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ON WELLPOSEDNESS QUADRATIC FUNCTION MINIMIZATION PROBLEM ON INTERSECTION OF TWO ELLIPSOIDS*

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Abstract: This paper deals with the existence of solutions and the conditions for the strong convergence of minimizing sequences towards the set of solutions of the quadratic function minimization problem on the intersection of two ellipsoids in Hilbert space.

Keywords: Quadratic functional, minimization, wellposedness.

1. INTRODUCTION

Suppose that H, F, G_1 and G_2 are Hilbert spaces; $A: H \to F$, $H \to G_1$ and $C: H \to G_2$ - bounded linear operators; $f \in F$ a fixed element; $r_1 > 0$ and $r_2 > 0$ are given real numbers; U_1 and U_2 ellipsoids in the space H defined by operators B and C:

$$U_1 = \{u \in H : \| Bu \| \le r_1\},\ U_2 = \{u \in H : \| Cu \| \le r_2\} .$$

This paper deals with the extremal problem:

$$J(u) = \| Au - f \|^2 \to \inf, \quad u \in U = U_1 \cap U_2 . \tag{1}$$

We study the existence of solutions and the wellposedness of the problem in the Tikhonov sense.

As an example of the problem of this type, we can quote the problem of minimization of the function

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$$J(u) = \| x(T,u) - z \|^2_{R^n} \ ,$$

where $z \in R^n$ and $x(t,u)$ is a solution of the system of differential equation

$$x'(t) = B(t)x(t) + D(t)u(t), \quad t \in (0,T), \ x(0) = 0 \in R^n \ ,$$

$$\| u(t) \|_{L_2^r} := \int_0^T | u(t) |_{R_n^2} \, dt \le r_1, \quad \| x(t,u) \| = \int_0^T | x(t,u(t)) |^2 \, dt \le r_2$$

with given matrices $B(\cdot) = (b_{ij}(\cdot))_{n \times n}$ and $D(\cdot) = (d_{ij}(\cdot))_{n \times r}$. These conditions guarantee the existence of the solution $x(t,u) \in H_n^1[0,T]$ of the previous system for each $u \in L_2^r[0,T]$.

The same problem with different set of constraints U, was considered in [1], [2] and [3]. In [3], the set of constraints U was a ball. In [2] the necessary and sufficient conditions for the existence of a solution have been considered in the case when U is a half-space, as have sufficient conditions in the case when U is an ellipsoid. Finally, the paper [1] contains sufficient and necessary conditions for these problems when the set of constraints U is a polyhedron.

It should be pointed out that this article deals with the wellposedness problem with the exact initial date which is also the case in the papers [1], [2] and [3]. Methods for approximate solving of problem (1) with errors in the initial data are considered, for example, in [3], [4], [5].

2. AUXILIARY RESULTS

Let us introduce the following notions: $R(A) = \{Au : u \in H\}$ - the set of operators values A, $A(U) = \{Au : u \in U\}$, $\operatorname{Ker} A = \{u \in H : Au = 0\}$ - kernel of A; $\bar{M}$ is the closure of the set M in the space H; $L^{\perp}$ is the orthogonal complement of the subspace $L \subseteq H$; P is the operator orthogonally projecting the space H on the closed subspace $\overline{R(A^*)}$; Pr - operator projecting the space F on the closed and convex set $\overline{A(U)}$; B_A - restriction on the operator B on the subspace $\operatorname{Ker} A$; C_{AB} - restriction of the operator C on the subspace $\operatorname{Ker} A \cap \operatorname{Ker} B$.

Generally, every linear bounded operator $A : H \to F$ generates the decomposition

$$H = \overline{R(A^*)} \oplus \operatorname{Ker} A \ . \tag{2}$$

Lemma 1. *The operators* A, B *and* C *generate the following orthogonal decompositions of the space* H*:*

$$H = \overline{R(A^*)} \oplus \overline{R(B_A^*)} \oplus \overline{R(C_{AB}^*)} \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C) \ , \tag{3}$$

$$H = \overline{R(B^*)} \oplus \overline{R(A_B^*)} \oplus \overline{R(C_{AB}^*)} \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C) . \tag{4}$$

Proof: By applying (2) on operators $B_A : \operatorname{Ker} A \to G_1$ and $C_{AB} : \operatorname{Ker} A \cap \operatorname{Ker} B \to G_2$ we get

$$\operatorname{Ker} A = \overline{R(B^*)} \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B) ,$$

and

$$\operatorname{Ker} A \cap (\operatorname{Ker} B)H = \overline{R(C_{AB}^*)} \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C) .$$

According to these relations and (2), the relation (3) follows. Relation (4) can be proved in the similar way. ♦

In order to formulate the next statements we need the following definition.

