Yugoslav Journal of Operations Research
31 (2021), Number 4, 429–453
DOI: https://doi.org/10.2298/YJOR200117024L

KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS AND DUALITY FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH EQUILIBRIUM CONSTRAINTS

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Received: January 2020 / Accepted: June 2020

Abstract: The purpose of this paper is to study multiobjective semi-infinite programming with equilibrium constraints. Firstly, the necessary and sufficient Karush-Kuhn-Tucker optimality conditions for multiobjective semi-infinite programming with equilibrium constraints are established. Then, we formulate types of Wolfe and Mond-Weir dual problems and investigate duality relations under convexity assumptions. Some examples are given to verify our results.

Keywords: Multiobjective Semi-Infinite Programming, Equilibrium Constraints, Constraint Qualifications, Karush-Kuhn-Tucker Optimality Conditions, Mond-Weir Duality, Wolfe Duality.

MSC: 90C46, 90C33, 49K10.

1. INTRODUCTION

The mathematical programming problems with equilibrium constraints could be served in reformulating many problems from economic equilibria, multilevel games [18], industrial engineering [2], healthcare management [16]. Among many other interesting research, optimality conditions and duality for mathematical programming problems with equilibrium constraints have been considered numerously by many researchers (see, e.g., [3, 6, 9, 11, 15, 22, 24, 26] and the references therein). On the other hand, a simultaneous minimization of a finite number of objective functions over an infinite number of constraints is called a multiobjective semi-infinite programming problem. Due to semi-infinite programming problems having application in many fields [7], they have attracted a lot of attention from many authors (see, e.g., [4, 5, 8, 12, 13, 14, 27, 28, 29, 30, 31, 32] and the references therein). Recently, semi-infinite programming problems with equilibrium constraints have been presented and investigated. In [21], strong Karush-Kuhn-Tucker (KKT) type sufficient optimality conditions for nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints were established via Clarke subdifferentials. By using convexificators, the paper [20] established necessary and sufficient optimality conditions and derived weak and strong duality theorems relating to the semi-infinite mathematical programming problems with equilibrium constraints. The Lagrange type dual model and saddle point optimality criteria of semi-infinite mathematical programming problems with equilibrium constraints were discussed in [25]. However, KKT necessary optimality conditions for multiobjective semi-infinite programming problems with equilibrium constraints have not yet been considered in [21]. Moreover, to the best of our knowledge, there is no paper dealing with duality for multiobjective semi-infinite programming problems with equilibrium constraints.

Inspired by the above observations, we concentrate on studying Karush-Kuhn-Tucker optimality conditions and duality results for the multiobjective semi-infinite programming with equilibrium constraints. The organization of this paper is as follows. In Section 2, some basic concepts and preliminaries are recollected. Section 3 is a discussion of the KKT necessary and sufficient optimality conditions for the multiobjective semi-infinite programming problems with equilibrium constraints. Then, we explore Mond-Weir and Wolfe dual problems of the multiobjective semi-infinite programming problems with equilibrium constraints in Section 4. Some examples are given to illuminate the results of the paper.

2. PRELIMINARIES

The following notations and definitions will be used throughout the paper. Let $\mathbb{R}^n$ be a finite-dimensional Euclidean space. The notation $\langle \cdot, \cdot \rangle$ is used to denote the inner product. By $B(x, \delta)$ we indicate the open ball centered at x with radius $\delta > 0$. For a given $\bar{x}$, $\mathcal{U}(\bar{x})$ is the system of the neighborhoods of $\bar{x}$. For $A \subseteq \mathbb{R}^n$, $\text{int}A$, $\text{cl}A$, $\text{aff}A$, $\text{span}A$ and $\text{co}A$ stand for its interior, closure, affine hull, linear hull, convex hull of A, respectively (resp). The cone and the convex cone (containing the origin) generated by A are denoted resp by $\text{cone}A$, $\text{pos}A$. It should be noted that, for the given sets A_1, A_2 in $\mathbb{R}^n$,

$$\text{span}(A_1 \cup A_2) = \text{span}A_1 + \text{span}A_2 \text{ and } \text{pos}(A_1 \cup A_2) = \text{pos}A_1 + \text{pos}A_2.$$

The negative polar cone, the strictly negative polar cone and the orthogonal complement of A are defined resp by

$$A^- := \{x^* \in \mathbb{R}^n | \langle x^*, x \rangle \leq 0, \ \forall x \in A\},$$

$$A^s := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle < 0,\ \forall x \in A \setminus \{0\}\},$$
$$A^{\perp} := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle = 0,\ \forall x \in A\}.$$

It is easy to check that $A^s \subset A^-$ and if $A^s \neq \emptyset$ then $\text{cl}A^s = A^-$. Moreover, the bipolar theorem, see e.g. [1], states that $A^{--} = \text{cl pos}A$. For a given nonempty subset A of $\mathbb{R}^n$, the contingent cone [1] of A at $\bar{x} \in \text{cl}A$ is

$$T(A, \bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \exists x_k \to x,\ \forall k \in \mathbb{N}, \bar{x} + \tau_k x_k \in A\}.$$

Note that if A is a convex set then $T(A, \bar{x}) = \text{clcone}(A - \bar{x})$. If $\langle x^*, x\rangle \geq 0$ for all $x^* \in A^*$, where A^* is a subset of the dual space of $\mathbb{R}^n$, we write $\langle A^*, x\rangle \geq 0$. The notion $o(\tau^k)$, for $\tau > 0$ and $k \in \mathbb{N}$, designates a moving point such that $o(\tau^k)/\tau^k \to 0$ as $\tau \to 0^+$. The cardinality of the index set I is denoted by $|I|$. For an index subset $I \subset \{1, ..., n\}$, $x_I = 0 (x_I \geq 0)$ stands for $x_i = 0$ ($x_i \geq 0$, resp) for all $i \in I$.

In the line of [21], we consider the following multiobjective semi-infinite programming with equilibrium constraints (P):

$$\begin{array}{rl} \mathbb{R}^m_+ - \min & f(x) = (f_1(x), ..., f_m(x)) \\ \text{s.t.} & g_t(x) \leq 0, t \in T, \\ & h_i(x) = 0, i = 1, ..., q, \\ & G_i(x) \geq 0, i = 1, ..., l, \\ & H_i(x) \geq 0, i = 1, ..., l, \\ & G_i(x)H_i(x) = 0, i = 1, ..., l, \end{array}$$

where $f_i (i = 1, ..., m)$, $g_t (t \in T)$, $h_i (i = 1, ..., q)$ and $G_i, H_i (i = 1, ..., l)$ are continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$. The index set T is an arbitrary nonempty set, not necessary finite. Let us denote $I := \{1, ..., m\}$, $I_h := \{1, ..., q\}$ and $I_l := \{1, ..., l\}$. The feasible solution set of (P) is

$$\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0 (t \in T), h_i(x) = 0 (i \in I_h),$$
$$G_i(x) \geq 0, H_i(x) \geq 0, G_i(x)H_i(x) = 0 (i \in I_l)\}.$$

Recall some types of efficient solutions, see e.g. [17], of the multiobjective semi-infinite programming as follows.

Definition 1. *Let $\bar{x} \in \Omega$.*

(i) $\bar{x}$ is a locally (Pareto) efficient solution of (P), denoted by $\bar{x} \in$ locE(P), if there exists a neighborhood $U \in \mathcal{U}(\bar{x})$ such that

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}, \forall x \in \Omega \cap U.$$

(ii) $\bar{x}$ is a locally weakly efficient solution of (P), denoted by $\bar{x} \in$ locWE(P), if there exists a neighborhood $U \in \mathcal{U}(\bar{x})$ such that

$$f(x) - f(\bar{x}) \notin -\text{int}\mathbb{R}^m_+, \forall x \in \Omega \cap U..$$

If $U = \mathbb{R}^n$, the word "locally" is omitted. In this case, the efficient solution sets/the weakly efficient solution sets are denoted by $E(P)/WE(P)$. It is straightforward that $E(P) \subset WE(P)$.

