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# NORMS ON INTUITIONISTIC FUZZY MULTIGROUPS

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**Abstract:** In this paper, we propose the notion of intuitionistic fuzzy multigroups under norms ($t$-norm $T$ and $t$-conorm $C$) and study the variety of their algebraic structure. Next, we define the inverse, product, intersection and the sum of them. Also, we explored some of their properties and got the related results. Finally, under group homomorphisms, image and pre image of them are introduced and investigated.

**Keywords:** Sets, Multisets, Fuzzy Set Theory, Group Theory, Intuitionistic Fuzzy Multigroups, Norms, Intersection Theory, Products, Homomorphisms.

**MSC:** 97E60, 03E72, 20XX, 47A30, 14C17, 08B25, 05C60.

## 1. INTRODUCTION

Modern set theory formulated (or invented) by a German mathematician George Cantor (1845-1918) is fundamental and indispensable for the whole of mathematics. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object are allowed in a set, then the mathematical structure is called a multiset. Thus, a multiset differs from a set in the sense that each element has a multiplicity. A complete account of the development of multiset theory can be seen in [4, 5, 25, 26]. The concept of fuzzy sets was proposed by Zaded [28] to capture uncertainty in a collection, which was neglected in crisp set. Fuzzy set theory has grown stupendously over the years giving birth to fuzzy groups introduced in [18]. Several works have been done on fuzzy groups since inception; some could be found in [7, 19]. Recently, Shinoj et al. [20] introduced a non-classical group called fuzzy multigroup, which generalized fuzzy group. In 1983, Atanassov [3, 2] introduced the concept of intuitionistic fuzzy sets. The concepts of intuitionistic fuzzy multiset and intuitionistic fuzzy multigroup are introduced in [22, 23], which have applications in medical diagnosis and robotics. The author by using norms, investigated some properties of fuzzy algebraic structures [8-17]. The author [11] defined fuzzy multigroups under t-norms and explored some of their properties and got the related results. In this paper, we introduce intuitionistic fuzzy multigroups under norms ($t$-norm $T$ and $t$-conorm $C$) as $IFMSN(G)$ and study some of their algebraic properties. Also, we introduce the inverse, product, intersection and sum of them and prove that the inverse, product, intersection and sum of any $A, B \in IFMSN(G)$. We investigated and discussed their properties, too. Finally, under group homomorphisms, image and pre image of them are introduced and investigated.

## 2. PRELIMINARIES

**Definition 1.** *([24]) Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset $A$ over $X$ is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$, denotes the number of times an object $x$ occur in $A$. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. The set $X$ is called the ground or generic set of the class of all multisets (for short, msets) containing objects from $X$.*
*A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$ or $A = [a^2, b^2, c^3]$ or $\{\frac{a}{2}, \frac{b}{2}, \frac{c}{3}\}$ Difierent forms of representing multiset exist other than this. See [10, 20, 30] for details. We denote the set of all multisets by $MS(X)$.*

**Definition 2.** *([27]) Let $A$ and $B$ be two multisets over $X$, then $A$ is called a submultiset of $B$ written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then $A$ is called a proper submultiset of $B$ and denoted as $A \subset B$. Note that a multiset is called the parent in relation to its submultiset. Also two multisets $A$ and $B$ over $X$ are comparable to each other if $A \subseteq B$ or $B \subseteq A$.*

**Definition 3.** *( [2]) For sets $X, Y$ and $Z$, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.*

**Definition 4.** *( [2]) Let $X$ be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, $IFS$) in $X$ if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to $A$, respectively. In particular $\emptyset_X$ and $U_X$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in $X$ defined by $\emptyset_X(x) = (0, 1)$ and $U_X(x) = (1, 0)$, respectively. We will denote the set of all $IFS$s in $X$ as $IFS(X)$.*

**Definition 5.** *( [2]) Let $X$ be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFS$s in $X$. Then*
*(1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.*
*(2) Equality:$A = B$ iff $A \subseteq B$ and $B \subseteq A$.*

**Definition 6.** *([6, 21]) Let $X$ be a set. An intuitionistic fuzzy multiset $A = (CM_A, CN_A) : X \to [0,1] \times [0,1]$ of $X$ is characterized by a count membership function $CM_A : X \to [0,1]$ and count non-membership function $CN_A : X \to [0,1]$ of which the values are multisets of the unit interval $I = [0,1]$. That is,*

$$CM_A(x) = \{\mu_A^1, \mu_A^2, ..., \mu_A^n, ...\} \text{ and } CN_A(x) = \{\nu_A^1, \nu_A^2, ..., \nu_A^n, ...\} \ \forall x \in X,$$

*where $\mu_A^1, \mu_A^2, ..., \mu_A^n, ... \in [0,1]$ and $\nu_A^1, \nu_A^2, ..., \nu_A^n, ... \in [0,1]$ such that $0 \leq \mu_A^i(x) + \nu_A^i(x) \leq 1$ for all $x \in X$. We arrange the membership sequence in decreasing order as*

$$\mu_A^1 \geq \mu_A^2 \geq ... \geq \mu_A^n \geq ...$$

*but the corresponding non membership sequence may not be in decreasing or increasing order. Whenever the intuitionistic fuzzy multiset is finite, we write*

$$CM_A(x) = \{\mu_A^1, \mu_A^2, ..., \mu_A^n\} \text{ and } CN_A(x) = \{\nu_A^1, \nu_A^2, ..., \nu_A^n\}$$

*where $\mu_A^1, \mu_A^2, ..., \mu_A^n \in [0,1]$ and $\nu_A^1, \nu_A^2, ..., \nu_A^n \in [0,1]$ such that*

$$\mu_A^1 \geq \mu_A^2 \geq ... \geq \mu_A^n,$$

*or simply*

$$CM_A(x) = \{\mu_A^i\} \text{ and } CN_A(x) = \{\nu_A^i\}$$

*for $\mu_A^i \in [0,1]$ and $\nu_A^i \in [0,1]$ with $i = 1, 2, ..., n$.*
*Now, an intuitionistic fuzzy multiset $A$ is given as*
$A = (\{\frac{CM_A(x)}{x}\}, \{\frac{CN_A(x)}{x}\} : x \in X)$.
*The set of all intuitionistic fuzzy multisets of $X$ is depicted by $IFMS(X)$.*

**Example 7.** *Assume that $X = \{a, b, c\}$ is a set. Define*

$$CM_A = \{\frac{0.9, 0.6, 0.4}{a}, \frac{0.8, 0.5}{b}, \frac{1, 0.9, 0.3}{c}\}$$

*and*

$$CN_A = \{\frac{0.1, 0.4, 0.6}{a}, \frac{0.2, 0.5}{b}, \frac{0, 0.1, 0.7}{c}\}$$

*then*

$$A = (CM_A, CN_A)$$

$$= (\{\frac{0.9, 0.6, 0.4}{a}, \frac{0.8, 0.5}{b}, \frac{1, 0.9, 0.3}{c}\}, \{\frac{0.1, 0.4, 0.6}{a}, \frac{0.2, 0.5}{b}, \frac{0, 0.1, 0.7}{c}\}) \in IFMS(X).$$

**Definition 8.** *([6, 21]) Let $A, B \in IFMS(X)$ as*

$$A = (CM_A, CN_A) = (\mu_A^1, \mu_A^2, ..., \mu_A^n, \nu_A^1, \nu_A^2, ..., \nu_A^n)$$

*and*

$$B = (CM_B, CN_B) = (\mu_B^1, \mu_B^2, ..., \mu_B^n, \nu_B^1, \nu_B^2, ..., \nu_B^n).$$

*Then $A$ is called an intuitionistic fuzzy submultiset of $B$ written as $A \subseteq B$ if $\mu_A^i(x) \leq \mu_B^i(x)$ and $\nu_A^i(x) \geq \nu_B^i(x)$ for all $x \in X$ and $i = 1, 2, ..., n$. Also, if $A \subseteq B$ and $A \neq B$, then $A$ is called a proper fuzzy submultiset of $B$ and denoted as $A \subset B$.*

**Definition 9.** *(1]) A t-norm $T$ is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
*(T1) $T(x,1) = x$ (neutral element)*
*(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)*
*(T3) $T(x,y) = T(y,x)$ (commutativity)*
*(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),*
*for all $x,y,z \in [0,1]$.*

**Definition 10.** *([1]) A t-conorm $C$ is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
*(C1) $C(x,0) = x$*
*(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$*
*(C3) $C(x,y) = C(y,x)$*
*(C4) $C(x,C(y,z)) = C(C(x,y),z)$ ,*
*for all $x,y,z \in [0,1]$.*

Recall that $t$-norm $T$( $t$-conorm $C$) is idempotent if for all $x \in [0,1]$, $T(x,x) = x(C(x,x) = x)$.