It is said that the operator A *is normal solvable, if the condition* $R(A) = \overline{R(A)}$ *is fulfilled. This condition is equivalent to* $\overline{R(A^*)} = R(A^*)$ ([4]).

Lemma 2. [4] *Linear bounded operator* $A : H \to F$ *is normal solvable if and only if*

$$m_A = \inf\{\| Au \| : u \perp \operatorname{Ker} A, \ \| u \| = 1\} > 0 ,$$

and than we have

$$(\forall x \in R(A^*)) \ \ m_A \| x \| \le \| Ax \| . \tag{5}$$

This Lemma Immediately implies

Lemma 3. *If a linear bounded operator* $A : H \to F$ *is not normal solvable then there exists a sequence* (p_n) *such that*

$$(\forall n \in N) \ \ p_n \in \overline{R(A^*)}, \ \| p_n \| = 1, \ p_n \to 0, \ Ap_n \to 0 \ \ (n \to \infty) .$$

3. EXISTENCE OF SOLUTION

It is obvious that for a given $f \in F$, the problem (1) has a solution, if and only if $Pr(f) \in A(U)$. Since $Pr(F) \in A(U)$, we have that problem (1) has a solution for every $f \in F$, if and only if $A(U) = \overline{A(U)}$.

Function J is weakly lower semicontinuous since it is convex and continuous. The set U is weakly closed since it is convex and closed in the norm of H. Suppose that (u_n) is minimizing sequence of problem (1), i.e. that

$$u_n \in U, \ n = 1,2,... ; \text{ and } \lim_{n \to \infty} J(u_n) = J_* := \inf\{J(u) : u \in U\} .$$

If for some $f \in F$ the sequence (u_n) is bounded, then for such f problem (1) has a solution. Namely, in that case there exists a subsequence (u_{n_k}) of the sequence (u_n) and a point $u_* \in U$ such that

$$u_{n_k} \to u_* \text{ as } k \to \infty .$$

Since the set U is weakly closed, it follows that $u_* \in U$. The function J is weakly lower semicontinuous and hence

$$J(u_*) \le \liminf_{k\to\infty} J(u_{n_k}) = J_* .$$

According to this we have that $J(u_*) = J_*$. It means that

$$u_* \in U_* := U_* := \{u \in U : J(u) = J_*\} .$$

If $U_* \neq \varnothing$, then it is easy to prove that for each $u_* \in U_*$ we have the equation

$$U_* = (u_* + \operatorname{Ker} A) \bigcap U . \tag{6}$$

By using the equation

$$J(u) = J(v) + \langle J'(v), u - v\rangle + \| A(u-v) \|^2, \quad u,v \in U ,$$

and the optimality conditions

$$(\forall u \in U)\langle J'(u_*), u - u_*\rangle \ge 0$$

we get the inequality

$$\| A(u-u_*) \|^2 \le J(u) - J_* .$$

From here we have that (u_n) is a minimizing sequence of problem (1) if and only if

$$Au_n \to Au_*, \quad n \to \infty .$$

If A is a normal solvable operator, then according to (5), we get

$$m_A \| Pu_n - Pu_* \| \le \| Au_n - Au_* \| \to 0, \quad n \to \infty ,$$

that is

$$Pu_n \to Pu_*, \quad n \to \infty . \tag{7}$$

Theorem 1. *If*

(i) A *is a normal solvable operator,*

(ii) $B(\operatorname{Ker} A)$ *- closed subspace of space* G_1 *,*

(iii) $C(\operatorname{Ker} A \bigcap \operatorname{Ker} B)$ *- closed subspace of space* G_2 *,*

then problem (1) has a solution for each $f \in F$ *.*

Proof: According to the theorem conditions, the equation (3) may be written down as

$$H = R(A^*) \oplus R(B_A^*) \oplus R(C_{AB}^*) \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C) .$$

Let (u_n) be a minimizing sequence. Then

$$u_n = Pu_n + x_n + y_n + z_n, \quad x_n \in R(B_A^*), \ y_n \in R(C_{AB}^*), \ z_n \in \operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C .$$

Sequence (v_n), $v_n = Pu_n + x_n + y_n$ is also a minimizing sequence. Besides, $Bv_n = P(Bu_n + x_n)$, that is

$$\| B(Pu_n + x_n) \| \le r_1, \quad \| C(Pu_n + x_n + y_n) \| \le r_2 .$$

According to (7) we have that sequence (Pu_n) is bounded. By using conditions ii) and iii) and applying relation (5) on the operators B_A and C_{AB}, we conclude that the sequences (x_n) and (y_n) are bounded. On the basis of this, the sequence (v_n) is a bounded minimizing sequence. ♦

By using the decomposition (4) and a similar decomposition, we may prove that operators A, B and C in Theorem 1 may mutually change their places. Let us mention one of these cases.