The notation $\mathbb{R}_+^{|T|}$ represents the collection of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x} \in \Omega$, we signify by $I_g(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|T|} | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $J \subset I_g(\bar{x})$ such that $\lambda_t > 0$ for all $t \in J$ and $\lambda_t = 0$ for all $t \in T \setminus J$. For each $\bar{x} \in \Omega$, let us define

$$I_{+0}(\bar{x}) := \{i \in I_l \mid G_i(\bar{x}) > 0, H_i(\bar{x}) = 0\},$$

$$I_{00}(\bar{x}) := \{i \in I_l \mid G_i(\bar{x}) = 0, H_i(\bar{x}) = 0\},$$

$$I_{0+}(\bar{x}) := \{i \in I_l \mid G_i(\bar{x}) = 0, H_i(\bar{x}) > 0\}.$$

Definition 2. *The point $\bar{x} \in \Omega$ is called a strong stationary point of (P) iff there exists $(\alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\lambda^G_{I_{+0}(\bar{x})} = 0$, $\lambda^G_{I_{00}(\bar{x})} \geq 0$, $\lambda^H_{I_{00}(\bar{x})} \geq 0$ and $\lambda^H_{I_{0+}(\bar{x})} = 0$ such that*

$$\sum_{i\in I} \alpha_i \nabla f_i(\bar{x}) + \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(\bar{x}) = 0.$$

For $\bar{x} \in \Omega$ and $(\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^{|T|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$, we define

$$I_g^+(\bar{x}) := \{t \in I_g(\bar{x}) \mid \lambda_t^g > 0\},$$

$$I_h^+(\bar{x}) := \{i \in I_h \mid \lambda_i^h > 0\}, I_h^-(\bar{x}) := \{i \in I_h \mid \lambda_i^h < 0\},$$

$$I_{00}^{++}(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G > 0, \lambda_i^H > 0\},$$

$$I_{0+}^+(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda_i^G > 0\}, I_{0+}^-(\bar{x}) := \{i \in I_{0+}(\bar{x}) \mid \lambda_i^G < 0\},$$

$$I_{00}^+(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G > 0, \lambda_i^H = 0\}, I_{00}^-(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G < 0, \lambda_i^H = 0\},$$

$$\hat{I}_{+0}^+(\bar{x}) := \{i \in I_{+0}(\bar{x}) \mid \lambda_i^H > 0\}, \hat{I}_{+0}^-(\bar{x}) := \{i \in I_{+0}(\bar{x}) \mid \lambda_i^H < 0\},$$

$$\hat{I}_{00}^+(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G = 0, \lambda_i^H > 0\}, \hat{I}_{00}^-(\bar{x}) := \{i \in I_{00}(\bar{x}) \mid \lambda_i^G = 0, \lambda_i^H < 0\}.$$

Definition 3. *[23] Let $X \subset \mathbb{R}^n$ be an open convex set and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\bar{x} \in X$.*

(i) φ is convex at $\bar{x}$ if $\varphi(\lambda\bar{x} + (1-\lambda)x) \leq \lambda\varphi(\bar{x}) + (1-\lambda)\varphi(x), \forall x \in X, \forall \lambda \in [0,1]$.

(ii) φ is strictly convex at $\bar{x}$ if

$$\varphi(\lambda\bar{x} + (1-\lambda)x) < \lambda\varphi(\bar{x}) + (1-\lambda)\varphi(x), \forall x \in X \setminus \{\bar{x}\}, \forall \lambda \in (0,1).$$

(iii) φ is quasiconvex at $\bar{x}$ if $\varphi(\lambda\bar{x} + (1-\lambda)x) \leq \max\{\varphi(\bar{x}), \varphi(x)\}, \forall x \in X, \forall \lambda \in [0,1]$.

(iv) φ *is pseudoconvex at* $\bar{x}$ *if, for all* $x \in X$,

$$\varphi(x) < \varphi(\bar{x}) \Rightarrow \langle \nabla\varphi(\bar{x}), x - \bar{x} \rangle < 0.$$

(v) φ *is strictly pseudoconvex at* $\bar{x}$ *if, for all* $x \in X \setminus \{\bar{x}\}$,

$$\varphi(x) \leq \varphi(\bar{x}) \Rightarrow \langle \nabla\varphi(\bar{x}), x - \bar{x} \rangle < 0.$$

(vi) φ *is convex on* X *if* φ *is convex on each point of* X. *Other concepts here introduced can be defined on a set in a similar way.*

Remark 4. *[23] Let* $X \subset \mathbb{R}^n$ *be an open convex set and* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *be differentiable at* $\bar{x} \in X$.

(i) If φ *is convex at* $\bar{x}$, *then*

$$\langle \nabla\varphi(\bar{x}), x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}), \textit{for all } x \in X.$$

(ii) If φ *is quasiconvex at* $\bar{x}$, *then, for all* $x \in X$,

$$\varphi(x) \leq \varphi(\bar{x}) \Rightarrow \langle \nabla\varphi(\bar{x}), x - \bar{x} \rangle \leq 0.$$

(iii) If φ *is convex at* $\bar{x}$ *then* φ *is pseudoconvex at* $\bar{x}$. *If* φ *is pseudoconvex at* $\bar{x}$ *then* φ *is quasiconvex at* $\bar{x}$.

Lemma 5. *[23] Let* $\{C_t | t \in \Gamma\}$ *be an arbitrary collection of nonempty convex sets in* $\mathbb{R}^n$ *and* $K = \text{pos}\left(\bigcup_{t\in\Gamma} C_t\right)$. *Then, every nonzero vector of* K *can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different* C_t.

Lemma 6. *[7] Suppose that* S, T, P *are arbitrary (possibly infinite) index sets,* $a_s = a(s) = (a_1(s), ..., a_n(s))$ *maps* S *onto* $\mathbb{R}^n$, *and so do* a_t *and* a_p. *Suppose that the set* $\text{co}\{a_s, s \in S\} + \text{pos}\{a_t, t \in T\} + \text{span}\{a_p, p \in P\}$ *is closed. Then the following statements are equivalent:*

$$I : \begin{cases} \langle a_s, x \rangle < 0, s \in S, S \neq \emptyset \\ \langle a_t, x \rangle \leq 0, t \in T \\ \langle a_p, x \rangle = 0, p \in P \end{cases} \quad \textit{has no solution } x \in \mathbb{R}^n;$$

$$II : \quad 0 \in \text{co}\{a_s, s \in S\} + \text{pos}\{a_t, t \in T\} + \text{span}\{a_p, p \in P\}.$$

Lemma 7. *[10] If* A *is a nonempty compact subset of* $\mathbb{R}^n$, *then,*

(i) $\text{co}A$ *is a compact set;*

(ii) *if* $0 \notin \text{co}A$, *then* $\text{pos}A$ *is a closed cone.*

3. KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

In this section, we write the index set I_g instead of $I_g(\bar{x})$ for the sake of convenience. The other index sets are expressed similarly.

Definition 8. *The linearized cone of (P) at $\bar{x} \in \Omega$ is*

$$L(\bar{x}) := \{d \in \mathbb{R}^n \mid \langle \nabla g_t(\bar{x}), d\rangle \leq 0 (t \in I_g), \langle \nabla h_i(\bar{x}), d\rangle = 0 (i \in I_h), \langle \nabla G_i(\bar{x}), d\rangle = 0 (i \in I_{0+}), \langle \nabla G_i(\bar{x}), d\rangle \geq 0 (i \in I_{00}), \langle \nabla H_i(\bar{x}), d\rangle \geq 0 (i \in I_{00}), \langle \nabla H_i(\bar{x}), d\rangle = 0 (i \in I_{+0})\}.$$

By the proof similar to the proof of Lemma 4 in [6], we can prove that $L(\bar{x})$ is the linearized cone of (P) in the sense of nonlinear programming.

Remark 9. *We can check that*

$$L(\bar{x}) = (\bigcup_{t\in I_g} \nabla g_t(\bar{x}))^- \cap (\bigcup_{i\in I_h} \nabla h_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{0+}} \nabla G_i(\bar{x}))^\perp$$

$$\cap (\bigcup_{i\in I_{00}} (-\nabla G_i(\bar{x}))^- \cap (\bigcup_{i\in I_{00}} (-\nabla H_i(\bar{x}))^- \cap (\bigcup_{i\in I_{+0}} \nabla H_i(\bar{x}))^\perp.$$

Now, we establish the KKT necessary optimality condition for locally weakly efficient solutions of (P) under the following constraint qualification:

$$(\mathrm{ACQ}) : L(\bar{x}) \subseteq T(\Omega, \bar{x}).$$

Proposition 10. *Let $\bar{x} \in$ locWE(P). If (ACQ) holds at $\bar{x}$ and the set*

$$\Delta := \mathrm{pos}\left(\bigcup_{t\in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i\in I_{00}} (-\nabla G_i(\bar{x})) \cup -\nabla H_i(\bar{x})\right)$$

$$+\mathrm{span}\left(\bigcup_{i\in I_h} \nabla h_i(\bar{x}) \cup \bigcup_{i\in I_{0+}} \nabla G_i(\bar{x}) \cup \bigcup_{i\in I_{+0}} \nabla H_i(\bar{x})\right)$$

is closed, then $\bar{x}$ is a strong stationary point of (P).