**Lemma 11.** *([1]) Let $T$ be a t-norm and $C$ be a t-conorm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

*and*

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

*for all $x,y,w,z \in [0,1]$.*

## 3. INTUITIONISTIC FUZZY MULTIGROUPS UNDER NORMS

**Definition 12.** *Let $A = (CM_A, CN_A) \in IFMS(G)$. Then $A$ is said to be an intuitionistic fuzzy multigroup of $G$ under norms( t-norm $T$ and t-conorm $C$) if it satisfies the following conditions:*
*(1) $CM_A(xy) \geq T(CM_A(x), CM_A(y))$,*
*(2) $CM_A(x^{-1}) \geq CM_A(x)$,*
*(3) $CN_A(xy) \leq C(CN_A(x), CN_A(y))$,*
*(4) $CN_A(x^{-1}) \leq CN_A(x)$,*
*for all $x,y \in G$.*
*The set of all intuitionistic fuzzy multigroups of $G$ under norms( t-norm $T$ and t-conorm $C$) is depicted by $IFMSN(G)$.*

**Example 13.** *Let $G = \mathbb{Z}_3 = \{\bar{0},\bar{1},\bar{2}\}$ be a group with respect to addition and let*

$$CM_A = \{\frac{0.9, 0.7, 0.3}{\bar{0}}, \frac{0.9, 0.6, 0.4}{\bar{1}}, \frac{0.9, 0.6, 0.4}{\bar{2}}\}$$

*and*

$$CN_A = \{\frac{0.1, 0.3, 0.7}{\bar{0}}, \frac{0.1, 0.4, 0.6}{\bar{1}}, \frac{0.1, 0.4, 0.6}{\bar{2}}\}$$

*Then* $A \in IFMS(G)$. *Let* $T$ *be an algebraic product t-norm* $T(x, y) = T_p(x, y) = xy$ *and* $C$ *be an algebraic sum t-conorm* $C(x, y) = C_p(x, y) = x + y - xy$ *for all* $x, y \in [0, 1]$. *Now*

$$CM_A(\bar{0} + \bar{0}) = CM_A(\bar{0}) = 0.9, 0.7, 0.3 \geq T(CM_A(\bar{0}), CM_A(\bar{0})) = 0.81, 0.49, 0.09$$

$$CM_A(\bar{0} + \bar{1}) = CM_A(\bar{1}) = 0.9, 0.6, 0.4 \geq T(CM_A(\bar{0}), CM_A(\bar{1})) = 0.81, 0.42, 0.12$$

$$CM_A(\bar{0} + \bar{2}) = CM_A(\bar{2}) = 0.9, 0.6, 0.4 \geq T(CM_A(\bar{0}), CM_A(\bar{2})) = 0.81, 0.42, 0.12$$

$$CM_A(\bar{1} + \bar{2}) = CM_A(\bar{0}) = 0.9, 0.7, 0.3 \geq T(CM_A(\bar{1}), CM_A(\bar{2})) = 0.81, 0.36, 0.16$$

$$CM_A(\bar{1} + \bar{1}) = CM_A(\bar{2}) = 0.9, 0.6, 0.4 \geq T(CM_A(\bar{1}), CM_A(\bar{1})) = 0.81, 0.36, 0.16$$

$$CM_A(\bar{2} + \bar{2}) = CM_A(\bar{1}) = 0.9, 0.6, 0.4 \geq T(CM_A(\bar{2}), CM_A(\bar{2})) = 0.81, 0.36, 0.16$$

$$CM_A((\bar{0})^{-1}) = CM_A(\bar{0}) = 0.9, 0.7, 0.3$$

$$CM_A((\bar{1})^{-1}) = CM_A(\bar{2}) = 0.9, 0.6, 0.4 \geq 0.9, 0.6, 0.4 = CM_A(\bar{1})$$

$$CM_A((\bar{2})^{-1}) = CM_A(\bar{1}) = 0.9, 0.6, 0.4 \geq 0.9, 0.6, 0.4 = CM_A(\bar{2}).$$

*Also*

$$CN_A(\bar{0} + \bar{0}) = CN_A(\bar{0}) = 0.1, 0.3, 0.7 \leq C(CN_A(\bar{0}), CN_A(\bar{0})) = 0.19, 0.51, 0.91$$

$$CN_A(\bar{0} + \bar{1}) = CN_A(\bar{1}) = 0.1, 0.4, 0.6 \leq C(CN_A(\bar{0}), CN_A(\bar{1})) = 0.19, 0.58, 0.88$$

$$CN_A(\bar{0} + \bar{2}) = CN_A(\bar{2}) = 0.1, 0.4, 0.6 \leq C(CN_A(\bar{0}), CN_A(\bar{2})) = 0.19, 0.58, 0.88$$

$$CN_A(\bar{1} + \bar{2}) = CN_A(\bar{0}) = 0.2, 0.3, 0.7 \leq C(CN_A(\bar{1}), CN_A(\bar{2})) = 0.19, 0.64, .084$$

$$CN_A(\bar{1} + \bar{1}) = CN_A(\bar{2}) = 0.1, 0.4, 0.6 \leq C(CN_A(\bar{1}), CN_A(\bar{1})) = 0.19, 0.64, .084$$

$$CN_A(\bar{2} + \bar{2}) = CN_A(\bar{1}) = 0.1, 0.4, 0.6 \leq C(CN_A(\bar{2}), CN_A(\bar{2})) = 0.19, 0.64, .084$$

$$CN_A((\bar{0})^{-1}) = CN_A(\bar{0}) = 0.1, 0.3, 0.7$$

$$CN_A((\bar{1})^{-1}) = CN_A(\bar{2}) = 0.1, 0.4, 0.6 \leq 0.1, 0.4, 0.6 = CN_A(\bar{1})$$

$$CM_A((\bar{2})^{-1}) = CM_A(\bar{1}) = 0.1, 0.4, 0.6 \leq 0.1, 0.4, 0.6 = CM_A(\bar{2}).$$

*Thus* $A = (CM_A, CN_A) \in IFMSN(G)$.

**Lemma 14.** *Let* $A = (CM_A, CN_A) \in IFMS(G)$ *and* $G$ *be a finite group such that* $T$ *and* $C$ *be idempotent. If* $A$ *satisfies conditions (1) and (3) of Definition 20, then* $A \in IFMSN(G)$.