Theorem 2. *If*

(i) B *is a normal solvable operator,*

(ii) $A(\operatorname{Ker} B)$ *- closed subspace of space* F *,*

(iii) $C(\operatorname{Ker} A \cap \operatorname{Ker} B)$ *- closed subspace of space* G_2*,*

then problem (1) has a solution for each $f \in F$.

4. WELLPOSEDNESS

Let in the following definition $U \subseteq H$ be an arbitrary closed and convex set, and J an arbitrary real function defined on the set U .

Definition. [*1*], [*4*], [*5*] *We say that the extremal problem*

$$J(u) \to \inf, \quad u \in U$$

is wellposed in the sense of Tikhonov if the following conditions are satisfied:

(i) $J_* = \inf\{J(u) : u \in U\} > -\infty$ *;*

(ii) $U_* = \inf\{u \in U : J(u) = J_*\} \ne \varnothing$ *;*

(iii) for each minimizing sequence (u_n) *we have*

$$d(u_n, U_*) = \inf\{\| u_n - u \| : u \in U_*\} \to 0 \text{ when } n \to \infty .$$

If at least one condition from this definition is not valid, we will say that the problem is *illposed*.

The following example shows that conditions of Theorem 1, in general, do not guarantee the wellposedness of the problem (1).

Example. Let $L=\{x\in l_2 : x=(0,x_2,x_2/3,x_3,x_3/5,...)\}$ and A be operator of the orthogonal projection on $L^{\perp}$. Operator A is normal solvable. Let operators $B,C:l_2\to l_2$ be defined as follows:

$$Bx=(0,x_2,x_3,...,),\quad Cx=(x_1,0,x_3,0,...,),\quad x=(x_1,x_2,x_3,...)\in l_2 .$$

Here we have that

$$\operatorname{Ker} A=L,\ \operatorname{Ker} B=\{x\in l_2 : x=(x_1,0,0,...)\},\ \operatorname{Ker} C=\{x\in l_2 : x=(0,x_2,0,x_4,...)\},$$

we can see that $B(\operatorname{Ker} A)=B(L)=L$ and $\operatorname{Ker} A\bigcap \operatorname{Ker} B=\{0\}$. It means that for sets

$$U_1=\{u\in l_2 :\| Bu\|\le 1\},\quad U_2=\{u\in l_2 :\| Cu\|\le 1\},$$

and for the element $f=(1,0,0,...)$ the conditions of the Theorem 1 are fulfilled. Let us prove that in this case the problem (1) is not wellposed.

Since $f\in L^{\perp}$, then $Af=f$. It is also $Bf=0$ and $Cf=f$. It means that $f\in U$ and then $u_*=f$ is a solution to problem (1). Let us consider the sequence $u_n=\alpha_n(u_*+v_n)$, where

$$v_n=\left(0,0,...,0,1,\frac{1}{2n+1},0,...\right) \text{ and } \alpha_n=\left(1+\frac{1}{(2n+1)^2}\right)^{-1/2}\to 1,\ n\to\infty .$$

Since $v_n\in L=\operatorname{Ker} A$, we have that

$$Au_n=\alpha_n u_*\to u_*=Au,\quad n\to\infty .$$

Besides, we also have

$$Bu_n=\alpha_n v_n \text{ and } Cu_n=\alpha_n\left(1,0,...,0,\frac{1}{2n+1},0,...\right).$$

Therefore, $\| Bu_n\|=1$ and $\| Cu_n\|=1$. According to this, the sequence (u_n) is the minimizing sequence.