Proof. Since $\bar{x} \in$ locWE(P), there exists $U \in \mathcal{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying

$$f_i(x) < f_i(\bar{x}), \forall i \in I. \tag{1}$$

First, we justify that

$$\left(\bigcup_{i\in I} \nabla f_i(\bar{x})\right)^s \cap T(\Omega, \bar{x}) = \emptyset. \tag{2}$$

On the contrary, assume that there exists $d \in \left(\bigcup_{i\in I} \nabla f_i(\bar{x})\right)^s \cap T(\Omega, \bar{x})$. Then, it is straightforward that

$$\langle \nabla f_i(\bar{x}), d\rangle < 0, \forall i \in I.$$

By $d \in T(\Omega, \bar{x})$, there exist $\tau_k \downarrow 0$ and $d_k \to d$ such that $\bar{x} + \tau_k d_k \in \Omega$ for all k. As $f_i (i \in I)$ is continuously differentiable at $\bar{x}$, one has

$$f_i(\bar{x} + \tau_k d_k) = f_i(\bar{x}) + \tau_k \langle \nabla f_i(\bar{x}), d_k\rangle + o(\tau_k \|d_k\|), \forall i \in I.$$

In consequence, for all $i \in I$,

$$\frac{f_i(\bar{x} + \tau_k d_k) - f_i(\bar{x})}{\tau_k} = \langle \nabla f_i(\bar{x}), d_k\rangle + \frac{o(\tau_k\|d_k\|)}{\tau_k\|d_k\|}.\|d_k\| \to \langle \nabla f_i(\bar{x}), d\rangle < 0, \text{ when } k \to \infty.$$

Hence, for each $i \in I$, there exists $\bar{k}_i > 0$ such that $\dfrac{f_i(\bar{x} + \tau_k d_k) - f_i(\bar{x})}{\tau_k} < 0$, for all $k > \bar{k}_i$. Setting $\bar{k} := \max\{\bar{k}_i \mid i \in I\}$, we guarantee the existence of $k > \bar{k}$ large enough such that $\bar{x} + \tau_k d_k \in \Omega \cap U$ and

$$f_i(\bar{x} + \tau_k d_k) < f_i(\bar{x}),\ \forall i \in I,$$

which contradicts (1). Therefore, the fulfillment of (2) follows. We get from (2) and (ACQ) that

$$\begin{aligned}&(\bigcup_{i\in I} \nabla f_i(\bar{x}))^s \cap (\bigcup_{t\in I_g} \nabla g_t(\bar{x}))^- \cap (\bigcup_{i\in I_h} \nabla h_i(\bar{x}))^\perp \cap (\bigcup_{i\in I_{0+}} \nabla G_i(\bar{x}))^\perp \\ &\cap (\bigcup_{i\in I_{00}} (-\nabla G_i(\bar{x}))^- \cap (\bigcup_{i\in I_{00}} (-\nabla H_i(\bar{x}))^- \cap (\bigcup_{i\in I_{+0}} \nabla H_i(\bar{x}))^\perp = \emptyset.\end{aligned}$$

This implies that there is no $d \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \nabla f_i(\bar{x}), d\rangle < 0, & \forall i \in I, \\ \langle \nabla g_t(\bar{x}), d\rangle \le 0, & \forall t \in I_g, \\ \langle \nabla h_i(\bar{x}), d\rangle = 0, & \forall i \in I_h, \\ \langle \nabla G_i(\bar{x}), d\rangle = 0, & \forall i \in I_{0+}, \\ \langle -\nabla G_i(\bar{x}), d\rangle \le 0, & \forall i \in I_{00}, \\ \langle -\nabla H_i(\bar{x}), d\rangle \le 0, & \forall i \in I_{00}, \\ \langle \nabla H_i(\bar{x}), d\rangle = 0, & \forall i \in I_{+0}. \end{cases}$$

In addition, it follows from Lemma 7 that $\mathrm{co}\{\bigcup_{i\in I} \nabla f_i(\bar{x})\}$ is a compact set, which in turn implies $\mathrm{co}\{\bigcup_{i\in I} \nabla f_i(\bar{x})\} + \Delta$ is closed. According to Lemma 6, one has

$$0 \in \mathrm{co} \bigcup_{i\in I} \nabla f_i(\bar{x}) + \mathrm{pos}\left(\bigcup_{t\in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i\in I_{00}} (-\nabla G_i(\bar{x})) \cup -\nabla H_i(\bar{x}))\right)$$

$$+\text{span}\left(\bigcup_{i\in I_h}\nabla h_i(\bar{x})\cup\bigcup_{i\in I_{0+}}\nabla G_i(\bar{x})\cup\bigcup_{i\in I_{+0}}\nabla H_i(\bar{x})\right).$$

This leads that

$$0\in \text{co}\bigcup_{i\in I}\nabla f(\bar{x})+\text{pos}\bigcup_{t\in I_g}\nabla g_t(\bar{x})+\text{span}\bigcup_{i\in I_h}\nabla h_i(\bar{x})+\text{pos}\bigcup_{i\in I_{00}}(-\nabla G_i(\bar{x}))$$
$$+\text{span}\bigcup_{i\in I_{0+}}\nabla G_i(\bar{x})+\text{pos}\bigcup_{i\in I_{00}}(-\nabla H_i(\bar{x}))+\text{span}\bigcup_{i\in I_{+0}}\nabla H_i(\bar{x}).$$

By Lemma 5, we know that there exists $(\alpha,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^m_+\times\Lambda(\bar{x})\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\lambda^G_{I_{+0}}=0$, $\lambda^G_{I_{00}}\geq 0$, $\lambda^H_{I_{00}}\geq 0$ and $\lambda^H_{I_{0+}}=0$ such that

$$\sum_{i\in I}\alpha_i\nabla f_i(\bar{x})+\sum_{t\in T}\lambda^g_t\nabla g_t(\bar{x})+\sum_{i\in I_h}\lambda^h_i\nabla h_i(\bar{x})-\sum_{i\in I_l}\lambda^G_i\nabla G_i(\bar{x})-\sum_{i\in I_l}\lambda^H_i\nabla H_i(\bar{x})=0.$$

So, $\bar{x}$ is a strong-stationary point of (P). □

Proposition 11. *Let $\bar{x}\in\Omega$ be a strong stationary point of (P). Suppose that $I^-_{0+}\cup\hat{I}^-_{+0}=\emptyset$ and $g_t(t\in I_g),h_i(i\in I^+_h),-h_i(i\in I^-_h),-G_i(i\in I^+_{0+}\cup I^+_{00}\cup I^{++}_{00}),-H_i(i\in\hat{I}^+_{00}\cup\hat{I}^+_{+0}\cup I^{++}_{00})$ are quasiconvex at $\bar{x}$.*

- *(i) If $f_i(i\in I)$ is pseudoconvex at $\bar{x}$, then $\bar{x}$ is a weakly efficient solution of (P).*
- *(ii) If $f_i(i\in I)$ is strictly pseudoconvex at $\bar{x}$, then $\bar{x}$ is an efficient solution of (P).*

Proof. Since $\bar{x}$ is a strong stationary point of (P), there exists $(\alpha,\lambda^g_J,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^m_+\times\mathbb{R}^{|J|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$, where J is a finite subset of I_g, with $\lambda^G_{I_{+0}}=0$, $\lambda^G_{I_{00}}\geq 0$, $\lambda^H_{I_{00}}\geq 0$ and $\lambda^H_{I_{0+}}=0$ such that

$$\sum_{i\in I}\alpha_i\nabla f_i(\bar{x})+\sum_{t\in J}\lambda^g_t\nabla g_t(\bar{x})+\sum_{i\in I_h}\lambda^h_i\nabla h_i(\bar{x})-\sum_{i\in I_{0+}\cup I_{00}}\lambda^G_i\nabla G_i(\bar{x})-\sum_{i\in I_{00}\cup I_{+0}}\lambda^H_i\nabla H_i(\bar{x})=0. \tag{3}$$

For an arbitrary $x\in\Omega$, one gets that $g_t(x)\leq 0=g_t(\bar{x})$ for each $t\in I_g$. Therefore, by the quasiconvexity at $\bar{x}$ of $g_t(t\in I_g)$, we have

$$\langle\nabla g_t(\bar{x}),x-\bar{x}\rangle\leq 0,\forall t\in J,$$

which in turn together with $\lambda^g_J\in\mathbb{R}^{|J|}_+$ derives that

$$\left\langle\sum_{t\in J}\lambda^g_t\nabla g_t(\bar{x}),x-\bar{x}\right\rangle\leq 0. \tag{4}$$

We deduce from $x,\bar{x}\in\Omega$ that $h_i(x)=h_i(\bar{x})=0,\forall i\in I_h$, and hence,

$$h_i(x)\leq h_i(\bar{x}),\forall i\in I^+_h \text{ and } -h_i(x)\leq -h(\bar{x}),\forall i\in I^-_h.$$

The above inequalities together with the quasiconvexity at $\bar{x}$ of $h_i(i \in I_h^+)$ and $-h_i(i \in I_h^-)$ ensures that

$$\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_h^+ \quad \text{and} \quad \langle -\nabla h_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_h^-.$$

This, taking into account the definitions of I_h^+, I_h^-, gives us

$$\left\langle \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \tag{5}$$

Again, we derive from $x \in \Omega$ that $-G_i(x) \leq 0, \forall i \in I_l$, and thus, $-G_i(x) \leq -G_i(\bar{x})(i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++})$. Therefore, by the quasiconvexity of $-G_i(i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++})$ at $\bar{x}$, one yields that

$$\langle -\nabla G_i(\bar{x}), x - \bar{x} \rangle \leq 0, \forall i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++},$$

which, along with the definitions of $I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++}$, leads that

$$-\left\langle \sum_{i \in I_{0+}^+ \cup I_{00}^+ \cup I_{00}^{++}} \lambda_i^G \nabla G_i(\bar{x}), x - \bar{x} \right\rangle \leq 0 \tag{6}$$

Similarly, we can justify that

$$-\left\langle \sum_{i \in \hat{I}_{00}^+ \cup \hat{I}_{+0}^+ \cup I_{00}^{++}} \lambda_i^H \nabla H_i(\bar{x}), x - \bar{x} \right\rangle \leq 0. \tag{7}$$

As $I_{0+}^- \cup \hat{I}_{+0}^- = \emptyset$, we infer from (3) - (7) that

$$\left\langle \sum_{i \in I} \alpha_i \nabla f_i(\bar{x}), x - \bar{x} \right\rangle$$

$$= -\left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(\bar{x}) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(\bar{x}), x - \bar{x} \right\rangle \geq 0, \tag{8}$$

for all $x \in \Omega$.

