Yugoslav Journal of Operations Research
31 (2021), Number 1, 75–94
DOI: https://doi.org/10.2298/YJOR191215041D

SET-VALUED OPTIMIZATION PROBLEMS VIA SECOND-ORDER CONTINGENT EPIDERIVATIVE

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Received: December 2019 / Accepted: December 2020

Abstract: In this paper, we establish second-order KKT conditions of a set-valued optimization problem and study second-order Mond-Weir, Wolfe, and mixed types duals with the help of second-order contingent epiderivative and second-order generalized cone convexity assumptions.

Keywords: Convex Cone, Set-valued Map, Contingent Epiderivative, Duality.

MSC: 26B25, 49N15.

1. INTRODUCTION

In last two decades, many authors extended results of vector optimization problems to set-valued optimization problems. A set-valued optimization problem is an optimization problem where the objective function and functions attached to constraints are set-valued maps. The notions of cone convexities and differentiability of set-valued maps play an important role in the theory of set-valued optimization problems. Various types of cone convexities and differentiability of set-valued maps have been introduced in the last few years to study set-valued optimization problems. Set-valued optimization problems are closely related to optimal control problems with differential inclusions. Some problems in mathematical economics, viability theory, image processing, and many more are set-valued optimization problems. Moreover, gap functions for vector variational inequalities, duality principles in vector optimization, and inverse problems for partial differential equations can be considered in the framework of set-valued optimization problems. Set-valued optimization problem makes a bridge between different areas in optimization theory. The analysis of set-valued maps is an important tool to investigate optimality conditions of set-valued optimization problems. A set-valued optimization problem being a new branch of optimization problem, attracts the attention of many researchers to an increasing extent in the last few years. The notion of cone convexity has an important role to establish the optimality conditions for existence of efficient points of set-valued optimization problems. Borwein [3] introduced the notion of cone convexity for set-valued maps. The concept of contingent derivative of set-valued maps was introduced by Aubin [1]. It is an extension of the concept of Frechet differentiability to set-valued maps. Jahn and Rauh [18] introduced another notion of differentiability of set-valued maps viz. the notion of contingent epiderivative. It is an extension of the notion of directional derivative to the set-valued case. Sheng and Liu [22] investigated the KKT conditions of set-valued optimization problems via generalized contingent epiderivative and preinvexity assumptions. Rodríguez-Marín and Sama [21] investigated the existence, uniqueness, and properties of contingent epiderivative. They also studied the relationship between contingent epiderivative and contingent derivative of set-valued maps. Li et al. [19] introduced higher-order Mond-Weir dual of set-valued optimization problems with the help of higher-order adjacent derivative and proved the corresponding duality theorems. Zhu et al. [23] established the second-order KKT necessary and sufficient conditions of set-valued optimization problems via second-order contingent derivative.

In this paper, we establish the second-order KKT conditions of a set-valued optimization problem and study the second-order Mond-Weir, Wolfe, and mixed types duality results of the said problem under the second-order contingent epiderivative and generalized cone convexity assumptions.

This paper is organized as follows. Section 2 deals with some definitions and preliminary concepts of set-valued maps. A set-valued optimization problem (P) is also considered in Section 2. The second-order sufficient KKT conditions are established for the problem (P) in Section 3. Various types of duality theorems are also proved under the second-order contingent epiderivative and generalized cone convexity assumptions.

2. DEFINITIONS AND PRELIMINARIES

Let Y be a real normed space and K be a nonempty subset of Y. Then K is called a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Furthermore, K is called non-trivial if $K \neq \{\theta_Y\}$, proper if $K \neq Y$, pointed if $K \cap (-K) = \{\theta_Y\}$, solid if $\mathrm{int}(K) \neq \emptyset$, closed if $\overline{K} = K$, and convex if $\lambda K + (1-\lambda)K \subseteq K$, for all $\lambda \in [0,1]$, where $\mathrm{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and θ_Y is the zero element of Y.

Let us define the non-negative orthant $\mathbb{R}^m_+$ of the m-dimensional Euclidean space $\mathbb{R}^m$ by

$$\mathbb{R}^m_+ = \{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, 2, ..., m\}.$$

Then $\mathbb{R}^m_+$ is a solid pointed closed convex cone and $\text{int}(\mathbb{R}^m_+) \cup \{\mathbf{0}_{\mathbb{R}^m}\}$ is a solid pointed convex cone in $\mathbb{R}^m$, where $\mathbf{0}_{\mathbb{R}^m}$ is the zero element of $\mathbb{R}^m$.

Let Y^* be the space of all continuous linear functionals on Y and K be a solid pointed convex cone in Y. Then the dual cone K^+ to K is defined as

$$K^+ = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in K\},$$

where $\langle ., .\rangle$ is the canonical bilinear form with respect to the duality between Y^* and Y.

Let K be a solid pointed convex cone in Y. There are two types of cone-orderings in Y with respect to K. For any two elements $y_1, y_2 \in Y$, we have

$$y_1 \leq y_2 \text{ if } y_2 - y_1 \in K$$

and

$$y_1 < y_2 \text{ if } y_2 - y_1 \in \text{int}(K).$$

The following notions of minimality are mainly used with respect to a solid pointed convex cone K in a real normed space Y.

Definition 2.1. *Let B be a nonempty subset of a real normed space Y. Then ideal minimal, minimal, and weakly minimal points of B are defined as*

(i) *$y' \in B$ is an ideal minimal point of B if $y' \leq y$, for all $y \in B$.*

(ii) *$y' \in B$ is a minimal point of B if there is no $y \in B \setminus \{y'\}$, such that $y \leq y'$.*

(iii) *$y' \in B$ is a weakly minimal point of B if there is no $y \in B$, such that $y < y'$.*

The sets of ideal minimal points, minimal points, and weakly minimal points of B are denoted by I-min(B), min(B), and w-min(B), respectively and characterized as

$$\text{I-min}(B) = \{y' \in B : B \subseteq \{y'\} + K\},$$

$$\min(B) = \{y' \in B : (y' - K) \cap B = \{y'\}\},$$

and

$$\text{w-min}(B) = \{y' \in B : (y' - \text{int}(K)) \cap B = \emptyset\}.$$

Similarly, the sets of ideal maximal points, maximal points, and weakly maximal points of B can be defined and characterized.

We recall the notions of contingent cone and second-order contingent set in a real normed space.