*Proof.* Let $x \in G, x \neq e$. As $G$ is finite, so $x$ has finite order, as $n > 1$. Then $x^n = e$ and $x^{-1} = x^{n-1}$. Now by using conditions (1) and (3) repeatedly, we have that

$$CM_A(x^{-1}) = CM_A(x^{n-1}) = CM_A(x^{n-2}x)$$
$$\geq T(CM_A(x^{n-2}), CM_A(x)) \geq T(\underbrace{CM_A(x), CM_A(x), ..., CM_A(x)}_{n})$$
$$= CM_A(x)$$

and

$$CN_A(x^{-1}) = CN_A(x^{n-1}) = CN_A(x^{n-2}x)$$
$$\leq C(CN_A(x^{n-2}), CN_A(x)) \leq C(\underbrace{CN_A(x), CN_A(x), ..., CN_A(x)}_{n})$$
$$= CN_A(x).$$

Then $A \in IFMSN(G)$. □

**Theorem 15.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $T, C$ be idempotent. Then*
*(1) $A(e) \supseteq A(x)$ for all $x \in G$.*
*(2) $A(x^n) \supseteq A(x)$ for all $x \in G$ and $n \geq 1$.*
*(3) $A(x) = A(x^{-1})$ for all $x \in G$.*

*Proof.* (1) Let $x \in G$ then

$$CM_A(e) = CM_A(xx^{-1}) \geq T(CM_A(x), CM_A(x^{-1}))$$
$$\geq T(CM_A(x), CM_A(x)) = CM_A(x)$$

and so

$$CM_A(e) \geq CM_A(x) \quad (a).$$

Also

$$CN_A(e) = CN_A(xx^{-1}) \leq C(CN_A(x), CN_A(x^{-1}))$$
$$\leq C(CN_A(x), CN_A(x)) = CN_A(x)$$

and then

$$CN_A(e) \leq CN_A(x) \quad (b).$$

Now from (a) and (b) we get that

$$A(e) = (CM_A(e), CN_A(e)) \supseteq (CM_A(x), CN_A(x)) = A(x).$$

(2) Let $x \in G$ and $n \geq 1$. Then

$$CM_A(x^n) = CM_A(\underbrace{xx...x}_{n}) \geq T(\underbrace{CM_A(x), CM_A(x), ..., CM_A(x)}_{n}) = CM_A(x) \quad (a)$$

and

$$CN_A(x^n) = CN_A(\underbrace{xx...x}_{n}) \leq C(\underbrace{CN_A(x), CN_A(x), ..., CN_A(x)}_{n}) = CN_A(x). \quad (b)$$

Thus from (a) and (b) we will have that

$$A(x^n) = (CM_A(x^n), CN_A(x^n)) \supseteq (CM_A(x), CN_A(x)) = A(x).$$

(3) If $x \in G$, then

$$CM_A(x) = CM_A((x^{-1}))^{-1} \geq CM_A(x^{-1}) \geq CM_A(x)$$

and then

$$CM_A(x) = CM_A(x^{-1}) \qquad (a)$$

and

$$CN_A(x) = CN_A((x^{-1}))^{-1} \leq CN_A(x^{-1}) \leq CN_A(x)$$

and so

$$CN_A(x) = CN_A(x^{-1}). \qquad (b)$$

Now (a) and (b) give us

$$A(x) = (CM_A(x), CN_A(x)) = (CM_A(x^{-1}), CN_A(x^{-1})) = A(x^{-1}).$$

□

**Proposition 16.** *Let $T$ and $C$ be idempotent. Then $A = (CM_A, CN_A) \in IFMSN(G)$ if and only if*

$$A(xy^{-1}) \supseteq (T(CM_A(x), CM_A(y)), C(CM_A(x), CM_A(y)))$$

*for all $x, y \in G$.*

*Proof.* Let $x, y \in G$. If $A = (CM_A, CN_A) \in IFMSN(G)$, then

$$CM_A(xy^{-1}) \geq T(CM_A(x), CM_A(y^{-1})) \geq T(CM_A(x), CM_A(y)) \quad (a)$$

and

$$CN_A(xy^{-1}) \leq C(CN_A(x), CN_A(y^{-1})) \leq C(CN_A(x), CN_A(y)). \quad (b)$$

Then (a) and (b) give us that

$$A(xy^{-1}) = (CM_A(xy^{-1}), CN_A(xy^{-1})) \supseteq (T(CM_A(x), CM_A(y)), C(CM_A(x), CM_A(y))).$$

Conversely, let

$$A(xy^{-1}) = (CM_A(xy^{-1}), CN_A(xy^{-1})) \supseteq (T(CM_A(x), CM_A(y)), C(CM_A(x), CM_A(y)))$$

for all $x, y \in G$. As $CM_A(xy^{-1}) \geq T(CM_A(x), CM_A(y))$ so

$$CM_A(x^{-1}) = CM_A(ex^{-1}) \geq T(CM_A(e), CM_A(x))$$

$$\geq T(CM_A(x), CM_A(x)) = CM_A(x). \text{ (\textit{Theorem 23 (part 1)})}$$

Also

$$CM_A(xy) = CM_A(x(y^{-1})^{-1}) \geq T(CM_A(x), CM_A(y^{-1})) \geq T(CM_A(x), CM_A(y)).$$

Since $CN_A(xy^{-1}) \leq C(CM_A(x), CM_A(y))$ so

$$CN_A(x^{-1}) = CN_A(ex^{-1}) \leq C(CN_A(e), CN_A(x))$$

$$\leq C(CN_A(x), CN_A(x)) = CN_A(x) \text{ (\textit{Theorem 23 (part 1)})}$$

and

$$CN_A(xy) = CN_A(x(y^{-1})^{-1}) \leq C(CN_A(x), CN_A(y^{-1})) \leq C(CN_A(x), CN_A(y)).$$

Therefore $A = (CM_A, CN_A) \in IFMSN(G)$. □

**Proposition 17.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $x \in G$. If $T$ and $C$ be idempotent, then $A(xy) = A(y)$ for all $y \in G$ if and only if $A(x) = A(e)$.*

*Proof.* Let $A(xy) = A(y)$ for all $y \in G$ and we let $y = e$, then we get that $A(x) = A(e)$.
Conversely,let $A(x) = A(e)$ and using Theorem 23 we obtain that $A(x) \supseteq A(xy)$ and $A(x) \supseteq A(y)$ which mean that $CM_A(x) \geq CM_A(xy)$ and $CM_A(x) \geq CM_A(y)$ and $CN_A(x) \leq CN_A(xy)$ and $CN_A(x) \leq CN_A(y)$. Then for all $y \in G$

$$CM_A(xy) \geq T(CM_A(x), CM_A(y)) \geq T(CM_A(y), CM_A(y))$$

$$= CM_A(y) = CM_A(x^{-1}xy) \geq T(CM_A(x), CM_A(xy))$$

$$\geq T(CM_A(xy), CM_A(xy)) = CM_A(xy)$$

and thus

$$CM_A(xy) = CM_A(y). \ (a)$$

Also

$$CN_A(xy) \leq C(CN_A(x), CN_A(y)) \leq C(CN_A(y), CN_A(y))$$

$$= CN_A(y) = CN_A(x^{-1}xy) \leq C(CN_A(x), CN_A(xy))$$

$$\leq C(CN_A(xy), CN_A(xy)) = CN_A(xy)$$

thus

$$CN_A(xy) = CN_A(y). \ (b)$$

Therefore from (a) and (b) we will have that

$$A(xy) = (CM_A(xy), CN_A(xy)) = (CM_A(y), CN_A(y)) = A(y).$$

□

**Proposition 18.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $T, C$ be idempotent. If $A(xy^{-1}) = A(e)$, then $A(x) = A(y)$ for all $x, y \in G$.*