(i) Suppose, to the contrary, that $\bar{x}$ is not a weakly efficient solution of (P). This leads to the existence of a feasible point $\tilde{x} \in \Omega$ satisfying

$$f_i(\tilde{x}) < f_i(\bar{x}), \forall i \in I.$$

The fact on $f_i(\tilde{x}) < f_i(\bar{x})$ for each i and the pseudoconvexity of $f_i(i \in I)$ give us the inclusions

$$\langle \nabla f_i(\bar{x}), \tilde{x} - \bar{x} \rangle < 0, i \in I.$$

Combining this with $\alpha \in \mathbb{R}^m_+$ and $\sum_{i=1}^m \alpha_i = 1$, we arrive at

$$\left\langle \sum_{i\in I} \alpha_i \nabla f_i(\bar{x}), \tilde{x} - \bar{x} \right\rangle < 0,$$

contradicting with (8).

(ii) Reasoning by contraposition, assume that $\bar{x}$ is not an efficient solution. Then there exists a feasible point $\tilde{x}$ and at least $i_0 \in I$ fulfilling

$$\begin{cases} f_i(\tilde{x}) \le f_i(\bar{x}), & \forall i \in I \setminus \{i_0\}, \\ f_{i_0}(\tilde{x}) < f_{i_0}(\bar{x}), \end{cases}$$

and hence, $\tilde{x} \neq \bar{x}$. It follows from the fact that $f_i (i \in I)$ are strictly pseudoconvex and $x \neq \bar{x}$, one has

$$\langle \nabla f_i, \tilde{x} - \bar{x} \rangle < 0, \ \forall i \in I.$$

Using this with $\alpha \in \mathbb{R}^m_+$ and $\sum_{i=1}^m \alpha_i = 1$ tells us that

$$\left\langle \sum_{i\in I} \alpha_i \nabla f_i(\bar{x}), \tilde{x} - \bar{x} \right\rangle < 0,$$

which contradicts (8). □

Example 12. *Let $m = 2$, $n = 2$ and $l = 1$. Let us consider the following (P):*

$$\begin{aligned} \mathbb{R}^2_+ - \min \quad & f(x) = (f_1(x), f_2(x)) = (x_1^2 + x_2^2 + 2x_1, x_1^2 + 2x_2^2), \\ s.t. \quad & g_t(x) = tx_1 \le 0, t \in T = \mathbb{N} = \{1, 2, ...\}, \\ & G_1(x) = x_1 \ge 0, \\ & H_1(x) = x_1 + x_2 \ge 0, \\ & G_1(x)H_1(x) = x_1(x_1 + x_2) = 0. \end{aligned}$$

Then, $\Omega = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \ge 0\}$. For $\bar{x} = (0, 0) \in \Omega$, direct calculations give us that

$$T(\Omega, \bar{x}) = \Omega, \nabla f_1(\bar{x}) = \{(2, 0)\}, \nabla f_2(\bar{x}) = \{(0, 0)\}, I_g = T = \mathbb{N},$$

$$\nabla g_t(\bar{x}) = \{(t, 0)\}, t \in T, (\bigcup_{t\in I_g} \nabla g_t(\bar{x}))^- = \{x \in \mathbb{R}^2 \mid x_1 \le 0\},$$

$$I_{+0} = I_{0+} = \emptyset, I_{00} = \{1\}, \nabla G_1(\bar{x}) = \{(1, 0)\}, \nabla H_1(\bar{x}) = \{(1, 1)\},$$

$$(\bigcup_{i\in I_{00}} (-\nabla H_i(\bar{x})))^- = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \ge 0\}, (\bigcup_{i\in I_{00}} -\nabla G_i(\bar{x}))^- = \{x \in \mathbb{R}^2 \mid x_1 \ge 0\},$$

$$(\bigcup_{t\in I_g} \nabla g_t(\bar{x}))^- \cap (\bigcup_{i\in I_{00}} (-\nabla H_i(\bar{x})))^- \cap (\bigcup_{i\in I_{00}} \nabla G_i(\bar{x}))^- = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \ge 0\}.$$

Hence,

$$(\bigcup_{t\in I_g} \nabla g_t(\bar{x}))^- \cap (\bigcup_{i\in I_{00}} (-\nabla G_i(\bar{x})))^- \cap (\bigcup_{i\in I_{00}} -\nabla H_i(\bar{x}))^- \subset T(\Omega, \bar{x}),$$

leading that (ACQ) holds at $\bar{x}$. *Moreover,*

$$\Delta = \text{pos}\left(\bigcup_{t\in I_g} \nabla g_t(\bar{x}) \cup \bigcup_{i\in I_{00}} (-\nabla G_i(\bar{x})) \cup \bigcup_{i\in I_{00}} (-\nabla H_i(\bar{x}))\right) = \{x \in \mathbb{R}^2 \mid x_2 \le 0\}$$

is closed. Due to the fact $f(x) - f(\bar{x}) \notin -\mathbb{R}^2_+ \setminus \{0\}, \forall x \in \Omega$, *we conclude that* $\bar{x} \in WE(P)$. *Thus, all assumptions in Proposition 10 are fulfilled. Now, let* $\alpha_1 = \alpha_2 = \frac{1}{2}, \lambda_1^G = 2$, $\lambda_1^H = 0$ *and* $\lambda^g : T \to \mathbb{R}$ *be defined by*

$$\lambda^g(t) = \begin{cases} 1, & if\, t = 1, \\ 0, & otherwise. \end{cases}$$

Then,

$$\frac{1}{2}(2,0) + \frac{1}{2}(0,0) + \sum_{t\in T} \lambda_t^g(t,0) - \lambda_1^G(1,0) - \lambda_1^H(1,1) = (0,0),$$

which means that $\bar{x}$ *is a strong stationary point of (P). Notice that, for the above* $\bar{x}$ *and* $(\lambda^g, \lambda_1^H, \lambda_1^G)$, *one has*

$$I_{00}^{++} = I_{00}^{-} = \hat{I}_{00}^{+} = \hat{I}_{00}^{-} = \emptyset, I_{00}^{+} = \{1\}.$$

Furthermore, we can check that $g_t(t \in I_g)$, $-G_1(1 \in I_{00}^+)$ *are convex at* $\bar{x}$ *and* $f_i(i \in I)$ *are strictly convex at* $\bar{x}$. *Hence, all assumptions in Proposition 11 (ii) are satisfied. Then, it follows Proposition 11 (ii) that* $\bar{x}$ *is an efficient solution of (P).*

4. DUALITY

In this section, we consider the Wolfe [33] and Mond-Weir [19] duality schemes for (P). For $\bar{x} \in \Omega$, the index sets with respect to $\bar{x}$ are denoted identically to Section 3. In what follows, for $u, v \in \mathbb{R}^m$, we use the notations:

$$u \prec v \Leftrightarrow u_i < v_i \text{ for all } i \in I,\ u \nprec v \text{ is the negation of } u \prec v,$$

$$u \preceq v \Leftrightarrow \begin{cases} u_i \le v_i, & \text{for all } i \in I, \\ u_i < v_i, & \text{for at least one } i_0 \in I, \end{cases} \quad u \npreceq v \text{ is the negation of } u \preceq v.$$

Note that $\bar{x} \in \text{loc}E(P)$ ($\bar{x} \in \text{loc}WE(P)$) if there exists $U \in \mathcal{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying $f(x) \preceq f(\bar{x})$ ($f(x) \prec f(\bar{x})$).