Definition 2.2. *[2, 1] Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, and $y' \in \overline{B}$. The contingent cone to B at y' is denoted by $T(B, y')$ and is defined as follows:*

An element $y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y.$$

Remark 2.1. *The contingent cone $T(B, y')$ is actually a local approximation of the set $B - y'$. If $y' \in int(B)$, then $T(B, y') = Y$.*

Proposition 2.1. *[2] The contingent cone $T(B, y')$ is a closed cone, but not necessarily convex and $T(B, y') \subseteq \overline{\bigcup_{h>0} \frac{B - y'}{h}}$.*

Definition 2.3. *[2, 1, 4] Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, $y' \in \overline{B}$, and $u \in Y$. The second-order contingent set to B at y' in the direction u is denoted by $T^2(B, y', u)$ and defined as*

An element $y \in T^2(B, y', u)$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n u + \frac{1}{2}{\lambda_n}^2 y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$, $\{t'_n\}$ in $\mathbb{R}$, with $t_n, t'_n > 0$, $t_n \to \infty$, $t'_n \to \infty$, $\frac{t'_n}{t_n} \to 2$, and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to u \text{ and } t'_n(t_n(y'_n - y') - u) \to y.$$

Proposition 2.2. *[23] The second-order contingent set $T^2(B, y', u)$ is a closed set, but not necessarily a cone. Even, $T^2(B, y', u)$ may not be convex, though B is convex. Also, $T^2(B, y', \theta_Y) = T(T(B, y'), \theta_Y) = T(B, y')$.*

Let X, Y be real normed spaces, 2^Y be the set of all subsets of Y, and K be a solid pointed convex cone in Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y, i.e., $F(x) \subseteq Y$, for all $x \in X$. The effective domain, image, graph, and epigraph of F are defined by

$$\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$F(A) = \bigcup_{x \in A} F(x), \text{ for any } A(\neq \emptyset) \subseteq X,$$

$$\text{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\},$$

and

$$\text{epi}(F) = \{(x,y) \in X \times Y : y \in F(x) + K\}.$$

Let A be a nonempty subset of X, $x' \in A$, $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$, and $y' \in F(x')$. Jahn and Rauh [18] introduced the notion of contingent epiderivative of set-valued maps which plays a vital role in various aspects of set-valued optimization problems.

Definition 2.4. *[18] A single-valued map $D_\uparrow F(x',y') : X \to Y$ whose epigraph coincides with the contingent cone to the epigraph of F at (x',y'), i.e.,*

$$\text{epi}(D_\uparrow F(x',y')) = T(\text{epi}(F),(x',y')),$$

is said to be the contingent epiderivative of F at (x',y').

When $f : X \to \mathbb{R}$ is a real-valued map, being continuous at $x_0 \in X$ and convex,

$$D_\uparrow f(x_0, f(x_0))(u) = f'(x_0)(u), \forall u \in X,$$

where $f'(x_0)(u)$ is the directional derivative of f at x_0 in the direction u.

Jahn et al. [17] introduced the notion of second-order contingent epiderivative of set-valued maps which also has a fundamental role in set-valued optimization problems.

Definition 2.5. *[17] A single-valued map $D^2_\uparrow F(x',y',u,v) : X \to Y$ whose epigraph coincides with the second-order contingent set to the epigraph of F at $(x',y') \in \text{gr}(F)$ in a direction $(u,v) \in X \times Y$, i.e.,*

$$\text{epi}(D^2_\uparrow F(x',y',u,v)) = T^2(\text{epi}(F),(x',y'),(u,v)),$$

is said to be the second-order contingent epiderivative of F at (x',y') in the direction (u,v).

The following theorem is a characterization of second-order contingent epiderivative of set-valued maps.

Theorem 2.1. *[16, 20] The second-order contingent epiderivative $D^2_\uparrow F(x',y',u,v)$ of a set-valued map $F : X \to 2^Y$ at $(x',y') \in \text{gr}(F)$ in a direction $(u,v) \in X \times Y$ exists if and only if the ideal minimal point of the set*

$$\{y \in Y : (x,y) \in T^2(\text{epi}(F),(x',y'),(u,v))\}$$

exists, for all $x \in L$, where L is the projection of $T^2(\text{epi}(F),(x',y'),(u,v))$ onto X. Since K is a pointed cone, the ideal minimal point of the set

$$\{y \in Y : (x,y) \in T^2(\text{epi}(F),(x',y'),(u,v))\},$$

if it exists, is unique, for all $x \in L$. In this case, the second-order contingent epiderivative $D^2_\uparrow F(x',y',u,v)$ is given by

$$D^2_\uparrow F(x',y',u,v)(x) = \text{I-min}\{y \in Y : (x,y) \in T^2(\text{epi}(F),(x',y'),(u,v))\}, \forall x \in L.$$

Proposition 2.3. *[2] Let* $\emptyset \neq A \subseteq X$, $x' \in A$, $u \in X$, *and* $f : X \to Y$ *be a single-valued map which is twice continuously differentiable around* x'. *The second-order contingent epiderivative* $D^2_{\uparrow} f(x', f(x'), u, f'(x')u)$ *of* f *at* $(x', f(x'))$ *in the direction* $(u, f'(x')u)$ *is given by*

$$D^2_{\uparrow} f(x', f(x'), u, f'(x')u)(x) = f'(x')x + \frac{1}{2} f''(x')(u, u), x \in T^2(A, x', u).$$

We now turn our attention to the notion of cone convexity of set-valued maps, introduced by Borwein [3].

Definition 2.6. *[3] Let* X, Y *be real normed spaces,* A *be a nonempty convex subset of* X, *and* K *be a solid pointed convex cone in* Y. *A set-valued map* $F : X \to 2^Y$, *with* $A \subseteq \mathrm{dom}(F)$, *is called* K*-convex on* A *if* $\forall x_1, x_2 \in A$ *and* $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

It is clear that if a set-valued map $F : X \to 2^Y$ is K-convex on A, then $\mathrm{epi}(F)$ is a convex subset of $X \times Y$.

The following lemma represents cone convex set-valued maps in terms of contingent epiderivative.