*Proof.* As $A(xy^{-1}) = A(e)$ so $CM_A(xy^{-1}) = CM_A(e)$ and $CN_A(xy^{-1}) = CN_A(e)$ for all $x, y \in G$. Now

$$CM_A(x) = CM_A(xy^{-1}y) \geq T(CM_A(xy^{-1}), CM_A(y))$$

$$= T(CM_A(e), CM_A(y)) \geq T(CM_A(y), CM_A(y)) = CM_A(y)$$

$$= CM_A(y^{-1}) = CM_A(x^{-1}xy^{-1}) \geq T(CM_A(x^{-1}), CM_A(xy^{-1}))$$

$$= T(CM_A(x^{-1}), CM_A(e)) \geq CM_A(x^{-1}) = CM_A(x)$$

and then

$$CM_A(x) = CM_A(y). \quad (a)$$

Also

$$CN_A(x) = CN_A(xy^{-1}y) \leq C(CN_A(xy^{-1}), CN_A(y))$$

$$= C(CN_A(e), CN_A(y)) \leq C(CN_A(y), CN_A(y)) = CN_A(y)$$

$$= CN_A(y^{-1}) = CN_A(x^{-1}xy^{-1}) \leq C(CN_A(x^{-1}), CN_A(xy^{-1}))$$

$$= C(CN_A(x^{-1}), CN_A(e)) \leq CN_A(x^{-1}) = CN_A(x)$$

which means that

$$CN_A(x) = CN_A(y). \quad (b)$$

Thus (a) and (b) give us that

$$A(x) = (CM_A(x), CN_A(x)) = (CM_A(y), CN_A(y)) = A(y).$$

□

**Proposition 19.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $A(x) \neq A(y)$ for all $x, y \in G$. Then*

$$A(xy) = (T(CM_A(x), CM_A(y)), C(CM_A(x), CM_A(y))).$$

*Proof.* Let $A(x) \supset A(y)$ for all $x, y \in G$. Then we get that $CM_A(x) > CM_A(y)$ and then $CM_A(x) > CM_A(xy)$ which give us

$$CM_A(y) = T(CM_A(x), CM_A(y))$$

and

$$CM_A(xy) = T(CM_A(x), CM_A(xy)).$$

Now

$$CM_A(xy) \geq T(CM_A(x), CM_A(y)) = CM_A(y)$$

$$= CM_A(x^{-1}xy) \geq T(CM_A(x^{-1}), CM_A(xy))$$

$$= T(CM_A(x), CM_A(xy)) = CM_A(xy)$$

and then

$$CM_A(xy) = CM_A(y) = T(CM_A(x), CM_A(y)). \quad (a)$$

Also as $A(x) \supset A(y)$ for all $x, y \in G$. Then we get that $CN_A(x) < CN_A(y)$ and then $CN_A(x) < CN_A(xy)$ so

$$CN_A(y) = C(CN_A(x), CN_A(y))$$

and

$$CN_A(xy) = C(CN_A(x), CN_A(xy)).$$

Then

$$\begin{aligned} CN_A(xy) &\leq C(CN_A(x), CN_A(y)) = CN_A(y) \\ &= CN_A(x^{-1}xy) \leq C(CN_A(x^{-1}), CN_A(xy)) \\ &= C(CN_A(x), CN_A(xy)) = CN_A(xy) \end{aligned}$$

and then

$$CN_A(xy) = CN_A(y) = C(CN_A(x), CN_A(y)). \quad (b)$$

Now from (a) and (b) we obtain that

$$A(xy) = (CM_A(xy), CN_A(xy)) = (T(CM_A(x), CM_A(y)), C(CM_A(x), CM_A(y))).$$

□

**Proposition 20.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$. Then*
*(1) If $T, C$ be idempotent, then $A^* = \{x \in G : A(x) = A(e)\}$ is a subgroup of $G$.*
*(2) If $T, C$ be idempotent, then*

$$A^{[\alpha]}_{[\beta]} = \{x \in G : A(x) \supseteq (\alpha, \beta)\}$$

*is a subgroup of $G$ for all $\alpha, \beta \in [0, 1]$.*

*Proof.* Let $x, y \in G$.
(1) If $x, y \in A^*$, then $A(x) = A(y) = A(e)$ which means that $CM_A(x) = CM_A(y) = CM_A(e)$ and $CN_A(x) = CN_A(y) = CN_A(e)$. Thus

$$\begin{aligned} CM_A(xy^{-1}) &\geq T(CM_A(x), CM_A(y^{-1})) \geq T(CM_A(x), CM_A(y)) \\ &= T(CM_A(e), CM_A(e)) = CM_A(e) = CM_A(xy^{-1}yx^{-1}) \\ &\geq T(CM_A(xy^{-1}), CM_A(yx^{-1})) = T(CM_A(xy^{-1}), CM_A(xy^{-1})^{-1}) \\ &\geq T(CM_A(xy^{-1}), CM_A(xy^{-1})) = CM_A(xy^{-1}) \end{aligned}$$

then

$$CM_A(xy^{-1}) = CM_A(e). \quad (a)$$

Also

$$CN_A(xy^{-1}) \leq C(CN_A(x), CN_A(y^{-1})) \leq C(CN_A(x), CN_A(y))$$

$$= C(CN_A(e), CN_A(e)) = CN_A(e) = CN_A(xy^{-1}yx^{-1})$$
$$\leq C(CN_A(xy^{-1}), CN_A(yx^{-1})) = C(CN_A(xy^{-1}), CN_A(xy^{-1})^{-1})$$
$$\leq C(CN_A(xy^{-1}), CN_A(xy^{-1})) = CN_A(xy^{-1})$$

so

$$CN_A(xy^{-1}) = CN_A(e). \quad (b)$$

Therefore using (a) and (b) we obtain that

$$A(xy^{-1}) = (CM_A(xy^{-1}), CN_A(xy^{-1})) = (CM_A(e), CN_A(e)) = A(e)$$

and then $xy^{-1} \in A^*$ and we get that $A^*$ is a subgroup of $G$.
(2) Let $x, y \in A_{[\beta]}^{[\alpha]}$ then $A(x) \supseteq (\alpha, \beta)$ and $A(y) \supseteq (\alpha, \beta)$ which mean that $CM_A(x) \geq \alpha$ and $CM_A(y) \geq \alpha$. Now

$$CM_A(xy^{-1}) \geq T(CM_A(x), CM_A(y^{-1}))$$

$$\geq T(CM_A(x), CM_A(y)) \geq T(\alpha, \alpha) = \alpha$$

and then

$$CM_A(xy^{-1}) \geq \alpha. \quad (a)$$

Also as $A(x) \supseteq (\alpha, \beta)$ and $A(y) \supseteq (\alpha, \beta)$ so $CN_A(x) \leq \beta$ and $CN_A(y) \leq \beta$. Now

$$CN_A(xy^{-1}) \leq C(CN_A(x), CN_A(y^{-1}))$$

$$\leq C(CN_A(x), CN_A(y)) \leq C(\beta, \beta) = \beta$$

and so

$$CN_A(xy^{-1}) \leq \beta. \quad (b)$$

Therefore from (a) and (b) we will have taht

$$A(xy^{-1}) = (CM_A(xy^{-1}), CN_A(xy^{-1})) \supseteq (\alpha, \beta)$$

and so $xy^{-1} \in A_{[\beta]}^{[\alpha]}$ and thus $A_{[\beta]}^{[\alpha]}$ is a subgroup of $G$. $\square$

## 4. INVERSE, PRODUCT, INTERSECTION AND SUM OF $IFMSN(G)$

**Definition 21.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$. Then $A^{-1} = (CM_{A^{-1}}, CN_{A^{-1}})$ is called inverse of $A = (CM_A, CN_A)$ and defined as*