4.1. The Wolfe type duality

For an arbitrary $\bar{x} \in \Omega$, $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\sum_{i\in I} \alpha_i = 1$, $\lambda^G_{I_{+0}(\bar{x})} = 0$, $\lambda^G_{I_{00}(\bar{x})} \ge 0$, $\lambda^H_{I_{00}(\bar{x})} \ge 0$ and $\lambda^H_{I_{0+}(\bar{x})} = 0$, we define

$$L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) := f(u)$$

$$+\left(\sum_{t\in T}\lambda_t g_t(u)+\sum_{i\in I_h}\lambda_i^h h_i(u)-\sum_{i\in I_l}\lambda_i^G G_i(u)-\sum_{i\in I_l}\lambda_i^H H_i(u)\right)e,$$

where $e := (1, ..., 1) \in \mathbb{R}^m$. In this paper, we consider the Wolfe type dual problem as follows:

$(\mathrm{D}_W(\bar{x}))$: $\mathbb{R}^m_+ - \max L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H)$

$$\text{s.t.} \sum_{i\in I}\alpha_i\nabla f_i(u)+\sum_{t\in T}\lambda_t^g\nabla g_t(u)+\sum_{i\in I_h}\lambda_i^h\nabla h_i(u)-\sum_{i\in I_l}\lambda_i^G\nabla G_i(u)-\sum_{i\in I_l}\lambda_i^H\nabla H_i(u)=0,$$

$$\sum_{i\in I}\alpha_i=1,\ \lambda^G_{I_{+0}(\bar{x})}=0,\ \lambda^G_{I_{00}(\bar{x})}\geq 0,\ \lambda^H_{I_{00}(\bar{x})}\geq 0,\ \lambda^H_{I_{0+}(\bar{x})}=0,$$

$$(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l.$$

The feasible set of $(\mathrm{D}_W(\bar{x}))$ is defined by

$$\Omega_W(\bar{x}) := \Big\{(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l \mid$$

$$\sum_{i\in I}\alpha_i=1, \lambda^G_{I_{+0}(\bar{x})}=0, \lambda^G_{I_{00}(\bar{x})}\geq 0, \lambda^H_{I_{00}(\bar{x})}\geq 0, \lambda^H_{I_{0+}(\bar{x})}=0,$$

$$\sum_{i\in I}\alpha_i\nabla f_i(u)+\sum_{t\in T}\lambda_t^g\nabla g_t(u)+\sum_{i\in I_h}\lambda_i^h\nabla h_i(u)-\sum_{i\in I_l}\lambda_i^G\nabla G_i(u)-\sum_{i\in I_l}\lambda_i^H\nabla H_i(u)=0\Big\}.$$

Definition 13. *Let $\bar{x} \in \Omega$.*

(i) $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$ is a locally efficient solution of $(\mathrm{D}_W(\bar{x}))$, denoted by $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in locE(\mathrm{D}_W(\bar{x}))$, if there exists $U \in \mathcal{N}(\bar{u})$ such that there is no $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x}) \cap U$ satisfying

$$L(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii) $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$ is a locally weakly efficient solution of $(\mathrm{D}_W(\bar{x}))$, denoted by $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in locWE(\mathrm{D}_W(\bar{x}))$, if there exists $U \in \mathcal{N}(\bar{u})$ such that there is no $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x}) \cap U$ fulfilling

$$L(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

If $U = \mathbb{R}^n$, the word "locally" is omitted.

Remark 14. *When $m = 1$ and f_1, $g_t(t \in T)$, $h_i(i = 1, ..., q)$ and $G_i, H_i(i = 1, ..., l)$ are continuously differentiable functions, $\mathrm{D}_W(\bar{x})$ becomes the Wolfe type dual model* WDSIMPEC$(\bar{x})$ *in [20].*

The following proposition describes weak duality relations between (P) and the dual problem $(\mathrm{D}_W(\bar{x}))$.

Proposition 15. *(weak duality) Let $x \in \Omega$ and $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$. Suppose that $I^-_{0+}(\bar{x}) \cup \hat{I}^-_{+0}(\bar{x}) = \emptyset$ and $g_t(t \in T), h_i(i \in I^+_h(\bar{x})), -h_i(i \in I^-_h(\bar{x}))$, $-G_i(i \in I^+_{0+}(\bar{x}) \cup I^+_{00}(\bar{x}) \cup I^{++}_{00}(\bar{x}))$, $-H_i(i \in \hat{I}^+_{+0}(\bar{x}) \cup \hat{I}^+_{00}(\bar{x}) \cup I^{++}_{00}(\bar{x}))$ are convex at u.*

(i) If $f_i(i \in I)$ *are convex at* u*, then*

$$f(x) \not< L(y, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii) If $f_i(i \in I)$ *are strictly convex at* u*, then*

$$f(x) \not\leq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

Proof. For $x \in \Omega$ and $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$, one gets

$$g_t(x) \leq 0 (t \in T), h_i(x) = 0 (i \in I_h), G_i(x) \geq 0, H_i(x) \geq 0, G_i(x)H_i(x) = 0 (i \in I_l), \tag{9}$$

and

$$\sum_{i \in I} \alpha_i \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0 \tag{10}$$

with

$$\sum_{i \in I} \alpha_i = 1, \lambda^G_{I_{+0}(\bar{x})} = 0, \lambda^G_{I_{00}(\bar{x})} \geq 0, \lambda^H_{I_{00}(\bar{x})} \geq 0, \lambda^H_{I_{0+}(\bar{x})} = 0. \tag{11}$$

Therefore, we infer from (9), the convexity of $g_t(t \in T), h_i(i \in I_h^+(\bar{x}))$, $-h_i(i \in I_h^-(\bar{x}))$, $-G_i(i \in I_{0+}^+(x) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$, $-H_i(i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$ at u and the definitions of the index sets that

$$g_t(u) + \langle \nabla g_t(u), x - u \rangle \leq g_t(x) \leq 0, \lambda_t^g \geq 0, \forall t \in T,$$

$$h_i(u) + \langle \nabla h_i(u), x - u \rangle \leq h_i(x) = 0, \lambda_i^h > 0, \forall i \in I_h^+(\bar{x}),$$

$$-h_i(u) + \langle -\nabla h_i(u), x - u \rangle \leq -h_i(x) = 0, \lambda_i^h < 0, \forall i \in I_h^-(\bar{x}),$$

$$-G_i(u) + \langle -\nabla G_i(u), x - u \rangle \leq -G_i(x) \leq 0, \lambda_i^G > 0, \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}),$$

$$-H_i(u) + \langle -\nabla H_i(u), x - u \rangle \leq -H_i(x) \leq 0, \lambda_i^H > 0, \forall i \in \hat{I}_{+0}^+(\bar{x}) \cup \hat{I}_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}).$$

The above inequalities together with $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$ imply that

$$\sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u)$$

$$+ \left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle \leq 0.$$

It follows from the above inequality and (10) that

$$\langle \sum_{i \in I} \alpha_i \nabla f_i(u), x - u \rangle$$

$$= -\left\langle \sum_{t\in T}\lambda_t^g \nabla g_t(u) + \sum_{i\in I_h}\lambda_i^h \nabla h_i(u) - \sum_{i\in I_l}\lambda_i^G \nabla G_i(u) - \sum_{i\in I_l}\lambda_i^H \nabla H_i(u), x-u \right\rangle$$
$$\geq \sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) - \sum_{i\in I_l}\lambda_i^G G_i(u) - \sum_{i\in I_l}\lambda_i^H H_i(u). \quad (12)$$

(i) Reasoning ad absurdum, suppose that

$$f(x) \prec L(u,\alpha,\lambda^g,\lambda^h,\lambda^G,\lambda^H). \quad (13)$$

It follows from (13), $\alpha \in \mathbb{R}^m_+$ and $\sum_{i=1}^m \alpha_i = 1$ that $\langle \alpha, f(x) - L(u,\alpha,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\rangle <$ 0, which is equivalent to

$$\sum_{i=1}^m \alpha_i(f_i(x)-f_i(u)) - \sum_{i=1}^m \alpha_i \left(\sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) - \sum_{i\in I_l}\lambda_i^G G_i(u) - \sum_{i\in I_l}\lambda_i^H H_i(u)\right) < 0.$$

The above inequality, together with $\sum_{i=1}^m \alpha_i = 1$, yields

$$\sum_{i=1}^m \alpha_i(f_i(x)-f_i(u)) < \left(\sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) - \sum_{i\in I_l}\lambda_i^G G_i(u) - \sum_{i\in I_l}\lambda_i^H H_i(u)\right). \quad (14)$$

The convexity of $f_i (i \in I)$ at u confirms that

$$\langle \nabla f_i(u), x-u\rangle \leq f_i(x) - f_i(u), \forall i \in I,$$

leading to

$$\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x-u\rangle \leq \sum_{i=1}^m \alpha_i (f_i(x) - f_i(u)). \quad (15)$$

We verify from (14) and (15) that

$$\langle \sum_{i=1}^m \alpha_i \nabla f_i(u), x-u\rangle < \left(\sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) - \sum_{i\in I_l}\lambda_i^G G_i(u) - \sum_{i\in I_l}\lambda_i^H H_i(u)\right),$$

contradicting with (12).
(ii) Reasoning by contraposition, assume that

$$f(x) \preceq L(u,\alpha,\lambda^g,\lambda^h,\lambda^G,\lambda^H), \quad (16)$$

We claim that $x \neq u$. If otherwise, we use (16) and $x = u$ to derive that

$$a := -\left(\sum_{t\in T}\lambda_t g_t(u) + \sum_{i\in I_h}\lambda_i^h h_i(u) - \sum_{i\in I_l}\lambda_i^G G_i(u) - \sum_{i\in I_l}\lambda_i^H H_i(u)\right) e \preceq 0. \quad (17)$$