Let us prove that u_* is the unique solution of the problem (1). Let $v_*\in U_*$. Then, according to relation (6) we have that there exists

$$z_*=\left(0,z_2,\frac{z_2}{3},z_3,\frac{z_3}{5},...,\right)\in L=\operatorname{Ker} A ,$$

such that

$$v_* = u_* + z_* = \left(1, z_2, \frac{z_2}{3}, z_3, \frac{z_3}{5}, \ldots, \right).$$

From here we have

$$\| Cv_* \|^2 = 1 + \frac{z_2^2}{3^2} + \frac{z_3^2}{5^2} + \cdots > 1 ,$$

for $z_* \neq 0$. In that way $U_* = \{u_*\}$. And now, we have

$$d(u_n, U_*) = \| u_n - u_* \| = \| \alpha_n v_n - (\alpha_n - 1) u_* \| \to 1, \quad n \to \infty . \qquad \blacklozenge$$

In the following theorem we are proving that if we add the condition $U_* \subseteq \Gamma_1$, where Γ_1 is the boudary of the ellipsoid U_1 , to the conditions from the previous theorem, then the problem (1) is wellposed.

Theorem 3. *If the conditions from the Theorem 1 are satisfied and if*

$$U_* \subseteq \Gamma_1 = \{u \in H : \| Bu \| = r_1\} , \tag{8}$$

then the problem (1) is wellposed.

Proof: Let us suppose that (u_n) is the arbitrary minimizing sequence. We have proved in Theorem 1 that there are bounded sequences $(x_n), (y_n)$ and (z_n), $x_n \in R(B_A^*),\ y_n \in R(C_{AB}^*),\ z_n \in \operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C$ such that

$$u_n = Pu_n + x_n + y_n + z_n .$$

Without a loss of generality, we can suppose that $x_n \to x_0 \in R(B_A^*)$ and $y_n \to y_0 \in R(C_{AB}^*),\ n \to \infty$. Then

$$Pu_n + x_n + y_n \to u_* = Pu_* + x_0 + y_0 \in U_* . \tag{9}$$

According to (8) we have $\| Bu_* \| = r_1$. Further, from (9) it follows that

$$r_1 = \| Bu_* \| \leq \liminf_{n\to\infty} \| B(Pu_n + x_n) \| \leq \limsup_{n\to\infty} \| B(Pu_n + x_n) \| \leq r_1 ,$$

that is

$$\lim_{n\to\infty} \| B(Pu_n + x_n) \| = r_1 .$$

Further on

$$\lim_{n\to\infty} \| B(Pu_n + x_n) - Bu_* \| = r_1^2 - 2r_1^2 + r_1^2 = 0 . \tag{10}$$

Operator B_A is normal solvable. By applying relation (5) on this operator we get

$$m_B \| x_n - x_0 \| \le \| B(x_n - x_0) \| = \| B(Pu_n + x_n) - Bu_* + B(Pu_* - Pu_n) \| .$$

On the basis of relations (7) and (10), we obtain the strong convergence

$$x_n \to x_0,\ Pu_n + x_n \to Pu_* + x_0,\ n \to \infty . \tag{11}$$

Let us consider the sequence (v_n), $v_n = Pu_* + x_0 + y_n + z_n$ and let us notice that $Av_n = Au_*$, $Bv_n = Bu_*$ and $v_n \to u_*$, $n \to \infty$.

(a) If $\| Cv_n \| \le r_2$, then $v_n \in U_*$ and in that case we have

$$d(u_n, U_*) \le \| u_n - v_n \| = \| Pu_n - Pu_* + x_n - x_0 \| \to 0,\ n \to \infty .$$

(b) We suppose here that $\| Cv_n \| > r_2$. Then from

$$\| Cv_n \| = \| C(Pu_n + x_n + y_n) - C(Pu_n - Pu_* + x_n - x_*) \| ,$$

and from relation (11), we can conclude

$$\lim_{n\to\infty} \| Cv_n \| = r_2 .$$

(b1) Let us first consider the case $\| Cu_* \| > r_2$. By an argument similar to the one used in proving the first relation in (11), the strong convergence may be proved:

$$y_n \to y_0,\ n \to \infty .$$

Then

$$Pu_n + x_n + y_n \to u_* = Pu_* + x_0 + y_0 \in U_* .$$

It follows

$$d(u_n, U_*) \le \| u_n - (u_* + z_n) \| = \| Pu_n - Pu_* + x_n - x_0 + y_n - y_0 \| \to 0,\ n \to \infty ,$$

so, in the case of (b1) the theorem is proved.