$$A^{-1}(x) = (CM_{A^{-1}}(x), CN_{A^{-1}}(x)) = (CM_A(x^{-1}), CN_A(x^{-1}))$$

*for all $x \in G$.*

**Proposition 22.** *$A = (CM_A, CN_A) \in IFMSN(G)$ if and only if $A^{-1} = (CM_{A^{-1}}, CN_{A^{-1}}) \in IFMSN(G)$.*

*Proof.* Let $x, y \in G$. If $A = (CM_A, CN_A) \in IFMSN(G)$, then
(1)
$$\begin{aligned} CM_{A^{-1}}(xy) &= CM_A(xy)^{-1} = CM_A(y^{-1}x^{-1}) \\ &\geq T(CM_A(y^{-1}), CM_A(x^{-1})) = T(CM_{A^{-1}}(y), CM_{A^{-1}}(x)) \\ &= T(CM_{A^{-1}}(x), CM_{A^{-1}}(y)). \end{aligned}$$
(2)
$$CM_{A^{-1}}(x^{-1}) = CM_A(x^{-1})^{-1} \geq CM_A(x^{-1}) = CM_{A^{-1}}(x).$$
(3)
$$\begin{aligned} CN_{A^{-1}}(xy) &= CN_A(xy)^{-1} = CN_A(y^{-1}x^{-1}) \\ &\leq C(CN_A(y^{-1}), CN_A(x^{-1})) = C(CN_{A^{-1}}(y), CN_{A^{-1}}(x)) \\ &= C(CN_{A^{-1}}(x), CN_{A^{-1}}(y)). \end{aligned}$$
(4)
$$CN_{A^{-1}}(x^{-1}) = CN_A(x^{-1})^{-1} \leq CN_A(x^{-1}) = CN_{A^{-1}}(x).$$
Thus $A^{-1} = (CM_{A^{-1}}, CN_{A^{-1}}) \in IFMSN(G)$.
Conversely, let $A^{-1} = (CM_{A^{-1}}, CN_{A^{-1}}) \in IFMSN(G)$. Then
(1)
$$\begin{aligned} CM_A(xy) &= CM_A((xy)^{-1})^{-1} = CM_{A^{-1}}((xy)^{-1}) \\ &= CM_{A^{-1}}(y^{-1}x^{-1}) \geq T(CM_{A^{-1}}(y^{-1}), CM_{A^{-1}}(x^{-1})) \\ &= T(CM_A(y), CM_A(x)) = T(CM_A(x), CM_A(y)). \end{aligned}$$
(2)
$$CM_A(x^{-1}) = CM_{A^{-1}}(x) = CM_{A^{-1}}(x^{-1})^{-1} \geq CM_{A^{-1}}(x^{-1}) = CM_A(x).$$
(3)
$$\begin{aligned} CN_A(xy) &= CN_A((xy)^{-1})^{-1} = CN_{A^{-1}}((xy)^{-1}) \\ &= CN_{A^{-1}}(y^{-1}x^{-1}) \leq C(CN_{A^{-1}}(y^{-1}), CN_{A^{-1}}(x^{-1})) \\ &= C(CN_A(y), CN_A(x)) = C(CN_A(x), CN_A(y)). \end{aligned}$$
(4)
$$CN_A(x^{-1}) = CN_{A^{-1}}(x) = CN_{A^{-1}}(x^{-1})^{-1} \leq CN_{A^{-1}}(x^{-1}) = CN_A(x).$$
Therefore $A = (CM_A, CN_A) \in IFMSN(G)$. □

**Definition 23.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$. Then the product of $A$ and $B$ denoted as*

$$A \circ B = (CM_A, CN_A) \circ (CM_B, CN_B) = (CM_A \circ CM_B, CN_A \circ CN_B) = (CM_{A\circ B}, CN_{A\circ B})$$

*is governed by*

$$CM_{A\circ B}(x) = \begin{cases} \sup_{x=yz} T(CM_A(y), CM_B(z)) & \textit{if } x = yz \\ 0 & \textit{otherwise} \end{cases}$$

*and*

$$CN_{A\circ B}(x) = \begin{cases} \inf_{x=yz} C(CN_A(y), CN_B(z)) & \textit{if } x = yz \\ 0 & \textit{otherwise.} \end{cases}$$

*Note that*

$$CM_{A\circ B}(x) = \sup_{y\in G} T(CM_A(y), CM_B(y^{-1}x)) = \sup_{y\in G} T(CM_A(xy^{-1}), CM_B(y))$$

*and*

$$CN_{A\circ B}(x) = \inf_{y\in G} C(CN_A(y), CN_B(y^{-1}x)) = \inf_{y\in G} C(CN_A(xy^{-1}), CN_B(y))$$

*and if $x = yz$ we can say that*

$$(A \circ B)(x) = (\sup_{x=yz} T(CM_A(y), CM_B(z)), \inf_{x=yz} C(CN_A(y), CN_B(z))).$$

**Definition 24.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$. Then the intersection of $A$ and $B$ denoted as*

$$A\cap B = (CM_A, CN_A)\cap(CM_B, CN_B) = (CM_A\cap CM_B, CN_A\cap CN_B) = (CM_{A\cap B}, CN_{A\cap B})$$

*is governed by*

$$(A\cap B)(x) = (CM_{A\cap B}(x), CN_{A\cap B}(x)) = (T(CM_A(x), CM_B(x)), C(CN_A(x), CN_B(x)))$$

*for all $x \in G$.*

**Proposition 25.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$. Then $A \cap B \in IFMSN(G)$.*

*Proof.* Let $x, y \in G$.
(1)

$$\begin{aligned} CM_{A\cap B}(xy) &= T(CM_A(xy), CM_B(xy)) \\ &\geq T(T(CM_A(x), CM_A(y)), T(CM_B(x), CM_B(y))) \\ &= T(T(CM_A(x), CM_B(x)), T(CM_A(y), CM_B(y))) \ (\textit{Lemma 15}) \\ &= T(CM_{A\cap B}(x), CM_{A\cap B}(y)). \end{aligned}$$

(2)

$$\begin{aligned} CM_{A\cap B}(x^{-1}) &= T(CM_A(x^{-1}), CM_B(x^{-1})) \\ &\geq T(CM_A(x), CM_B(x)) = CM_{A\cap B}(x). \end{aligned}$$

(3)

$$\begin{aligned} CN_{A\cap B}(xy) &= C(CN_A(xy), CN_B(xy)) \\ &\leq C(C(CN_A(x), CN_A(y)), C(CN_B(x), CN_B(y))) \\ &= C(C(CN_A(x), CN_B(x)), C(CN_A(y), CN_B(y))) \text{ (Lemma 11)} \\ &= C(CN_{A\cap B}(x), CN_{A\cap B}(y)). \end{aligned}$$

(4)

$$\begin{aligned} CN_{A\cap B}(x^{-1}) &= C(CN_A(x^{-1}), CN_B(x^{-1})) \\ &\leq C(CN_A(x), CN_B(x)) = CN_{A\cap B}(x). \end{aligned}$$

Therefore $A \cap B \in IFMSN(G)$. □

**Corollary 26.** *Let $I_n = \{1, 2, ..., n\}$. If $\{A_i \mid i \in I_n\} \subseteq IFMSN(G)$, Then $A = \cap_{i\in I_n} A_i \in IFMSN(G)$.*

**Proposition 27.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$. Then the following assertions hold:*
*(1) $(A^{-1})^{-1} = A$.*
*(2) If $A \subseteq B$, then $A^{-1} \subseteq B^{-1}$.*
*(3) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$.*
*(4) $(A \cap B)^{-1} = A^{-1} \cap B^{-1}$.*

*Proof.* Let $x, y, z \in G$. Then
(1)

$$CM_{(A^{-1})^{-1}}(x) = CM_{A^{-1}}(x^{-1}) = CM_A((x^{-1})^{-1}) = CM_A(x)$$

and

$$CN_{(A^{-1})^{-1}}(x) = CN_{A^{-1}}(x^{-1}) = CN_A((x^{-1})^{-1}) = CN_A(x)$$

then

$$(A^{-1})^{-1}(x) = (CM_{(A^{-1})^{-1}}(x), CN_{(A^{-1})^{-1}}(x)) = (CM_A(x), CN_A(x)) = A(x)$$

then $(\mathrm{A}^{-1})^{-1} = A$.
(2) As $A \subseteq B$ so $CM_A(x^{-1}) \leq CM_B(x^{-1})$ and $CN_A(x^{-1}) \geq CN_B(x^{-1})$. Then

$$CM_{A^{-1}}(x) = CM_A(x^{-1}) \leq CM_B(x^{-1}) = CM_{B^{-1}}(x)$$

and

$$CN_{A^{-1}}(x) = CN_A(x^{-1}) \geq CN_B(x^{-1}) = CN_{B^{-1}}(x)$$

which mean that

$$A^{-1}(x) = (CM_{A^{-1}}(x), CN_{A^{-1}}(x)) \subseteq (CM_{B^{-1}}(x), CN_{B^{-1}}(x)) = B^{-1}(x)$$

and then $A^{-1} \subseteq B^{-1}$.
(3)