Observe by $u = x \in \Omega(\bar{x})$ and (11) that

$$g_t(u) = g_t(x) \leq 0, \forall t \in T,\ \lambda \in \mathbb{R}_+^{|T|},$$

$$h_i(u) = h_i(x) = 0, \forall i \in I_h, \lambda_i^h \in \mathbb{R},$$

$$-G_i(u) = -G_i(x) \leq 0, \forall i \in I_{0+}^{+}(\bar{x}) \cup I_{00}^{+}(\bar{x}) \cup I_{00}^{++}(\bar{x}), \lambda^H_{I_{0+}^{+}(\bar{x}) \cup I_{00}^{+}(\bar{x}) \cup I_{00}^{++}(\bar{x})} \geq 0,$$

$$-H_i(u) = -H_i(x) \leq 0, \forall i \in \hat{I}_{+0}^{+}(\bar{x}) \cup \hat{I}_{00}^{+}(\bar{x}) \cup I_{00}^{++}(\bar{x}), \lambda^H_{\hat{I}_{+0}^{+}(\bar{x}) \cup \hat{I}_{00}^{+}(\bar{x}) \cup I_{00}^{++}(\bar{x})} \geq 0.$$

The above inequalities together with $I_{0+}^{-}(\bar{x}) \cup \hat{I}_{+0}^{-}(\bar{x}) = \emptyset$ imply that

$$\sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \leq 0.$$

Hence, $a_i \geq 0, \forall i \in I$, contradicts with (17), which in turn leads to $x \neq u$. On the other hand, we deduce from (16) and $\alpha \in \mathbb{R}_+^m$ that $\langle \alpha, f(x) - L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \rangle \leq 0$, in other words,

$$\sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)) - \sum_{i=1}^{m} \alpha_i \left(\sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right) \leq 0.$$

Employing this, together with $\sum_{i=1}^{m} \alpha_i = 1$, bring us the inequality

$$\sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)) \leq \sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u). \quad (18)$$

Since $f_i (i \in I)$ are strictly convex at u and $x \neq u$, we have

$$\langle \nabla f_i(u), x - u \rangle < f_i(x) - f_i(u), \forall i \in I,$$

leading that

$$\langle \sum_{i=1}^{m} \alpha_i \nabla f_i(u), x - u \rangle < \sum_{i=1}^{m} \alpha_i (f_i(x) - f_i(u)). \quad (19)$$

It follows from (18) and (19) that

$$\langle \sum_{i=1}^{m} \alpha_i \nabla f_i(u), x - u \rangle < \left(\sum_{t \in T} \lambda_t g_t(u) + \sum_{i \in I_h} \lambda_i^h h_i(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right),$$

contradicting with (12). □

Proposition 16. *(strong duality) Let $\bar{x} \in \Omega$ be a locally weakly efficient solution of (P). If (ACQ) holds at $\bar{x}$ and the set Δ is closed, then there exists $(\bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\sum_{i\in I} \bar{\alpha}_i = 1, \bar{\lambda}^G_{I_{+0}(\bar{x})} = 0, \bar{\lambda}^G_{I_{00}(\bar{x})} \geq 0, \bar{\lambda}^H_{I_{00}(\bar{x})} \geq 0$ and $\bar{\lambda}^H_{I_{0+}(\bar{x})} = 0$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$ and*

$$f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H).$$

Assume further that $I^-_{0+}(\bar{x}) \cup \hat{I}^-_{+0}(\bar{x}) = \emptyset$ and $g_t(t \in T), h_i(i \in I^+_h(\bar{x})), -h_i(i \in I^-_h(\bar{x}))$, $-G_i(i \in I^+_{0+}(\bar{x}) \cup I^+_{00}(\bar{x}) \cup I^{++}_{00}(\bar{x}))$, $-H_i(i \in \hat{I}^+_{+0}(\bar{x}) \cup \hat{I}^+_{00}(\bar{x}) \cup I^{++}_{00}(\bar{x}))$ are convex at $\bar{x}$.

- *(i) If $f_i(i \in I)$ are convex at $\bar{x}$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is a weakly efficient solution of $D_W(\bar{x})$.*
- *(ii) If $f_i(i \in I)$ are strictly convex at $\bar{x}$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is an efficient solution of $D_W(\bar{x})$.*

Proof. In view of Proposition 10, there exists $(\bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^m_+ \times \Lambda(\bar{x}) \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\sum_{i\in I} \bar{\alpha}_i = 1, \bar{\lambda}^G_{I_{+0}(\bar{x})} = 0, \bar{\lambda}^G_{I_{00}(\bar{x})} \geq 0, \bar{\lambda}^H_{I_{00}(\bar{x})} \geq 0$ and $\bar{\lambda}^H_{I_{0+}(\bar{x})} = 0$ such that

$$\sum_{i\in I} \bar{\alpha}_i \nabla f_i(\bar{x}) + \sum_{t\in T} \bar{\lambda}^g_t \nabla g_t(\bar{x}) + \sum_{i\in I_h} \bar{\lambda}^h_i \nabla h_i(\bar{x}) - \sum_{i\in I_l} \bar{\lambda}^G_i \nabla G_i(\bar{x}) - \sum_{i\in I_l} \bar{\lambda}^H_i \nabla H_i(\bar{x}) = 0.$$

Since $\bar{\lambda}^g \in \Lambda(\bar{x})$, one has $\bar{\lambda}^g_t g_t(\bar{x}) = 0$ for all $t \in T$, and thus, $\sum_{t\in T} \bar{\lambda}^g_t g_t(\bar{x}) = 0$. The fact that $\bar{x} \in \Omega$ ensures that $\sum_{i\in I_h} \bar{\lambda}^h_i h_i(\bar{x}) = 0$. Moreover, as $\lambda^G_{I_{+0}(\bar{x})} = 0$ and $G_i(\bar{x}) = 0$ for all $i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})$, we know that $\sum_{i\in I_l} \bar{\lambda}^G_i G_i(\bar{x}) = 0$. Analogously, we observe by $\bar{\lambda}^H_{I_{0+}(\bar{x})} = 0$ and $H_i(\bar{x}) = 0$ for all $i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x})$ that $\sum_{i\in I_l} \bar{\lambda}^H_i H_i(\bar{x}) = 0$. Thus, $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_W(\bar{x})$ and

$$\sum_{t\in T} \lambda_t g_t(\bar{x}) + \sum_{i\in I_h} \lambda^h_i h_i(\bar{x}) - \sum_{i\in I_l} \lambda^G_i G_i(\bar{x}) - \sum_{i\in I_l} \lambda^H_i H_i(\bar{x}) = 0,$$

which is nothing else but the following equality $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$.
(i) Now, arguing by contradiction, let us suppose that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is not a weakly efficient solution of $D_W(\bar{x})$. By definition, there exists $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$ such that

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

This shows that

$$f(\bar{x}) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

which contradicts with Proposition 15 (i). So, $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is a weakly efficient solution to $(D_W(\bar{x}))$.

(ii) Reasoning to the contrary, let us assume that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is not an efficient solution to $D_W(\bar{x})$. Then, it guarantees the existence of $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$ such that

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

Consequently,

$$f(\bar{x}) \preceq L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

which contradicts with Proposition 15 (ii). So, $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is an efficient solution to $(D_W(\bar{x}))$. □

Example 17. *Let $m = n = 2$ and $l = 1$. Consider the following (P):*

$$\begin{aligned} \mathbb{R}^2_+ - \min \quad & f(x) = (x_1^2 + x_2^2 + 4x_2, x_1 - x_2), \\ s.t. \quad & g_t(x) = tx_1 \leq 0, t \in T = \mathbb{N}, \\ & G_1(x) = x_1 \geq 0, \\ & H_1(x) = x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) = x_1(x_1 + x_2) = 0. \end{aligned}$$

Then, $\Omega = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$. For any $\bar{x} \in \Omega$,

$$\begin{aligned} (\mathrm{D}_{MW}(\bar{x})) : \mathbb{R}^2_+ - \max L(u, \alpha, \lambda^g, \lambda^G, \lambda^H) &= (u_1^2 + u_2^2 + 4u_2, u_1 - u_2) + \left(\sum_{t \in T} tu_1 - \lambda_1^G u_1 - \lambda_1^H (u_1 + u_2) \right)(1, 1) \end{aligned}$$

$$s.t.\ \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1) + \sum_{t \in T} \lambda_t^g(t, 0) - \lambda_1^G(1, 0) - \lambda_1^H(1, 1) = (0, 0),$$

$$\alpha_1 + \alpha_2 = 1,\ \lambda_1^G \begin{cases} = 0, & \text{if } 1 \in I_{+0}(\bar{x}), \\ \geq 0, & \text{if } 1 \in I_{00}(\bar{x}), \\ \in \mathbb{R}, & \text{if } 1 \in I_{0+}(\bar{x}), \end{cases} \quad \lambda_1^H \begin{cases} \in \mathbb{R}, & \text{if } 1 \in I_{+0}(\bar{x}), \\ \geq 0, & \text{if } 1 \in I_{00}(\bar{x}), \\ = 0, & \text{if } 1 \in I_{0+}(\bar{x}), \end{cases}$$

$$(u, \alpha, \lambda^g, \lambda_1^G, \lambda_1^H) \in \mathbb{R}^2 \times \mathbb{R}^2_+ \times \mathbb{R}^{|T|}_+ \times \mathbb{R} \times \mathbb{R}.$$