(b2) Let us now suppose that $\| Cu_* \| < r_2$. Let us denote with $g_n = y_n - y_0 \in R(C_{AB}^*)$ and let us define the sequence (α_n) such that

$$\| C(u_* + \alpha_n g_n) \|^2 = r_2^2 . \tag{12}$$

For α_n defined in this way, we have

$$u_* + \alpha_n g_n \in U_* .$$

Considering that $Pu_n \to Pu_*$, $x_n \to x_0$ and $g_n \to g_0$ as $n \to \infty$, it is easy to prove that

$$\lim_{n\to\infty} \| Cg_n \|^2 = r_2^2 - \| Cu_* \|^2 > 0 . \tag{13}$$

From (12) and (13) it follows that $\lim_{n\to\infty} \alpha_n = 1$. And finally,

$$d(u_n, U_*) \le \| u_n - (u_* + \alpha_n g_n + z_n) \| = \| Pu_n - Pu_* + x_n - x_0 + (1-\alpha_n) g_n \| \to 0, \ n \to \infty ,$$

which proves the theorem. ♦

In the following four theorems we will prove that if some of the conditions from the previous theorem are violated, then problem (1) does not have to be wellposed.

Theorem 4. *If*

(i) $R(A) \ne \overline{R(A)}$,

(ii) $U_* \cap \operatorname{int} U \ne \varnothing$

then problem (1) is illposed.

Proof: From (i) and Lemma 3 we have that there exists a sequence (p_n) such that

$$p_n \in \overline{R(A^*)}, \ \| p_n \| = 1, \ Ap_n \to 0 \ \text{ as } \ n \to \infty .$$

According to (ii) we can conclude that there are $\alpha > 0$ and the element $u_* \in U_* \cap \operatorname{int} U$ such that

$$(\forall n \in N) \ v_n = u_* + \alpha p_n \in U .$$

The sequence (v_n) is minimizing, since $\| Av_n - Au_* \| = \alpha \| Ap_n \| \to 0$ as $n \to \infty$.

Let $v_* \in U$ be an arbitrary element. According to (6) we have that $u_* - v_* \in \operatorname{Ker} A$. Then

$$\| v_n - v_* \|^2 = \| \alpha p_n + u_* - v_* \|^2 = \alpha^2 + \| u_* - v_* \|^2 \ge \alpha^2$$

and it means that the sequence $(d(v_n, u_*))$ does not converge to zero. ♦

Theorem 5. *If*

(i) $B(\operatorname{Ker} A) \ne \overline{B(\operatorname{Ker} A)}$,

(ii) $U_* \subseteq \Gamma_1 = \{u \in H : \| Bu \| = r_1\}$,

(iii) $U_* \cap \operatorname{int} U_2 \ne \varnothing$

then problem (1) is illposed.

Proof: Let $u_* \in U_* \cap \operatorname{int} U_2$. According to the condition (ii) and relation (6), we have that

$$U_* \subseteq u_* + (\operatorname{Ker} A \cap \operatorname{Ker} B) . \tag{14}$$

The set $\operatorname{Ker} A$ may be presented as

$$\operatorname{Ker} A = \overline{R(B_A^*)} \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B) . \tag{15}$$

According to the condition (i) and Lemma 3 we have that there exists a sequence (q_n) such that

$$q_n \in \overline{R(B_A^*)},\ \ \| q_n \| = 1,\ \ Bq_n \to 0\ ,\ \text{as}\ \ n \to \infty\ . \tag{16}$$

By taking into account the condition (iii), there is $\varepsilon > 0$ such that

$$(\forall n \in N)\ \ \| C(u_* + \varepsilon q_n) \| < r_2\ .$$

Let us consider the sequence (v_n), $v_n = u_* + \varepsilon q_n$. According (14)-(16), we have that

$$\| Bv_n \| > r_1\ ,\ \text{and}\ \lim_{n\to\infty} \| Bv_n \| = r_2\ .$$

If we take

$$u_n = \alpha_n v_n = \alpha_n u_* + \alpha_n \varepsilon q_n,\ \ \alpha_n = \frac{r_1}{\| Bv_n \|}\ ,$$

we can see that $\alpha_n < 1$ and $\alpha_n \to 1$ as $n \to \infty$. And now

$$Au_n = \alpha_n Au_* \to Au_*\ \ \text{as}\ \ n \to \infty,\ \| Bu_n \| = r_1,\ \| Cu_n \| \le r_2\ .$$

According to this, the sequence (u_n) is minimizing. On the basis of (14), we have that for each $v_* \in U_*$, there is $x(v_*) \in \operatorname{Ker} A \cap \operatorname{Ker} B$ such that $v_* = u_* + x(v_*)$. That is why the following holds

$$d(u_n, U) = \inf\{\| u_n - v_* \| : v_* \in U_*\} = \inf\{\| (\alpha_n - 1)u_* + \alpha_n \varepsilon q_n + x(v_*) \| : v_* \in U_*\} \ge$$
$$\ge \sqrt{\alpha_n^2 \varepsilon^2 - (1 - \alpha_n) \| u_* \|} \to \varepsilon\ \ \text{as}\ \ n \to \infty\ . \qquad \blacklozenge$$

In a similar way, we can prove the following theorem.