$$CM_{(A\circ B)^{-1}}(x) = CM_{(A\circ B)}(x^{-1}) = \sup_{x^{-1}=y^{-1}z^{-1}} T(CM_A(y^{-1}), CM_B(z^{-1}))$$

$$= \sup_{x^{-1}=(zy)^{-1}} T(CM_A(y^{-1}), CM_B(z^{-1})) = \sup_{x^{-1}=(zy)^{-1}} T(CM_B(z^{-1}), CM_A(y^{-1}))$$

$$= \sup_{x=zy} T(CM_{B^{-1}}(z), CM_{A^{-1}}(y)) = CM_{(B^{-1}\circ A^{-1})}(x)$$

and then

$$CM_{(A\circ B)^{-1}}(x) = CM_{(B^{-1}\circ A^{-1})}(x). \quad (a)$$

Also

$$CN_{(A\circ B)^{-1}}(x) = CN_{(A\circ B)}(x^{-1}) = \inf_{x^{-1}=y^{-1}z^{-1}} C(CN_A(y^{-1}), CN_B(z^{-1}))$$

$$= \inf_{x^{-1}=(zy)^{-1}} C(CN_A(y^{-1}), CN_B(z^{-1})) = \inf_{x^{-1}=(zy)^{-1}} C(CN_B(z^{-1}), CN_A(y^{-1}))$$

$$= \inf_{x=zy} C(CN_{B^{-1}}(z), CN_{A^{-1}}(y)) = CN_{(B^{-1}\circ A^{-1})}(x)$$

and then

$$CN_{(A\circ B)^{-1}}(x) = CN_{(B^{-1}\circ A^{-1})}(x). \quad (b)$$

Now from (a) and (b) we get that

$$(A\circ B)^{-1}(x) = (CM_{(A\circ B)^{-1}}(x), CN_{(A\circ B)^{-1}}(x))$$

$$= (CM_{(B^{-1}\circ A^{-1})}(x), CN_{(B^{-1}\circ A^{-1})}(x)) = (B^{-1}\circ A^{-1})(x)$$

and thus $(A\circ B)^{-1} = B^{-1}\circ A^{-1}$.
(4)

$$CM_{(A\cap B)^{-1}}(x) = CM_{(A\cap B)}(x^{-1}) = T(CM_A(x^{-1}), CM_B(x^{-1}))$$

$$= T(CM_{A^{-1}}(x), CM_{B^{-1}}(x)) = CM_{(A^{-1}\cap B^{-1})}(x)$$

and so

$$CM_{(A\cap B)^{-1}}(x) = CM_{(A^{-1}\cap B^{-1})}(x). \quad (a)$$

Also

$$CN_{(A\cap B)^{-1}}(x) = CN_{(A\cap B)}(x^{-1}) = C(CN_A(x^{-1}), CN_B(x^{-1}))$$

$$= C(CN_{A^{-1}}(x), CN_{B^{-1}}(x)) = CN_{(A^{-1}\cap B^{-1})}(x)$$

and then

$$CN_{(A\cap B)^{-1}}(x) = CN_{(A^{-1}\cap B^{-1})}(x). \quad (b)$$

Using (a) and (b) give us

$$(A\cap B)^{-1}(x) = (CM_{(A\cap B)^{-1}}(x), CN_{(A\cap B)^{-1}}(x))$$

$$= (CM_{(A^{-1}\cap B^{-1})}(x), CN_{(A^{-1}\cap B^{-1})}(x)) = (A^{-1}\cap B^{-1})(x)$$

and therefore $(A\cap B)^{-1} = A^{-1}\cap B^{-1}$. $\square$

**Proposition 28.** *$A = (CM_A, CN_A) \in IFMSN(G)$ if and only if $A$ satisfies the following conditions:*
*(1) $A \supseteq A \circ A$;*
*(2) $A^{-1} = A$.*

*Proof.* Let $x, y, z \in G$ such that $x = yz$. If $A = (CM_A, CN_A) \in IFMSN(G)$, then
(1)

$$CM_A(x) = CM_A(yz) \geq T(CM_A(y), CM_A(z)) = CM_{A\circ A}(x)$$

and

$$CN_A(x) = CN_A(yz) \leq C(CN_A(y), CN_A(z)) = CN_{A\circ A}(x)$$

and so

$$A(x) = (CM_A(x), CN_A(x)) \supseteq (CM_{A\circ A}(x), CN_{A\circ A}(x)) = (A \circ A)(x)$$

which means that $A \supseteq A \circ A$.
(2) As

$$CM_{A^{-1}}(x) = CM_A(x^{-1}) = CM_A(x)$$

and

$$CN_{A^{-1}}(x) = CN_A(x^{-1}) = CN_A(x)$$

so

$$A(x) = (CM_A(x), CN_A(x)) = (CM_{A^{-1}}(x), CN_{A^{-1}}(x)) = A^{-1}(x)$$

and then $A^{-1} = A$.
Conversely, let $x \in G$. As $A \supseteq A \circ A$ so

$$A(x) = (CM_A(x), CN_A(x)) \supseteq (A \circ A)(x) = (CM_{A\circ A}(x), CN_{A\circ A}(x))$$

so $CM_A(x) \geq CM_{A\circ A}(x)$ and $CN_A(x) \leq CN_{A\circ A}(x)$. Now

$$CM_A(yz) = CM_A(x) \geq (CM_{A\circ A})(x) = \sup_{x=yz} T(CM_A(y), CM_A(z)) \geq T(CM_A(y), CM_A(z))$$

and

$$CN_A(yz) = CN_A(x) \leq (CN_{A\circ A})(x) = \inf_{x=yz} C(CN_A(y), CN_A(z)) \leq C(CN_A(y), CN_A(z)).$$

Also since $A^{-1} = A$ so

$$CM_A(x^{-1}) = CM_{A^{-1}}(x) = CM_A(x)$$

and

$$CN_A(x^{-1}) = CN_{A^{-1}}(x) = CN_A(x).$$

Therefore $A = (CM_A, CN_A) \in IFMSN(G)$. □

**Proposition 29.** *Let $A = (CM_A, CN_A) \in IFMSN(G)$. Then $(A \circ B) \in IFMSN(G)$ if and only if $A \circ B = B \circ A$.*

*Proof.* Let $A, B \in TFMS(G)$ then from Proposition 28 we get that $A \circ A \subseteq A$ and $B \circ B \subseteq B$ and $A^{-1} = A$ and $B^{-1} = B$.
If $(A \circ B) \in TFMS(G)$, then from Proposition 28 and Proposition 29 we get that

$$B \circ A = B^{-1} \circ A^{-1} = (A \circ B)^{-1} = A \circ B.$$

Conversely, let $A \circ B = B \circ A$. As
(1)

$$(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B$$
$$= A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) \subseteq A \circ B$$

and
(2)

$$(A \circ B)^{-1} = B^{-1} \circ A^{-1} = B \circ A = A \circ B$$

so Proposition 28 gives us that $(A \circ B) \in IFMSN(G)$. $\square$

**Proposition 30.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$ and $T, C$ be idempotent. Then $A(e) \subseteq B(e)$ if and only if $A \subseteq A \circ B$.*