Lemma 2.1. *[18] If* $F : X \to 2^Y$ *is* K*-convex on a nonempty convex subset* A *of a real normed space* X, *then for all* $x, x' \in A$ *and* $y' \in F(x')$,

$$F(x) - y' \subseteq D_{\uparrow} F(x', y')(x - x') + K.$$

Definition 2.7. *[23] Let* X, Y *be real normed spaces,* A *be a nonempty subset of* X, K *be a solid pointed convex cone in* Y, *and* $F : X \to 2^Y$ *be a set-valued map, with* $A \subseteq \mathrm{dom}(F)$. *Let* $x', u \in A$, $y' \in F(x')$, *and* $v \in F(u) + K$. *Assume that* F *is second-order contingent epiderivable at* (x', y') *in the direction* $(u - x', v - y')$. *Then* F *is said to be second-order* K*-convex at* (x', y') *in the direction* $(u - x', v - y')$ *on* A *if*

$$F(x) - y' \subseteq D^2_{\uparrow} F(x', y', u - x', v - y')(x - x') + K, \forall x \in A.$$

Let X, Y, Z be real normed spaces and A be a nonempty subset of X. Let K and L be solid pointed convex cones of Y and Z, respectively. Suppose that $F : X \to 2^Y$ and $G : X \to 2^Z$ are two set-valued maps, with

$$A \subseteq \mathrm{dom}(F) \cap \mathrm{dom}(G).$$

We consider a set-valued optimization problem (P).

$$\begin{aligned} &\underset{x \in A}{\text{minimize}} && F(x) \\ &\text{subject to} && G(x) \cap (-L) \neq \emptyset. \end{aligned} \tag{P}$$

The feasible set of the problem (P) is given by

$$S = \{x \in A : G(x) \cap (-L) \neq \emptyset\}.$$

The minimizer and weak minimizer of (P) are defined in the following ways.

Definition 2.8. *A point* $(x', y') \in X \times Y$*, with* $x' \in S$ *and* $y' \in F(x')$*, is called a minimizer of the problem* (P) *if for all* $(x, y) \in X \times Y$*, with* $x \in S$ *and* $y \in F(x)$*,*

$$y - y' \notin (-K) \setminus \{\theta_Y\}.$$

Definition 2.9. *A point* $(x', y') \in X \times Y$*, with* $x' \in S$ *and* $y' \in F(x')$*, is called a weak minimizer of the problem* (P) *if for all* $(x, y) \in X \times Y$*, with* $x \in S$ *and* $y \in F(x)$*,*

$$y - y' \notin (-\mathrm{int}(K)).$$

3. MAIN RESULTS

Das and Nahak [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] introduced the notion of ρ-cone convex set-valued maps. They establish the sufficient KKT conditions and study the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions. For $\rho = 0$, we have the usual notion of cone convexity of set-valued maps introduced by Borwein [3].

Definition 3.1. *[5, 8] Let* X, Y *be real normed spaces,* A *be a nonempty convex subset of* X*,* K *be a solid pointed convex cone in* Y*,* $e \in \mathrm{int}(K)$*, and* $F : X \to 2^Y$ *be a set-valued map, with* $A \subseteq \mathrm{dom}(F)$*. Then* F *is said to be* ρ*-*K*-convex with respect to* e *on* A *if there exists* $\rho \in \mathbb{R}$ *such that*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho\lambda(1 - \lambda)\|x_1 - x_2\|^2 e + K, \quad \forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

Das and Nahak [8] constructed an example of ρ-cone convex set-valued map, which is not cone convex. They also characterized ρ-cone convex set-valued maps in terms of contingent epiderivative of set-valued maps.

Theorem 3.1. *[8] Let* A *be a nonempty convex subset of* X*,* $e \in \mathrm{int}(K)$*, and* $F : X \to 2^Y$ *be* ρ*-*K*-convex with respect to* e *on* A*. Let* $x' \in A$ *and* $y' \in F(x')$*. Then,*

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in A.$$

We introduce second-order ρ-cone convexity of set-valued maps via second-order contingent epiderivative.

Definition 3.2. *[6] Let* X, Y *be real normed spaces,* A *be a nonempty subset of* X*,* K *be a solid pointed convex cone in* Y*,* $e \in \mathrm{int}(K)$*, and* $F : X \to 2^Y$ *be a set-valued map, with* $A \subseteq \mathrm{dom}(F)$*. Let* $x', u \in A$*,* $y' \in F(x')$*, and* $v \in F(u) + K$*. Assume that* F *is second-order contingent epiderivable at* (x', y') *in the direction* $(u - x', v - y')$*. Then* F *is said to be second-order* ρ*-*K*-convex with respect to* e *at* (x', y') *in the direction* $(u - x', v - y')$ *on* A *if there exists* $\rho \in \mathbb{R}$ *such that*

$$F(x) - y' \subseteq D^2_{\uparrow}F(x', y', u - x', v - y')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in A.$$

Remark 3.1. *For $u = x'$ and $v = y'$, we have*

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in A.$$

In this case, we have the first order ρ-K-convexity via contingent epiderivative.

If $\rho > 0$, then F is said to be strongly second-order ρ-K-convex, if $\rho = 0$, we have the usual notion of second-order K-convexity, and if $\rho < 0$, then F is said to be weakly second-order ρ-K-convex.
Obviously, strongly second-order ρ-K-convexity $\Rightarrow$ second-order K-convexity $\Rightarrow$ weakly second-order ρ-K-convexity.

We construct the following set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$, which is second-order ρ-$\mathbb{R}^2_+$-convex for some ρ but is not second-order $\mathbb{R}^2_+$-convex.

Example 3.1. *Let a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ be defined by*

$$F(\lambda) = \begin{cases} \{(x - \lambda^2, \sqrt{x} - \lambda^2) : x \geq \lambda^2, \sqrt{x} \geq \lambda^2\}, & \text{if } \lambda \neq 2, \\ \{(x, \sqrt{-x}) : x \in [-4, 0]\}, & \text{if } \lambda = 2. \end{cases}$$

Let $K = \mathbb{R}^2_+$. We have the epigraph of the set-valued map F as

$$\begin{aligned} \text{epi}(F) = & \{(\lambda, (x, y)) : \lambda \in \mathbb{R}, x \geq 0, y \geq 0\} \\ & \cup \{(\lambda, (x, y)) : \lambda \geq 0, x \in [-4, 0], y \geq \sqrt{-x}\}. \end{aligned}$$

Take $(\lambda', (x', y')) = (0, (0, 0))$ and $(\lambda, (u, v)) = (-1, (0, 0))$.
Obviously, $(x', y') \in F(\lambda')$ and $(u, v) \in F(\lambda) + \mathbb{R}^2_+$. Therefore,

$$T^2(\text{epi}(F), (0, (0, 0)), (-1, (0, 0))) = \{(\lambda, (x, y)) : \lambda \in \mathbb{R}, x \geq 0, y \geq 0\}.$$

Hence,

$$\begin{aligned} & D^2_{\uparrow}F(0, (0, 0), -1, (0, 0))(\lambda) \\ & = I\text{-}min\ \{(x, y) : (\lambda, (x, y)) \in T^2(\text{epi}(F), (0, (0, 0)), (-1, (0, 0)))\}. \end{aligned}$$

Therefore,

$$\text{dom}(D^2_{\uparrow}F(0, (0, 0), -1, (0, 0))) = \mathbb{R}$$

and

$$D^2_{\uparrow}F(0, (0, 0), -1, (0, 0))(\alpha) = \{(0, 0)\}, \forall \alpha \in \mathbb{R}.$$

We have

$$(x, \sqrt{-x}) \notin \mathbb{R}^2_+, \text{ when } x \in [-4, 0].$$

Hence,

$$F(\lambda) - (0,0) \nsubseteq D^2_{\uparrow}F(0,(0,0),-1,(0,0))(2) + \mathbb{R}^2_+.$$

Consequently, F is not second-order $\mathbb{R}^2_+$-convex on $\mathbb{R}$.