Theorem 6. *If*

(i) $C(\operatorname{Ker} A) \ne \overline{C(\operatorname{Ker} A)}$,

(ii) $U_* \subseteq \Gamma_2 := \{u \in H : \| Cu \| = r_2\}$,

(iii) $U_* \cap \operatorname{int} U_1 \ne \varnothing$

then problem (1) is illposed.

Theorem 7. *If*

(i) $C(\operatorname{Ker} A \cap \operatorname{Ker} B) \ne \overline{C(\operatorname{Ker} A \cap \operatorname{Ker} B)}$,

(ii) $U_* \subseteq \Gamma := \{u \in H : \| Bu \| = r_1,\ \| Cu \| = r_2\}$,

then problem (1) is illposed.

Proof: According to the condition (ii) and relation (6), we have that

$$U_* = u_* + (\operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C)\ .$$

where $u_* \in U_*$ is an arbitrary element. The set $\operatorname{Ker} A \cap \operatorname{Ker} B$ may be presented as

$$\operatorname{Ker} A \cap \operatorname{Ker} B = \overline{R(C_{AB}^*)} \oplus (\operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C)\,.$$

According to the condition (ii) and Lemma 3 we can conclude that there exists a sequence (q_n) whose elements satisfy the following conditions

$$q_n \in \overline{R(C_{AB}^*)},\ \| q_n \| = 1,\ Cq_n \to 0, \quad \text{as} \quad n \to \infty\,.$$

The further argument is similar to one in the proof of Theorem 5. ♦

In the proof of Theorem 3, we used relation (7). The conditions of Theorem 2, do not guarantee this relation.

Theorem 8. *If the conditions of Theorem 2 are satisfied and if*

$$U_* \subseteq \Gamma_1 := \{u \in H : \| Bu \| = r_1\}\,, \tag{17}$$

then problem (1) is illposed.

Proof: Let us suppose that (u_n) is a minimizing sequence. By using the relation (4) and the conditions of the Theorem, the elements of this sequence may be represented as

$$u_n = s_n + x_n + y_n + z_n\,,$$

where

$$s_n \in R(B^*),\ x_n \in R(A_B^*),\ y_n \in R(C_{A_B}^*),\ z_n \in \operatorname{Ker} A \cap \operatorname{Ker} B \cap \operatorname{Ker} C\,.$$

Then

$$Bu_n = Bs_n,\ Au_n = A(s_n + x_n),\ Cu_n = C(s_n + x_n + y_n)\,.$$

By an argument similar to the one used in the proof of Theorem 3, we can prove that the sequences $(s_n), (x_n)$ and (y_n) are bounded. Without a loss of generality we can suppose that

$$s_n \to s_0 \in R(B^*),\ x_n \to x_0 \in R(A_B^*),\ y_n \to y_0 \in R(C_{AB}^*)\,, \text{ as } \quad n \to \infty\,.$$

Then

$$s_n + x_n + y_n \to u_* = s_0 + x_0 + y_0 \in U_* \quad \text{as} \quad n \to \infty\,.$$

As in Theorem 3, using relation (17) and weak convergence of (s_n) to s_0, strong convergence

$$s_n \to s_0 \quad \text{as} \quad n \to \infty \tag{18}$$

is proved.

Regarding the fact that the sequence (u_n) is minimizing we have that

$$Au_n = A(s_n + x_n) \to Au_* = As_0 + Ax_0, \quad \text{as} \quad n \to \infty .$$

Then from (18), we have that

$$Ax_n \to Ax_0 , \text{ as } \quad n \to \infty .$$

If we suppose that $A(\operatorname{Ker} B) = \overline{A(\operatorname{Ker} B)}$, applying relation (15) on operator A_B , we get strong convergence

$$x_n \to x_0 , \text{ as } \quad n \to \infty .$$

Consideration of the sequence (y_n) and proof of the wellposedness of problem (1) are the same as in the points a) and b) in Theorem 3. ♦

Here we can also prove that if any of the conditions from the previous theorem is not respected, then the problem (1) generally is not wellposed.

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