*Proof.* Let $x, y, z \in G$ and $A(e) \subseteq B(e)$ then $CM_A(e) \leq CM_B(e)$ and $CN_A(e) \geq CN_B(e)$. Now

$$CM_{(A \circ B)}(x) = \sup_{x=yz} T(CM_A(y), CM_B(z))$$
$$= \sup_{x=xe} T(CM_A(x), CM_B(e)) \geq \sup_{x=xe} T(CM_A(x), CM_A(e))$$
$$\geq T(CM_A(x), CM_A(e)) \geq T(CM_A(x), CM_A(x))$$
$$= CM_A(x)$$

and so

$$CM_{(A \circ B)}(x) \geq CM_A(x). \qquad (a)$$

Also

$$CN_{(A \circ B)}(x) = \inf_{x=yz} C(CN_A(y), CN_B(z))$$
$$= \inf_{x=xe} C(CN_A(x), CN_B(e)) \leq \inf_{x=xe} C(CN_A(x), CN_A(e))$$
$$\leq C(CN_A(x), CN_A(e)) \leq C(CN_A(x), CN_A(x))$$
$$= CN_A(x)$$

and so

$$CN_{(A \circ B)}(x) \leq CN_A(x). \qquad (b)$$

Now from (a) and (b) we get that

$$A(x) = (CM_A(x), CN_A(x)) \subseteq (CM_{(A \circ B)}(x), CN_{(A \circ B)}(x)) = (A \circ B)(x)$$

and thus $A \subseteq A \circ B$.
Conversely, let $A \subseteq A \circ B$. If $A(e) \supset B(e)$, then $CM_A(e) > CM_B(e)$ and $CN_A(e) < CN_B(e)$. Thus

$$CM_{(A\circ B)}(e) = \sup_{e=ee} T(CM_A(e), CM_B(e))$$

$$< \sup_{e=ee} T(CM_A(e), CM_A(e)) = T(CM_A(e), CM_A(e)) = CM_A(e)$$

which means that

$$CM_{(A\circ B)}(e) < CM_A(e). \qquad (a)$$

Also

$$CN_{(A\circ B)}(e) = \inf_{e=ee} C(CN_A(e), CN_B(e))$$

$$> \inf_{e=ee} C(CN_A(e), CN_A(e)) = C(CN_A(e), CN_A(e)) = CN_A(e)$$

then

$$CN_{(A\circ B)}(e) > CN_A(e). \qquad (b)$$

Using (a) and (b) give us

$$A(e) = (CM_A(e), CN_A(e)) \supset (CM_{(A\circ B)}(e), CN_{(A\circ B)}(e)) = (A \circ B)(e)$$

and then $A \supset A \circ B$ and this is a contradiction with $A \subseteq A \circ B$. Therefore $A(e) \subseteq B(e)$. $\square$

**Proposition 31.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$ and $A(e) = B(e)$ and $T, C$ be idempotent. Then $A \subseteq A \circ B$ and $B \subseteq A \circ B$.*

*Proof.* Let $x, y, z \in G$ and $A(e) = B(e)$ then $CM_A(e) = CM_B(e)$ and $CN_A(e) = CN_B(e)$. Thus

$$CM_{(A\circ B)}(x) = \sup_{x=yz} T(CM_A(y), CM_B(z))$$

$$\geq T(CM_A(x), CM_B(e)) = T(CM_A(x), CM_A(e))$$

$$\geq T(CM_A(x), CM_A(x)) = CM_A(x)$$

and then

$$CM_{(A\circ B)}(x) \geq CM_A(x). \qquad (a)$$

Also

$$CN_{(A\circ B)}(x) = \inf_{x=yz} C(CN_A(y), CN_B(z))$$

$$\leq C(CN_A(x), CN_B(e)) = C(CN_A(x), CN_A(e))$$

$$\leq C(CN_A(x), CN_A(x)) = CN_A(x)$$

so

$$CN_{(A\circ B)}(x) \leq CN_A(x). \qquad (b)$$

Then from (a) and (b) we will have that

$$(A \circ B)(x) = (CM_{(A\circ B)}(x), CN_{(A\circ B)}(x)) \supseteq (CM_A(x), CN_A(x)) = A(x)$$

that is $A \circ B \supseteq A$.
Now as

$$CM_{(A\circ B)}(x) = \sup_{x=yz} T(CM_A(y), CM_B(z))$$
$$\geq T(CM_A(e), CM_B(x)) = T(CM_B(e), CM_B(x))$$
$$\geq T(CM_B(x), CM_B(x)) = CM_B(x)$$

so

$$CM_{(A\circ B)}(x) \geq CM_B(x). \quad (a)$$

Also

$$CN_{(A\circ B)}(x) = \inf_{x=yz} C(CN_A(y), CN_B(z))$$
$$\leq C(CN_A(e), CN_B(x)) = C(CN_B(e), CN_B(x))$$
$$\leq C(CN_B(x), CN_B(x)) = CN_B(x)$$

therefore

$$CN_{(A\circ B)}(x) \leq CN_B(x). \quad (b)$$

Thus as (a) and (b)

$$(A \circ B)(x) = (CM_{(A\circ B)}(x), CN_{(A\circ B)}(x)) \supseteq (CM_B(x), CN_B(x)) = B(x)$$

so $A \circ B \supseteq B$. $\square$

**Proposition 32.** *Let $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$ and $A(e) = B(e)$ and $T, C$ be idempotent. If $A \circ B \in IFMSN(G)$, then $A \circ B$ is generated by $A$ and $B$.*

*Proof.* Suppose that $A \circ B \in IFMSN(G)$. Then we show that $A \circ B$ is the smallest containing $A$ and $B$. As Proposition 31 we get that $A \subseteq A \circ B$ and $B \subseteq A \circ B$. Let $C = (CM_C, CN_C) \in IFMSN(G)$ such that $A, B \subseteq C$ and $x, y, z \in G$. Then

$$CM_{(A\circ B)}(x) = \sup_{x=yz} T(CM_A(y), CM_B(z))$$
$$\leq \sup_{x=yz} T(CM_C(y), CM_C(z))$$
$$= CM_{(C\circ C)}(x) = CM_C(x)$$

and

$$CN_{(A\circ B)}(x) = \inf_{x=yz} C(CN_A(y), CN_B(z))$$
$$\geq \inf_{x=yz} C(CN_C(y), CN_C(z))$$
$$= CN_{(C\circ C)}(x) = CN_C(x)$$

and then

$$(A \circ B)(x) = (CM_{(A\circ B)}(x), CN_{(A\circ B)}(x)) \subseteq (CM_C(x), CN_C(x)) = C(x)$$

and then $A \circ B \subseteq C$. Thus $A \circ B$ is generated by $A$ and $B$. $\square$

**Definition 33.** *Let* $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMS(G)$. *Then the sum of* $A$ *and* $B$ *denoted as*

$$A+B = (CM_A, CN_A)+(CM_B, CN_B) = (CM_A+CM_B, CN_A+CN_B) = (CM_{A+B}, CN_{A+B})$$

*is defined by the addition operation in* $X \to [0,1]$ *for crisp multiset. That is,*

$$CM_{A+B}(x) = CM_A(x) + CM_B(x)$$

*and*

$$CN_{A+B}(x) = CN_A(x) + CN_B(x)$$

*for all* $x \in X$. *The meaning of the addition operation here is not as in the case of crisp multiset.*

**Example 34.** *Assume that* $X = \{a, b, c\}$ *is a set. Define*

$$A = (CM_A, CN_A)$$
$$= (\{\frac{0.9, 0.6, 0.4}{a}, \frac{0.8, 0.5}{b}, \frac{1, 0.9, 0.3}{c}\}, \{\frac{0.1, 0.4, 0.6}{a}, \frac{0.2, 0.5}{b}, \frac{0, 0.1, 0.7}{c}\})$$

*and*

$$B = (CM_B, CN_B)$$
$$= (\{\frac{0.8, 0.5, 0.3}{a}, \frac{0.9, 0.6}{b}, \frac{0.3, 0.7, 0.6}{c}\}, \{\frac{0.2, 0.5, 0.7}{a}, \frac{0.1, 0.4}{b}, \frac{0.7, 0.3, 0.4}{c}\}).$$

*Then* $A + B = (CM_{A+B}, CN_{A+B})$ *such that*

$$CM_{A+B} = CM_A + CM_B$$
$$= \{\frac{0.9, 0.8, 0.6, 0.5, 0.4, 0.3}{a}, \frac{0.9, 0.8, 0.6, 0.5}{b}, \frac{0.7, 0.7, 0.4, 0.3, 0.1, 0.0}{c}\}$$