Choose $\rho = -1$. We have

$$(x - \lambda^2, \sqrt{x} - \lambda^2) \in (-\lambda^2, -\lambda^2) + \mathbb{R}^2_+, \text{ for } x \geq \lambda^2 \geq 0$$

and

$$(x, \sqrt{-x}) \in -(4,4) + \mathbb{R}^2_+, \text{ for } x \in [-4, 0].$$

So,

$$F(\lambda) - (0,0) \subseteq D^2_{\uparrow}F(0,(0,0),-1,(0,0))(\lambda) + \rho|\lambda - 0|^2(1,1) + \mathbb{R}^2_+, \forall \lambda \in \mathbb{R}.$$

Therefore, F is second-order ρ-$\mathbb{R}^2_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^2}$ at $(0,(0,0))$ in the direction $(-1,(0,0))$ on $\mathbb{R}$. Similarly, it can be shown that F is second-order ρ-$\mathbb{R}^2_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^2}$ at $(0,(0,0))$ in the direction $(1,(0,0))$ on $\mathbb{R}$. □

Remark 3.2. *For the case of single-valued map, Definition 3.2 coincides with the existing one. Let X, Y be real normed spaces, K be a solid pointed convex cone in Y, $e \in \text{int}(K)$, $u \in X$, and $v \in Y$. Let $f : X \to Y$ be second-order continuously differentiable function at $x' \in X$. By considering $F(x) = \{f(x)\}$, from Definition 3.2 and Proposition 2.3, we can conclude that f is called second-order ρ-K-convex with respect to e at $(x', f(x'))$ in the direction $(u - x', v - f(x'))$ if there exists $\rho \in \mathbb{R}$ such that*

$$f(x) - f(x') \in f'(x')(x - x') + \frac{1}{2}f''(x')(u - x', u - x') + \rho\|x - x'\|^2 e + K, \forall x \in X,$$

where $v - f(x') = f'(x')(u - x')$.
The followings are some special cases.
When $Y = \mathbb{R}^m$, $K = \mathbb{R}^m_+$, $f = (f_1, f_2, ..., f_m)$, and $e = (1, 1, ..., 1)$, we have

$$f_i(x) - f_i(x') \geq f_i'(x')(x - x') + \frac{1}{2}f_i''(x')(u - x', u - x') + \rho\|x - x'\|^2,$$
$$\forall x \in X \text{ and } i = 1, 2, ..., m.$$

When $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x) - f(x') \geq f'(x')(x - x') + \frac{1}{2}f''(x')(u - x', u - x') + \rho\|x - x'\|^2, \forall x \in X.$$

When $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x) - f(x') \geq (x - x')^T \nabla f(x') + \frac{1}{2}(u - x')^T H(x')(u - x') + \rho\|x - x')\|^2, \forall x \in X,$$

where $\nabla f(x')$ and $H(x')$ are the gradient and Hessain matrix of f at x', respectively.

3.1. SECOND-ORDER OPTIMALITY CONDITIONS

We establish the second-order KKT sufficient optimality conditions of the problem (P) via second-order contingent epiderivative and second-order ρ-cone convexity assumptions.

Theorem 3.2. *(Second-order sufficient optimality conditions) Let x' be a feasible point of the problem* (P)*, $y' \in F(x')$, and $z' \in G(x') \cap (-L)$. Let $u \in A$, $v \in F(u) + K$, $w \in G(u) + L$, $e \in \text{int}(K)$, and $e' \in \text{int}(L)$. Let $\rho_1, \rho_2 \in \mathbb{R}$. Assume that F is second-order ρ_1-K-convex with respect to e at (x', y') in the direction $(u - x', v - y')$ and G is second-order ρ_2-L-convex with respect to e' at (x', z') in the direction $(u - x', w - z')$, on A. Suppose that there exists $(y^*, z^*) \in K^+ \times L^+$, with $y^* \neq \theta_{Y^*}$, satisfying*

$$\rho_1 \langle y^*, e \rangle + \rho_2 \langle z^*, e' \rangle \geq 0, \tag{3.1}$$

such that

$$\begin{aligned} &\langle y^*, D_{\uparrow}^2 F(x', y', u - x', v - y')(x - x') \rangle \\ &+ \langle z^*, D_{\uparrow}^2 G(x', z', u - x', w - z')(x - x') \rangle \geq 0, \forall x \in A \end{aligned} \tag{3.2}$$

and

$$\langle z^*, z' \rangle = 0. \tag{3.3}$$

Then (x', y') is a weak minimizer of the problem (P)*.*

Proof. We prove the theorem by the method of contradiction.
Suppose that (x', y') is not a weak minimizer of the problem (P).
Then there exists $x \in S$ and $y \in F(x)$, such that

$$y - y' \in (-\text{int}(K)).$$

Hence,

$$\langle y^*, y - y' \rangle < 0, \text{ since } \theta_{Y^*} \neq y^* \in K^+.$$

Since $x \in S$, there exists an element $z \in G(x) \cap (-L)$.
Therefore,

$$\langle z^*, z \rangle \leq 0, \text{ as } z^* \in L^+.$$

We have

$$\langle z^*, z - z' \rangle \leq 0, \text{ as } \langle z^*, z' \rangle = 0.$$

Hence,

$$\langle y^*, y - y' \rangle + \langle z^*, z - z' \rangle < 0. \tag{3.4}$$

As F is second-order ρ_1-K-convex with respect to e at (x', y') in the direction $(u - x', v - y')$ and G is second-order ρ_2-L-convex with respect to e' at (x', z') in the direction $(u - x', w - z')$, on A, we have

$$F(x) - y' \in D^2_{\uparrow}F(x', y', u - x', v - y')(x - x') + \rho_1\|x - x'\|^2 e + K$$

and

$$G(x) - z' \in D^2_{\uparrow}G(x', z', u - x', w - z')(x - x') + \rho_2\|x - x'\|^2 e' + L.$$

So,

$$y - y' \in D^2_{\uparrow}F(x', y', u - x', v - y')(x - x') + \rho_1\|x - x'\|^2 e + K$$

and

$$z - z' \in D^2_{\uparrow}G(x', z', u - x', w - z')(x - x') + \rho_2\|x - x'\|^2 e' + L.$$

Hence, from (3.1) and (3.2), we have

$$\langle y^*, y - y'\rangle + \langle z^*, z - z'\rangle \geq 0,$$

which contradicts (3.4).
Consequently, (x', y') is a weak minimizer of (P). □

We illustrate Theorem 3.2 by the following example.

