality, using the fact that  $\psi(0) = 0$ , we have

$$fx_n \rightarrow fx^* \text{ as } n \rightarrow \infty.$$

From (3.3) and the above limit, for all  $t > 0$

$$M(A, B, t) = M(x_{n+1}, fx_n, t) = M(x^*, fx^*, t) \text{ as } n \rightarrow \infty.$$

Therefore, for all  $t > 0, M(x^*, fx^*, t) = M(A, B, t)$ . This completes the proof.

#### 4. ILLUSTRATION

**Example 15.** Suppose that  $X = \mathbb{R}^2$  with fuzzy metric space

$$M((x, y), (x', y'), t) = \frac{t}{t + |x-x'| + |y-y'|} \text{ and minimum } t\text{-norm } *.$$

Consider the closed subsets  $A$  and  $B$  in the topology induced by the fuzzy metric as

$$\begin{aligned} A &= \{(0, x) : x \in \mathbb{R}\}, \\ B &= \{(1, x) : x \in \mathbb{R}\}. \end{aligned}$$

Let  $\psi(t) = ct$  and  $\phi(t) = t^2$ , where  $0 < c < 1$ . Let  $f : A \rightarrow B$  be the mapping defined by

$$f((0, x)) = (1, 1 - e^{-c^3x}).$$

Here  $M(A, B, t) = \frac{t}{1+t}$  for all  $t > 0$ .

Here  $A_0 = A$  and  $B_0 = B$  and  $f(A_0) \subseteq B_0$ .

Now we show that  $f$  satisfies fuzzy  $P$ -property.

Let  $u_1 = (0, x_1)$ ,  $u_2 = (0, x_2) \in A$  and  $v_1 = (1, y_1)$ ,  $v_2 = (1, y_2) \in B$  with

$$M(u_1, v_1, t) = M(A, B, t) \text{ for all } t > 0 \quad (4.1)$$

and

$$M(u_2, v_2, t) = M(A, B, t) \text{ for all } t > 0 \quad (4.2)$$

From (4.1), we get for all  $t > 0$

$$\frac{t}{t+1+|x_1-y_1|} = \frac{t}{t+1},$$

which implies that  $x_1 = y_1$ .

Similarly from (4.2), we get for all  $t > 0$

$$x_2 = y_2.$$

Now for all  $t > 0$

$$\begin{aligned} M(u_1, u_2, t) &= \frac{t}{t+|x_1-x_2|} \\ &= \frac{t}{t+|y_1-y_2|} \\ &= M(v_1, v_2, t). \end{aligned}$$

Hence  $f$  satisfies fuzzy  $P$ -property.

Let  $u = (0, x)$ ,  $v = (0, y) \in A$ . Without loss of generality, we may assume that  $x < y$ .

Now for all  $t > 0$ ,

$$\begin{aligned} \left( \frac{1}{M(fu, fv, \phi(ct))} - 1 \right) &= \frac{|e^{-c^3x} - e^{-c^3y}|}{c^2t^2} \\ &= \frac{c^3 e^{-c^3[x+\theta(y-x)]} |x-y|}{c^2t^2} \quad (\text{Using MVT, where } 0 < \theta < 1) \\ &\leq \frac{c|x-y|}{t^2} \\ &= c \left( \frac{1}{M(u, v, \phi(t))} - 1 \right) \\ &= \psi \left( \frac{1}{M(u, v, \phi(t))} - 1 \right). \end{aligned}$$

Hence  $f$  satisfies  $(\phi - \psi)$ -proximal contraction.

Here  $(0, 0) \in A$  is the best proximity point of  $f$ .

**Note:** The above illustration indicates that our result is an effective generalization of the fuzzy Banach contraction mapping principle given by Gregori and Sapena [13] in complete fuzzy metric space since the latter is not applicable to the above example.

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