*and*

$$CN_{A+B} = CN_A + CN_B$$
$$= \{\frac{0.1, 0.2, 0.4, 0.5, 0.6, 0.7}{a}, \frac{0.1, 0.2, 0.4, 0.5}{b}, \frac{0.3, 0.3, 0.6, 0.7, 0.9, 1.0}{c}\}$$

**Proposition 35.** *Let* $A = (CM_A, CN_A), B = (CM_B, CN_B) \in IFMSN(G)$. *Then*

$$A + B = (CM_{A+B}, CN_{A+B}) \in IFMSN(G).$$

*Proof.* Let $x, y \in G$. Then
(1)

$$\begin{aligned} CM_{(A+B)}(xy) &= CM_A(xy) + CM_B(xy) \\ &\geq T(CM_A(x), CM_A(y)) + T(CM_B(x), CM_B(y)) \\ &= T(CM_A(x) + CM_B(x), CM_A(y) + CM_B(y)) \\ &= T(CM_{A+B}(x), CM_{A+B}(y)). \end{aligned}$$

(2)

$$\begin{aligned} CM_{(A+B)}(x^{-1}) &= CM_A(x^{-1}) + CM_B(x^{-1}) \\ &\geq CM_A(x) + CM_B(x) \\ &= CM_{(A+B)}(x). \end{aligned}$$

(3)

$$\begin{aligned} CN_{(A+B)}(xy) &= CN_A(xy) + CN_B(xy) \\ &\leq C(CN_A(x), CN_A(y)) + C(CN_B(x), CN_B(y)) \\ &= C(CN_A(x) + CN_B(x), CN_A(y) + CM_B(y)) \\ &= C(CN_{A+B}(x), CN_{A+B}(y)). \end{aligned}$$

(4)

$$\begin{aligned} CN_{(A+B)}(x^{-1}) &= CN_A(x^{-1}) + CN_B(x^{-1}) \\ &\leq CN_A(x) + CN_B(x) \\ &= CN_{(A+B)}(x). \end{aligned}$$

Thus $A + B = (CM_{A+B}, CN_{A+B}) \in IFMSN(G)$. $\square$

**Remark 36.** *Let $\{A_i\}_{i\in I} \in IFMSN(G)$. Then $\sum_{i\in I} A_i \in IFMSN(G)$.*

## 5. GROUP HOMOMORPHISMS OVER $IFMSN(G)$

**Definition 37.** *Let $G$ and $H$ be groups and $f : G \to H$ be a homomorphism. Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$. Define $f(A) \in IFMS(H)$ as*

$$f(A) = f(CM_A, CN_A) = (f(CM_A), f(CN_A)) = (CM_{f(A)}, CN_{f(A)})$$

*such that for all $h \in H$*

$$f(CM_A)(h) = (CM_{f(A)})(h) = \begin{matrix} \sup\{CM_A(g) \mid g \in G, f(g) = h\} & \text{if } f^{-1}(h) \neq \emptyset \\ 0 & \text{otherwise} \end{matrix}$$

*and*

$$f(CN_A)(h) = (CN_{f(A)})(h) = \begin{matrix} \inf\{CN_A(g) \mid g \in G, f(g) = h\} & \text{if } f^{-1}(h) \neq \emptyset \\ 0 & \text{otherwise.} \end{matrix}$$

*Note that if $f^{-1}(h) \neq \emptyset$, we can write*

$$f(A)(h) = (CM_{f(A)}(h), CN_{f(A)}(h))$$

$$= (\sup\{CM_A(g) \mid g \in G, f(g) = h\}, \inf\{CN_A(g) \mid g \in G, f(g) = h\}).$$

*Also define* $f^{-1}(B) \in IFMS(G)$ *as*

$$f^{-1}(B) = f^{-1}(CM_B, CN_B) = (f^{-1}(CM_B), f^{-1}(CN_B)) = (CM_{f^{-1}(B)}, CN_{f^{-1}(B)})$$

*such that for all* $g \in G$

$$f^{-1}(B)(g) = (CM_{f^{-1}(B)}(g), CN_{f^{-1}(B)}(g)) = (CM_B(f(g)), CN_B(f(g))).$$

**Proposition 38.** *Let* $G$ *and* $H$ *be groups and* $f : G \to H$ *be an epimorphism. If* $A = (CM_A, CN_A) \in IFMSN(G)$, *then* $f(A) = (CM_{f(A)}, CN_{f(A)}) \in IFMSN(H)$.

*Proof.* Let $u, v \in H$ and $x, y \in G$ such that $u = f(x)$ and $v = f(y)$ then
(1)

$$\begin{aligned} CM_{f(A)}(uv) &= \sup\{CM_A(xy) \mid u = f(x), v = f(y)\} \\ &\geq \sup\{T(CM_A(x), CM_A(y)) \mid u = f(x), v = f(y)\} \\ &= T(\sup\{CM_A(x) \mid u = f(x)\}, \sup\{CM_A(y) \mid v = f(y)\}) \\ &= T(CM_{f(A)}(u), CM_{f(A)}(v)). \end{aligned}$$

(2)

$$\begin{aligned} CM_{f(A)}(u^{-1}) &= \sup\{CM_A(x^{-1}) \mid u^{-1} = f(x^{-1})\} \\ &= \sup\{CM_A(x^{-1}) \mid u^{-1} = f^{-1}(x)\} \\ &\geq \sup\{CM_A(x) \mid u = f(x)\} = CM_{f(A)}(u). \end{aligned}$$

(3)

$$\begin{aligned} CN_{f(A)}(uv) &= \inf\{CN_A(xy) \mid u = f(x), v = f(y)\} \\ &\leq \inf\{C(CN_A(x), CN_A(y)) \mid u = f(x), v = f(y)\} \\ &= C(\inf\{CN_A(x) \mid u = f(x)\}, \inf\{CM_A(y) \mid v = f(y)\}) \\ &= C(CN_{f(A)}(u), CN_{f(A)}(v)). \end{aligned}$$

(4)

$$\begin{aligned} CN_{f(A)}(u^{-1}) &= \inf\{CN_A(x^{-1}) \mid u^{-1} = f(x^{-1})\} \\ &= \inf\{CN_A(x^{-1}) \mid u^{-1} = f^{-1}(x)\} \\ &\leq \inf\{CN_A(x) \mid u = f(x)\} = CN_{f(A)}(u). \end{aligned}$$

Therefore $f(A) = (CM_{f(A)}, CN_{f(A)}) \in IFMSN(H)$. □

**Proposition 39.** *Let* $G$ *and* $H$ *be groups and* $f : G \to H$ *be a homomorphism. If* $B = (CM_B, CN_B) \in IFMSN(H)$, *then* $f^{-1}(B) = (CM_{f^{-1}(B)}, CN_{f^{-1}(B)}) \in IFMSN(G)$.

*Proof.* Let $x, y \in G$. Then
(1)
$$CM_{f^{-1}(B)}(xy) = CM_B(f(xy)) = CM_B(f(x)f(y))$$
$$\geq T(CM_B(f(x)), CM_B(f(y))) = T(CM_{f^{-1}(B)}(x), CM_{f^{-1}(B)}(y)).$$
(2)
$$CM_{f^{-1}(B)}(x^{-1}) = CM_B(f(x^{-1})) = CM_B(f^{-1}(x))$$
$$\geq CM_B(f(x)) = CM_{f^{-1}(B)}(x).$$
(3)
$$CN_{f^{-1}(B)}(xy) = CM_B(f(xy)) = CM_B(f(x)f(y))$$
$$\leq C(CM_B(f(x)), CM_B(f(y))) = C(CN_{f^{-1}(B)}(x), CN_{f^{-1}(B)}(y)).$$
(4)
$$CN_{f^{-1}(B)}(x^{-1}) = CN_B(f(x^{-1})) = CN_B(f^{-1}(x))$$
$$\leq CN_B(f(x)) = CN_{f^{-1}(B)}(x).$$
Thus $f^{-1}(B) = (CM_{f^{-1}(B)}, CN_{f^{-1}(B)}) \in IFMSN(G)$. □

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