Example 3.2. *We consider a primal problem* (P)*, where* $X = \mathbb{R}$*, the set-valued map* $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ *is given in Example 3.1, and* $G : \mathbb{R} \to 2^{\mathbb{R}^2}$ *is defined as*

$$G(\lambda) = \begin{cases} \{(x^2 + \lambda^2, x^2 + \lambda^2) : x \in \mathbb{R}\}, & \text{if } \lambda \neq 2, \\ \{(x + 4, x + 4) : x \geq 0\}, & \text{if } \lambda = 2. \end{cases}$$

Let $K = \mathbb{R}^2_+$ *and* $L = \mathbb{R}^2_+$*. We have the epigraph of the set-valued map* G *as*

$$\begin{aligned} \text{epi}(G) = &\{(\lambda, (x, y)) : \lambda \in \mathbb{R}, x \geq 0, y \geq 0\} \\ &\cup \{(\lambda, (x, y)) : \lambda \geq 0, x \geq 4, y \geq 4\}. \end{aligned}$$

Suppose $(\lambda', (x', y')) = (0, (0, 0))$*,* $(\lambda', (w', z')) = (0, (0, 0))$*,* $(\overline{\lambda}, (u, v)) = (1, (0, 0))$*, and* $(\overline{\lambda}, (w, z)) = (1, (1, 1))$*. Obviously,* $(x', y') \in F(\lambda')$*,* $(w', z') \in G(\lambda') \cap (-\mathbb{R}^2_+)$*,* $(u, v) \in F(\overline{\lambda}) + \mathbb{R}^2_+$*, and* $(w, z) \in G(\overline{\lambda}) + \mathbb{R}^2_+$*. Therefore,*

$$T^2(\text{epi}(G), (0, (0, 0)), (1, (1, 1))) = \{(\lambda, (x, y)) : \lambda \in \mathbb{R}, x \geq 0, y \geq 0\}.$$

Hence,

$$D^2_{\uparrow}G(0, (0, 0), 1, (1, 1))(\lambda)$$

$$= I\text{-}min \; \{(x, y) : (\lambda, (x, y)) \in T^2(\text{epi}(G), (0, (0, 0)), (1, (1, 1)))\}.$$

Therefore, $\mathrm{dom}(D^2_{\uparrow}G(0,(0,0),1,(1,1))) = \mathbb{R}$ *and* $D^2_{\uparrow}G(0,(0,0),1,(1,1))(\alpha) = \{(0,0)\}$, *for all* $\alpha \in \mathbb{R}$.
Choose $\rho_1 = -1$ *and* $\rho_2 = 1$.
We have,

$$(x^2+\lambda^2, x^2+\lambda^2) \in (\lambda^2,\lambda^2) + \mathbb{R}^2_+, \text{ for } x \in \mathbb{R}$$

and

$$(x+4, x+4) \in (4,4) + \mathbb{R}^2_+, \text{ for } x \geq 0.$$

Hence,

$$G(\lambda) - (0,0) \subseteq D^2_{\uparrow}G(0,(0,0),1,(1,1))(\lambda) + \rho_2|\lambda - 0|^2(1,1) + \mathbb{R}^2_+, \forall \lambda \in \mathbb{R}.$$

Therefore, G *is second-order* ρ_2-$\mathbb{R}^2_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^2}$ *at* $(0,(0,0))$ *in the direction* $(1,(1,1))$ *on* $\mathbb{R}$. *Again, from Example 3.1,* F *is second-order* ρ_1-$\mathbb{R}^2_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^2}$ *at* $(0,(0,0))$ *in the direction* $(1,(0,0))$ *on* $\mathbb{R}$. *It is clear that for* $y^* = z^* = (1,1)$, *Eqs. (3.2) and (3.3) are satisfied. Therefore,* $(\lambda',(x',y')) = (0,(0,0))$ *is a weak minimizer of the problem* (P). □

3.2. SECOND-ORDER MOND-WEIR TYPE DUAL

We consider a second-order Mond-Weir type dual (MWD) of (P), where F and G are second-order contingent epiderivable set-valued maps.
Let $u \in A, v \in F(u) + K$, and $w \in G(u) + L$.

$$\begin{aligned}
&\text{maximize} \quad y' && (MWD)\\
&\text{subject to}\\
&\quad \langle y^*, D^2_{\uparrow}F(x',y',u-x',v-y')(x-x')\rangle\\
&\quad + \langle z^*, D^2_{\uparrow}G(x',z',u-x',w-z')(x-x')\rangle \geq 0, \forall x \in A,\\
&\quad \langle z^*, z'\rangle \geq 0,\\
&\quad x' \in A, y' \in F(x'), z' \in G(x'), y^* \in K^+, z^* \in L^+, \text{ and } \langle y^*, e\rangle = 1.
\end{aligned}$$

Definition 3.3. *A feasible point* (x',y',z',y^*,z^*) *of the problem* (MWD) *is called a weak maximizer of* (MWD) *if for all feasible points* (x,y,z,y_1^*,z_1^*) *of* (MWD),

$$y - y' \notin \mathrm{int}(K).$$

Theorem 3.3. *(Second-order weak duality) Let* $x_0 \in S$ *and* (x',y',z',y^*,z^*) *be a feasible point of* (MWD). *Assume that* F *is second-order* ρ_1-K*-convex with respect to* e *at* (x',y') *in the direction* $(u-x',v-y')$ *and* G *is second-order* ρ_2-L*-convex with respect to* e' *at* (x',z') *in the direction* $(u-x',w-z')$, *on* A, *satisfying* (3.1). *Then,*

$$F(x_0) - y' \subseteq Y \setminus (-\mathrm{int}(K)).$$

Proof. We prove the theorem by the method of contradiction.
Suppose for some $y_0 \in F(x_0)$,

$$y_0 - y' \in (-\text{int}(K)).$$

Therefore,

$$\langle y^*, y_0 - y' \rangle < 0, \text{ as } \theta_{Y^*} \neq y^* \in K^+.$$

Again, since $x_0 \in S$, we have

$$G(x_0) \cap (-L) \neq \emptyset.$$

Choose $z_0 \in G(x_0) \cap (-L)$.
So,

$$\langle z^*, z_0 \rangle \leq 0, \text{ as } z^* \in L^+.$$

Again, from the constraints of (MWD), we have

$$\langle z^*, z' \rangle \geq 0.$$

So,

$$\langle z^*, z_0 - z' \rangle = \langle z^*, z_0 \rangle - \langle z^*, z' \rangle \leq 0.$$

Hence,

$$\langle y^*, y_0 - y' \rangle + \langle z^*, z_0 - z' \rangle < 0. \tag{3.5}$$

As F is second-order ρ_1-K-convex with respect to e at (x', y') in the direction $(u - x', v - y')$ and G is second-order ρ_2-L-convex with respect to e' at (x', z') in the direction $(u - x', w - z')$, on A, we have

$$F(x_0) - y' \in D^2_\uparrow F(x', y', u - x', v - y')(x_0 - x') + \rho_1 \|x_0 - x'\|^2 e + K$$

and

$$G(x_0) - z' \in D^2_\uparrow G(x', z', u - x', w - z')(x_0 - x') + \rho_2 \|x_0 - x'\|^2 e' + L.$$

So,

$$y_0 - y' \in D^2_\uparrow F(x', y', u - x', v - y')(x_0 - x') + \rho_1 \|x_0 - x'\|^2 e + K$$

and

$$z_0 - z' \in D^2_\uparrow G(x', z', u - x', w - z')(x_0 - x') + \rho_2 \|x_0 - x'\|^2 e' + L.$$

Hence, from the constraints of (MWD) and (3.1),

$$\langle y^*, y_0 - y' \rangle + \langle z^*, z_0 - z' \rangle \geq 0,$$

which contradicts (3.5).
Therefore,

$$y_0 - y' \notin (-\text{int}(K)).$$

Hence,

$$F(x_0) - y' \subseteq Y \setminus (-\text{int}(K)),$$

which completes the proof of the theorem. □

Theorem 3.4. *(Second-order strong duality) Let (x', y') be a weak minimizer of the problem* (P) *and $z' \in G(x') \cap (-L)$. Assume that for some $(y^*, z^*) \in K^+ \times L^+$, with $\langle y^*, e\rangle = 1$, Eqs. (3.2) and (3.3) are satisfied at (x', y', z', y^*, z^*). Then (x', y', z', y^*, z^*) is a feasible solution of (MWD). If the second-order weak duality Theorem 3.3 holds between* (P) *and (MWD), then (x', y', z', y^*, z^*) is a weak maximizer of (MWD).*

Proof. Since the Eqs. (3.2) and (3.3) are satisfied at (x', y', z', y^*, z^*), we have

$$\langle y^*, D^2_{\uparrow}F(x', y', u - x', v - y')(x - x')\rangle$$
$$+ \langle z^*, D^2_{\uparrow}G(x', z', u - x', w - z')(x - x')\rangle \geq 0, \forall x \in A$$

and

$$\langle z^*, z'\rangle = 0.$$

Hence, (x', y', z', y^*, z^*) is a feasible solution of (MWD).
Suppose that the second-order weak duality Theorem 3.3 holds between (P) and (MWD) and (x', y', z', y^*, z^*) is not a weak maximizer of (MWD).
So there exists a feasible point (x, y, z, y_1^*, z_1^*) of (MWD), such that

$$y' - y \in (-\text{int}(K)).$$

It contradicts the second-order weak duality Theorem 3.3 between (P) and (MWD).
Consequently, (x', y', z', y^*, z^*) is a weak maximizer of (MWD). □

Theorem 3.5. *(Second-order converse duality) Let (x', y', z', y^*, z^*) be a feasible point of the problem (MWD). Assume that F is second-order ρ_1-K-convex with respect to e at (x', y') in the direction $(u - x', v - y')$ and G is second-order ρ_2-L-convex with respect to e' at (x', z') in the direction $(u - x', w - z')$, on A, satisfying* (3.1)*. If $x' \in S$, then (x', y') is a weak minimizer of* (P).

Proof. We prove the theorem by the method of contradiction.
Suppose that (x', y') is not a weak minimzer of the problem (P).
Then there exists $x \in S$ and $y \in F(x)$, such that

$$y - y' \in (-\text{int}(K)).$$

Therefore,

$$\langle y^*, y - y' \rangle < 0, \text{ as } \theta_{Y^*} \neq y^* \in K^+.$$

Again, since $x \in S$,

$$G(x) \cap (-L) \neq \emptyset.$$

Let

$$z \in G(x) \cap (-L).$$

So,

$$\langle z^*, z \rangle \leq 0, \text{ as } z^* \in L^+.$$

From the constraints of (MWD), we have

$$\langle z^*, z' \rangle \geq 0.$$

Therefore,

$$\langle z^*, z - z' \rangle = \langle z^*, z \rangle - \langle z^*, z' \rangle \leq 0.$$

Hence,

$$\langle y^*, y - y' \rangle + \langle z^*, z - z' \rangle < 0. \tag{3.6}$$

As F is second-order ρ_1-K-convex with respect to e at (x', y') in the direction $(u - x', v - y')$ and G is second-order ρ_2-L-convex with respect to e' at (x', z') in the direction $(u - x', w - z')$, on A, we have

$$F(x) - y' \in D^2_\uparrow F(x', y', u - x', v - y')(x - x') + \rho_1 \|x - x'\|^2 e + K$$

and

$$G(x) - z' \in D^2_\uparrow G(x', z', u - x', w - z')(x - x') + \rho_2 \|x - x'\|^2 e' + L.$$

So,

$$y - y' \in D^2_\uparrow F(x', y', u - x', v - y')(x - x') + \rho_1 \|x - x'\|^2 e + K$$

and

$$z - z' \in D^2_\uparrow G(x', z', u - x', w - z')(x - x') + \rho_2 \|x - x'\|^2 e' + L.$$

Hence, from the constraints of (MWD) and (3.1), we have

$$\langle y^*, y - y' \rangle + \langle z^*, z - z' \rangle \geq 0,$$

which contradicts (3.6).
Consequently, (x', y') is a weak minimizer of (P). □

The second-order Mond-Weir type duality results are verified by the following example.

Example 3.3. *We consider a primal problem* (P) *defined in Example 3.2. Let* $K = \mathbb{R}^2_+$ *and* $L = \mathbb{R}^2_+$. *Here* $S = \mathbb{R}_+$. *Take* $\overline{\lambda} = 0$, $(u, v) = (0, 0)$, *and* $(w, z) = (0, 0)$. *Obviously,* $(0, 0) \in F(0) + \mathbb{R}^2_+$ *and* $(0, 0) \in G(0) + \mathbb{R}^2_+$.
For $(\lambda', (x', y')) \in \mathrm{gr}(F)$, *we have*

$$T^2(\mathrm{epi}(F), (\lambda', (x', y')), (-\lambda', (-x', -y')))$$
$$= \begin{cases} \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, & \text{if } \lambda' \in \mathbb{R}, (x', y') = (0, 0), \\ \mathbb{R} \times \mathbb{R} \times \mathbb{R}, & \text{if } \lambda' \neq 2, x' = t - \lambda'^2 > 0, y' = \sqrt{t} - \lambda'^2 > 0, t \in \mathbb{R}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

As I-min $\{(x, y) : (\lambda, (x, y)) \in T^2(\mathrm{epi}(F), (\lambda', (x', y')), (-\lambda', (-x', -y')))\} = \{(0, 0)\}$, *for* $\lambda' \in \mathbb{R}$ *and* $(x', y') = (0, 0)$,

$$D^2_{\uparrow} F(\lambda', (0, 0), -\lambda', (0, 0))(\lambda) = \{(0, 0)\}, \forall \lambda \in \mathbb{R}.$$

Therefore,

$$\mathrm{dom}(D^2_{\uparrow} F(\lambda', (0, 0), -\lambda', (0, 0))) = \mathbb{R}, \forall \lambda' \in \mathbb{R}.$$

Also, for $(\lambda', (w', z')) \in \mathrm{gr}(G)$, *we have*

$$T^2(\mathrm{epi}(G), (\lambda', (w', z')), (-\lambda', (-w', -z')))$$
$$= \begin{cases} \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, & \text{if } (\lambda', (w', z')) = (0, (0, 0)), \\ \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, & \text{if } \lambda' = 0, 0 < w' \leq 4, z' = w' = t^2, t \in \mathbb{R}, \\ \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, & \text{if } \lambda' \neq 2, (w', z') = (0, 0), \\ \emptyset, & \text{if } \lambda' = 2, (w', z') = (4, 4), \\ \mathbb{R} \times \mathbb{R} \times \mathbb{R}, & \text{otherwise.} \end{cases}$$

As I-min $\{(x, y) : (\lambda, (x, y)) \in T^2(\mathrm{epi}(G), (\lambda', (w', z')), (-\lambda', (-w', -z')))\} = \{(0, 0)\}$, *for* $(\lambda', (w', z')) = (0, (0, 0))$ *and for* $\lambda' \neq 2, (w', z') = (0, 0)$,

$$D^2_{\uparrow} G(0, (0, 0), 0, (0, 0))(\lambda) = \{(0, 0)\}, \forall \lambda \geq 0$$

and

$$D^2_{\uparrow} G(\lambda', (0, 0), -\lambda', (0, 0))(\lambda) = \{(0, 0)\}, \forall \lambda \in \mathbb{R}, \text{ for } \lambda' \neq 2.$$

Therefore, $\mathrm{dom}(D^2_{\uparrow} G(0, (0, 0), 0, (0, 0))) = \mathbb{R}_+$ *and*

$$\mathrm{dom}(D^2_{\uparrow} G(\lambda', (0, 0), -\lambda', (0, 0))) = \mathbb{R}, \forall \lambda' \neq 2.$$

We have the second-order Mond-Weir type dual problem as

$$\begin{aligned}
&\textit{maximize } (x', y') \\
&\textit{subject to} \\
&\qquad \langle y^*, D^2_{\uparrow} F(\lambda', (-x', -y'), -\lambda', (x', y'))(x - x')\rangle \\
&\qquad + \langle z^*, D^2_{\uparrow} G(\lambda', (w', z'), -\lambda', (-w', -z'))(x - x')\rangle \geq 0, \forall x \in \mathbb{R}, \\
&\qquad \langle z^*, (w', z')\rangle \geq 0, \\
&\qquad \lambda' \in \mathbb{R}, (x', y') \in F(\lambda'), (w', z') \in G(\lambda'), \\
&\qquad \langle y^*, (1, 1)\rangle = 1, y^* \in \mathbb{R}^2_+ \setminus \{(0, 0)\}, \text{ and } z^* \in \mathbb{R}^2_+.
\end{aligned}$$

Here, the feasible set of the Mond-Weir type dual problem is $\{(\lambda', (0,0), (0,0), y^*, z^*) : \lambda' \geq 0, \langle y^*, (1,1)\rangle = 1\}$. *Therefore, the only weak maximal point of the Mond-Weir type dual problem is* $(0, 0)$. □

3.3. SECOND-ORDER WOLFE TYPE DUAL

We consider a second-order Wolfe type dual (WD) of (P), where F and G are second-order contingent epiderivable set-valued maps.
Let $u \in A, v \in F(u) + K$, and $w \in G(u) + L$.

$$\begin{aligned}
&\text{maximize} \quad y' + \langle z^*, z'\rangle e \qquad\qquad (WD) \\
&\text{subject to} \\
&\qquad \langle y^*, D^2_{\uparrow} F(x', y', u - x', v - y')(x - x')\rangle \\
&\qquad + \langle z^*, D^2_{\uparrow} G(x', z', u - x', w - z')(x - x')\rangle \geq 0, \forall x \in A, \\
&\qquad x' \in A, y' \in F(x'), z' \in G(x'), y^* \in K^+, z^* \in L^+, \text{ and } \langle y^*, e\rangle = 1.
\end{aligned}$$

Definition 3.4. *A feasible point* (x', y', z', y^*, z^*) *of the problem* (WD) *is called a weak maximizer of* (WD) *if for all feasible points* (x, y, z, y_1^*, z_1^*) *of* (WD),

$$(y + \langle z_1^*, z\rangle e) - (y' + \langle z^*, z'\rangle e) \notin \text{int}(K).$$

We prove the duality results of second-order Wolfe type of the problem (P). The proofs are very similar to Theorems 3.3 - 3.5, and hence omitted.

Theorem 3.6. *(Second-order weak duality) Let* $x_0 \in S$ *and* (x', y', z', y^*, z^*) *be a feasible point of the problem* (WD). *Assume that* F *is second-order* ρ_1-K*-convex with respect to* e *at* (x', y') *in the direction* $(u - x', v - y')$ *and* G *is second-order* ρ_2-L*-convex with respect to* e' *at* (x', z') *in the direction* $(u - x', w - z')$, *on* A, *satisfying* (3.1). *Then,*

$$F(x_0) - (y' + \langle z^*, z'\rangle e) \subseteq Y \setminus (-\text{int}(K)).$$

Theorem 3.7. *(Second-order strong duality) Let* (x', y') *be a weak minimizer of the problem* (P) *and* $z' \in G(x') \cap (-L)$. *Assume that for some* $(y^*, z^*) \in K^+ \times L^+$, *with* $\langle y^*, e\rangle = 1$, *Eqs. (3.2) and (3.3) are satisfied at* (x', y', z', y^*, z^*). *Then* (x', y', z', y^*, z^*) *is a feasible solution of* (WD). *If the second-order weak duality Theorem 3.6 holds between* (P) *and* (WD), *then* (x', y', z', y^*, z^*) *is a weak maximizer of* (WD).

Theorem 3.8. *(Second-order converse duality) Let* (x', y', z', y^*, z^*) *be a feasible point of the problem* (WD) *and* $\langle z^*, z'\rangle \geq 0$. *Assume that* F *is second-order* ρ_1-K-*convex with respect to* e *at* (x', y') *in the direction* $(u - x', v - y')$ *and* G *is second-order* ρ_2-L-*convex with respect to* e' *at* (x', z') *in the direction* $(u - x', w - z')$, *on* A, *satisfying* (3.1). *If* $x' \in S$, *then* (x', y') *is a weak minimizer of* (P).

The second-order Wolfe type duality results are verified by the following example.

Example 3.4. *Take the primal problem* (P) *of Example 3.3. Here, we have the second-order Wolfe type dual problem as*

$$\text{maximize } (x', y') + \langle z^*, (w', z')\rangle(1,1)$$
$$\text{subject to}$$
$$\langle y^*, D^2_{\uparrow}F(\lambda', (-x', -y'), -\lambda', (x', y'))(x - x')\rangle$$
$$+ \langle z^*, D^2_{\uparrow}G(\lambda', (w', z'), -\lambda', (-w', -z'))(x - x')\rangle \geq 0, \forall x \in \mathbb{R},$$
$$\lambda' \in \mathbb{R}, (x', y') \in F(\lambda'), (w', z') \in G(\lambda'),$$
$$\langle y^*, (1,1)\rangle = 1, y^* \in \mathbb{R}^2_+ \setminus \{(0,0)\}, \text{ and } z^* \in \mathbb{R}^2_+.$$

Here, the feasible set of the Wolfe type dual problem is

$$\{(\lambda', (0,0), (0,0), y^*, z^*) : \lambda' \geq 0, \langle y^*, (1,1)\rangle = 1\}.$$

Therefore, the only weak maximal point of the Wolfe type dual problem is $(0,0)$. □

3.4. SECOND-ORDER MIXED TYPE DUAL

We consider a second-order mixed dual (MD) of (P), where F and G are second-order contingent epiderivable set-valued maps.
Let $u \in A, v \in F(u) + K$, and $w \in G(u) + L$.

$$\text{maximize} \quad y' + \langle z^*, z'\rangle e \tag{MD}$$
$$\text{subject to}$$
$$\langle y^*, D^2_{\uparrow}F(x', y', u - x', v - y')(x - x')\rangle$$
$$+ \langle z^*, D^2_{\uparrow}G(x', z', u - x', w - z')(x - x')\rangle \geq 0, \forall x \in A,$$
$$\langle z^*, z'\rangle \geq 0,$$
$$x' \in A, y' \in F(x'), z' \in G(x'), y^* \in K^+, z^* \in L^+, \text{ and } \langle y^*, e\rangle = 1.$$

Definition 3.5. *A feasible point* (x', y', z', y^*, z^*) *of the problem* (MD) *is called a weak maximizer of* (MD) *if for all feasible points* (x, y, z, y_1^*, z_1^*) *of* (MD),

$$(y + \langle z_1^*, z\rangle e) - (y' + \langle z^*, z'\rangle e) \notin \text{int}(K).$$

We prove the duality results of second-order mixed type of the problem (P). The proofs are very similar to Theorems 3.3 - 3.5, and hence omitted.

Theorem 3.9. *(Second-order weak duality) Let* $x_0 \in S$ *and* (x', y', z', y^*, z^*) *be a feasible point of the problem* (MD)*. Assume that* F *is second-order* ρ_1*-*K*-convex with respect to e at* (x', y') *in the direction* $(u - x', v - y')$ *and* G *is second-order* ρ_2*-*L*-convex with respect to* e' *at* (x', z') *in the direction* $(u - x', w - z')$*, on* A*, satisfying* (3.1)*. Then,*

$$F(x_0) - (y' + \langle z^*, z' \rangle e) \subseteq Y \setminus (-\text{int}(K)).$$

Theorem 3.10. *(Second-order strong duality) Let* (x', y') *be a weak minimizer of the problem* (P) *and* $z' \in G(x') \cap (-L)$*. Assume that for some* $(y^*, z^*) \in K^+ \times L^+$*, with* $\langle y^*, e \rangle = 1$*, Eqs. (3.2) and (3.3) are satisfied at* (x', y', z', y^*, z^*)*. Then* (x', y', z', y^*, z^*) *is a feasible solution of* (MD)*. If the second-order weak duality Theorem 3.9 holds between* (P) *and* (MD)*, then* (x', y', z', y^*, z^*) *is a weak maximizer of* (MD)*.*

Theorem 3.11. *(Second-order converse duality) Let* (x', y', z', y^*, z^*) *be a feasible point of the problem* (MD)*. Assume that* F *is second-order* ρ_1*-*K*-convex with respect to e at* (x', y') *in the direction* $(u - x', v - y')$ *and* G *is second-order* ρ_2*-*L*-convex with respect to* e' *at* (x', z') *in the direction* $(u - x', w - z')$*, on* A*, satisfying* (3.1)*. If* $x' \in S$*, then* (x', y') *is a weak minimizer of* (P)*.*

4. CONCLUSIONS

In this paper, we establish the sufficient second-order KKT conditions of a set-valued optimization problem (P) via the second-order contingent epiderivative and second-order ρ-cone convexity assumptions. We also formulate the second-order Mond-Weir, Wolfe, and mixed types duals of the problem (P) and prove the corresponding duality results.

Acknowledgement: The authors are very thankful to referees for their valuable comments which improved the presentation of the paper.

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