By taking $\bar{x} = (0, 0) \in \Omega$, we invoke from Example 12 that all hypotheses of Proposition 16 (i) are fulfilled. Since $f(x) - f(\bar{x}) = (x_2^2 + 4x_2, -x_2) \notin -\mathrm{int}\mathbb{R}^2_+, \forall x \in \Omega$, one has $\bar{x} \in WE(P)$. Now, if we select $\bar{\alpha}_1 = \bar{\alpha}_2 = \frac{1}{2}, \bar{\lambda}_1^G = 0, \bar{\lambda}_1^H = \frac{3}{2}$ and

$$\bar{\lambda}^g(t) = \begin{cases} 1, & \text{if } t = 1, \\ 0, & \text{otherwise}, \end{cases}$$

then we get

$$\frac{1}{2}(0, 4) + \frac{1}{2}(1, -1) + \sum_{t \in T} \bar{\lambda}_t^g(t, 0) - \bar{\lambda}_1^G(1, 0) - \bar{\lambda}_1^H(1, 1) = (0, 0),$$

and,

$$I_{0+}(\bar{x}) = I_{0+}(\bar{x}) = \emptyset, I_{00}(\bar{x}) = \{1\},$$
$$\bar{\lambda}_1^H = 1 \geq 0, \bar{\lambda}_1^G = 0 \geq 0, 1 \in I_{00}(\bar{x}),$$

which gives the result $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H) \in \Omega_W(\bar{x})$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$. Note that, for the above $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$,

$$\hat{I}_{00}^+(\bar{x}) = \{1\}, \hat{I}_{00}^-(\bar{x}) = I_{00}^+(\bar{x}) = I_{00}^-(\bar{x}) = I_{00}^{++}(\bar{x}) = \emptyset.$$

Moreover, we can verify that $f_1, f_2, g_t(t \in T), -H_i(i \in \hat{I}_{00}^{+}(\bar{x}))$ *are convex at* $\bar{x}$. *Hence, Proposition 16 (i) asserts that* $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$ *is a weakly efficient solution to* $(\mathrm{D}_W(\bar{x}))$.

We can check directly that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$ *is a weakly efficient solution to* $(\mathrm{D}_W(\bar{x}))$ *as follows. Firstly, we conclude from* $\bar{x} = (0,0)$ *and* $I_{0+}(\bar{x}) = I_{0+}(\bar{x}) = \emptyset, I_{00}(\bar{x}) = \{1\}$ *that*

$$\Omega_W(\bar{x}) = \Big\{(u, \alpha, \lambda^g, \lambda_1^G, \lambda_1^H) \in \mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{R}_+^{|T|} \times \mathbb{R} \times \mathbb{R} \mid \alpha_1 + \alpha_2 = 1, \lambda_1^G \geq 0, \lambda_1^H \geq 0$$

$$\alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1) + \sum_{t \in T} \lambda_t^g (t, 0) - \lambda_1^G (1, 0) - \lambda_1^H (1, 1) = (0, 0)\Big\}.$$

Now, for an arbitrary $u \in \Omega_W(\bar{x})$, *the convexity of* $g_t(t \in T), -G_i(i \in I_{00}^{+}), -H_i(i \in \hat{I}_{00}^{+}(\bar{x}))$ *at* u *and the definitions of the index sets deduce the inequalities*

$$g_t(u) + \langle (t, 0), \bar{x} - u \rangle \leq g_t(\bar{x}) \leq 0, \lambda_t^g \geq 0, \forall t \in T,$$

$$-G_1(u) + \langle -(1, 0), \bar{x} - u \rangle \leq -G_1(\bar{x}) = 0, \lambda_1^G > 0, \mathit{if}\, 1 \in I_{00}^{+}(\bar{x}) \cup I_{00}^{++}(\bar{x}),$$

$$-H_1(u) + \langle -(1, -1), \bar{x} - u \rangle \leq -H_1(\bar{x}) = 0, \lambda_1^H > 0, \mathit{if}\, 1 \in \hat{I}_{00}^{+}(\bar{x}) \cup I_{00}^{++}(\bar{x}).$$

We deduce from the above inequalities, $u \in \Omega_W(\bar{x})$ *and* $I_{00}^{-}(\bar{x}) = \hat{I}_{00}^{-}(\bar{x}) = \emptyset$ *that*

$$\langle \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1), \bar{x} - u \rangle = -\left\langle \sum_{t \in T} \lambda_t^g (t, 0) - \lambda_1^G (1, 0) - \lambda_1^H (1, 1), \bar{x} - u \right\rangle$$

$$\geq \sum_{t \in T} \lambda_t g_t(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u). \tag{20}$$

Reasoning by contraposition, suppose that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$ *is not a weakly efficient solution to* $(\mathrm{D}_W(\bar{x}))$. *Then there exists* $(u, \alpha, \lambda^g, \lambda^G, \lambda^H) \in \Omega_W(\bar{x})$ *such that*

$$L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H) \prec L(u, \alpha, \lambda^g, \lambda_1^G, \lambda_1^H).$$

This along with $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}_1^G, \bar{\lambda}_1^H)$, $\alpha \in \mathbb{R}_+^2$ *and* $\sum_{i=1}^{2} \alpha_i = 1$ *gives us that* $\langle \alpha, f(\bar{x}) - L(u, \alpha, \lambda^g, \lambda^G, \lambda^H) \rangle < 0$, *which is equivalent to*

$$\sum_{i=1}^{2} \alpha_i (f_i(\bar{x}) - f_i(u)) - \left(\sum_{t \in T} \lambda_t g_t(u) - \sum_{i \in I_l} \lambda_i^G G_i(u) - \sum_{i \in I_l} \lambda_i^H H_i(u) \right) < 0.$$

From the above relation together with (20), we derive

$$\sum_{i=1}^{2} \alpha_i (f_i(\bar{x}) - f_i(u)) < \langle \alpha_1(2u_1, 2u_2 + 4) + \alpha_2(1, -1), \bar{x} - u \rangle\,. \tag{21}$$

On the other hand, since f_1, f_2 are convexity at u, this yields

$$\langle (2u_1, 2u_2+4), \bar{x}-u\rangle \leq f_1(\bar{x}) - f_1(u),$$

$$\langle (1,-1), \bar{x}-u\rangle \leq f_2(\bar{x}) - f_2(u),$$

which, taking into account $\alpha \in \mathbb{R}^m_+$, justifies that

$$\langle \alpha_1(2u_1, 2u_2+4) + \alpha_2(1,-1), \bar{x}-u\rangle \leq \sum_{i=1}^{2} \alpha_i (f_i(\bar{x}) - f_i(u)),$$

contradicting with (21).

4.2. The Mond-Weir type duality

For an arbitrary $\bar{x} \in \Omega$ and $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ with $\sum_{i\in I} \alpha_i = 1$, $\lambda^g_{T\setminus I_g(\bar{x})} = 0$, $\lambda^G_{I_{+0}(\bar{x})} = 0$, $\lambda^G_{I_{00}(\bar{x})} \geq 0$, $\lambda^H_{I_{00}(\bar{x})} \geq 0$ and $\lambda^H_{I_{0+}(\bar{x})} = 0$, we define

$$\widetilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) := f(u).$$

Now, we consider the Mond-Weir type dual problem as follows:

($\mathrm{D}_{MW}(\bar{x})$): $\max \widetilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) = f(u)$

$$\text{s.t.} \sum_{i\in I} \alpha_i \nabla f_i(u) + \sum_{t\in T} \lambda^g_t \nabla g_t(u) + \sum_{i\in I_h} \lambda^h_i \nabla h_i(u) - \sum_{i\in I_l} \lambda^G_i \nabla G_i(u) - \sum_{i\in I_l} \lambda^H_i \nabla H_i(u) = 0,$$

$$g_t(u) \geq 0 (t \in I_g(\bar{x})), h_i(u) = 0 (i \in I_h(\bar{x})),$$

$$G_i(u) \geq 0 (i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x}), H_i(u) \geq 0 (i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}))$$

$$\sum_{i\in I} \alpha_i = 1, \lambda^g_{T\setminus I_g(\bar{x})} = 0, \lambda^G_{I_{+0}(\bar{x})} = 0, \lambda^G_{I_{00}(\bar{x})} \geq 0, \lambda^H_{I_{00}(\bar{x})} \geq 0, \lambda^H_{I_{0+}(\bar{x})} = 0,$$

$$(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l.$$

The feasible set of ($\mathrm{D}_{MW}(\bar{x})$) is defined by

$$\Omega_{MW}(\bar{x}) := \Big\{ (u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l \mid$$

$$\sum_{i\in I} \alpha_i \nabla f_i(u) + \sum_{t\in T} \lambda^g_t \nabla g_t(u) + \sum_{i\in I_h} \lambda^h_i \nabla h_i(u) - \sum_{i\in I_l} \lambda^G_i \nabla G_i(u) - \sum_{i\in I_l} \lambda^H_i \nabla H_i(u) = 0,$$

$$g_t(u) \geq 0 (t \in I_g(\bar{x})), h_i(u) = 0 (i \in I_h(\bar{x})),$$

$$G_i(u) \geq 0 (i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x}), H_i(u) \geq 0 (i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}))$$

$$\sum_{i\in I} \alpha_i = 1, \lambda^g_{T\setminus I_g(\bar{x})} = 0, \lambda^G_{I_{+0}(\bar{x})} = 0, \lambda^G_{I_{00}(\bar{x})} \geq 0, \lambda^H_{I_{00}(\bar{x})} \geq 0, \lambda^H_{I_{0+}(\bar{x})} = 0, \Big\}.$$

Definition 18. *Let $\bar{x} \in \Omega$.*

(i) $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$ *is said to be a locally efficient solution to* $(\mathrm{D}_{MW}(\bar{x}))$*, denoted by* $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in locE(\mathrm{D}_{MW}(\bar{x}))$*, if there exists* $U \in \mathcal{N}(\bar{u})$ *such that there is no* $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x}) \cap U$ *fulfilling*

$$\widetilde{L}(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq \widetilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii) $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$ *is called a locally weakly efficient solution to* $(\mathrm{D}_{MW}(\bar{x}))$*, denoted by* $(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in locWE(\mathrm{D}_{MW}(\bar{x}))$*, if there exists* $U \in \mathcal{N}(\bar{u})$ *such that there is no* $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x}) \cap U$ *satisfying*

$$\widetilde{L}(\bar{u}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec \widetilde{L}(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

If $U = \mathbb{R}^n$*, the word "locally" is dropped.*

Remark 19. *When* $m = 1$ *and* f_1*,* $g_t (t \in T)$*,* $h_i (i = 1, ..., q)$ *and* $G_i, H_i (i = 1, ..., l)$ *are continuously differentiable functions,* $\mathrm{D}_{MW}(\bar{x})$ *becomes the Mond-Weir type dual model* MWDSIMEC$(\bar{x})$ *in [20].*

Proposition 20. *(weak duality) Let* $x \in \Omega$ *and* $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$*. Suppose that* $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$ *and* $g_t (t \in T), h_i (i \in I_h^+(\bar{x})), -h_i (i \in I_h^-(\bar{x})), -G_i (i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i (i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$ *are quasiconvex at* u*.*

(i) If $f_i (i \in I)$ *are pseudoconvex at* u*, then*

$$f(x) \not\prec \widetilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

(ii) If $f_i (i \in I)$ *are strictly pseudoconvex at* u*, then*

$$f(x) \not\preceq \widetilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H).$$

Proof. For $x \in \Omega$ and $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$, we have

$$g_t(x) \le 0 (t \in T), h_i(x) = 0 (i \in I_h), G_i(x) \ge 0, H_i(x) \ge 0, G_i(x) H_i(x) = 0 (i \in I_l), \tag{22}$$

$$\sum_{i \in I} \nabla f_i(u) + \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u) = 0, \tag{23}$$

and

$$\begin{gathered} g_t(u) \ge 0 (t \in I_g(\bar{x})), h_i(u) = 0 (i \in I_h), \\ G_i(u) \ge 0 (i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})), H_i(u) \ge 0 (i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x})) \end{gathered} \tag{24}$$

with $\sum\limits_{i \in I} \alpha_i = 1, \lambda_{T \setminus I_g(\bar{x})}^g = 0, \lambda_{I_{+0}(\bar{x})}^G = 0, \lambda_{I_{00}(\bar{x})}^G \ge 0, \lambda_{I_{00}(\bar{x})}^H \ge 0, \lambda_{I_{0+}(\bar{x})}^H = 0.$

It follows from the above inequalities that

$$g_t(x) \leq 0 \leq g_t(u), \forall t \in I_g(\bar{x}),$$

$$h_i(x) = h_i(u) = 0, \forall i \in I_h^+(\bar{x}) \cup I_h^-(\bar{x}),$$
$$-G_i(x) \leq 0 \leq -G_i(u), \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}),$$
$$-H_i(x) \leq 0 \leq -H_i(u), \forall i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}).$$

Therefore, we deduce from the quasiconvexity of $g_t (t \in T), h_i(i \in I_h^+(\bar{x})), -h_i(i \in I_h^-(\bar{x})), -G_i(i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x})), -H_i(i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}))$ at u and the definitions of the index sets that

$$\langle \nabla g_i(u), x - u \rangle \leq 0, \lambda_i^g \geq 0, \forall i \in I_g(\bar{x}),$$

$$\langle \nabla h_i(u), x - u \rangle \leq 0, \lambda_i^h > 0, \forall i \in I_h^+(\bar{x}),$$
$$\langle -\nabla h_i(u), x - u \rangle \leq 0, \lambda_i^h < 0, \forall i \in I_h^-(\bar{x}),$$
$$\langle -\nabla G_i(u), x - u \rangle \leq 0, \lambda_i^G > 0, \forall i \in I_{0+}^+(\bar{x}) \cup I_{00}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}),$$
$$\langle -\nabla H_i(u), x - u \rangle \leq 0, \lambda_i^H > 0, \forall i \in \hat{I}_{00}^+(\bar{x}) \cup \hat{I}_{+0}^+(\bar{x}) \cup I_{00}^{++}(\bar{x}).$$

It follows from the above inequalities, $I_{0+}^-(\bar{x}) \cup \hat{I}_{+0}^-(\bar{x}) = \emptyset$, $\lambda_{T \setminus I_g(\bar{x})}^g = 0$ and (23) that

$$\langle \sum_{i \in I} \alpha_i \nabla f_i(u), x - u \rangle$$

$$= -\left\langle \sum_{t \in I_g(\bar{x})} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle$$

$$= -\left\langle \sum_{t \in T} \lambda_t^g \nabla g_t(u) + \sum_{i \in I_h} \lambda_i^h \nabla h_i(u) - \sum_{i \in I_l} \lambda_i^G \nabla G_i(u) - \sum_{i \in I_l} \lambda_i^H \nabla H_i(u), x - u \right\rangle \geq 0. \tag{25}$$

(i) Suppose by contradiction that

$$f(x) \prec L(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

equivalently,

$$f_i(x) < f_i(u), \forall i \in I.$$

The above inequalities and the pseudoconvexity of $f_i (i \in I)$ at u tell us that

$$\langle \nabla f_i(u), x - u \rangle < 0, \forall i \in I,$$

which, along with $\sum_{i\in I}\alpha_i = 1$, lead to

$$\langle\sum_{i=1}^{m}\alpha_i\nabla f_i(u), x-u\rangle < 0,$$

contradicting with (25).
(ii) Assume by contradiction that

$$f(x) \preceq L(y,\alpha,\lambda^g,\lambda^h,\lambda^G,\lambda^H).$$

This is equivalent to saying that

$$\begin{cases} f_i(x) \le f_i(u), & \forall i \in I, \\ f_{i_0}(x) < f_{i_0}(u), & \text{for at least one } i_0 \in I, \end{cases}$$

which imply $x \neq u$. Granting this, we can deduce from the strictly pseudoconvexity of $f_i(i\in I)$ at u that

$$\langle\nabla f_i(u), x-u\rangle < 0, \forall i \in I.$$

This, taking into account $\sum_{i\in I}\alpha_i = 1$, yields

$$\langle\sum_{i=1}^{m}\alpha_i\nabla f_i(u), x-u\rangle < 0,$$

contradicting with (25). □

Proposition 21. *(strong duality) Let $\bar{x}\in\Omega$ be a local weakly efficient solution to (P). If (ACQ) holds at $\bar{x}$ and the set Δ is closed, then there exist $(\bar\alpha,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)\in\mathbb{R}^m_+\times\mathbb{R}^{|T|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\sum_{i\in I}\bar\alpha_i = 1, \lambda^g_{T\setminus I_g(\bar x)} = 0, \bar\lambda^G_{I_{+0}(\bar x)} = 0, \bar\lambda^G_{I_{00}(\bar x)} \geq 0, \bar\lambda^H_{I_{00}(\bar x)} \geq 0$ and $\bar\lambda^H_{I_{0+}(\bar x)} = 0$ such that $(\bar x,\bar\alpha,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)\in\Omega_{MW}(\bar x)$. Assume further that $I^-_{0+}(\bar x)\cup\hat{I}^-_{+0}(\bar x)=\emptyset$ and $g_t(t\in T), h_i(i\in I^+_h(\bar x)), -h_i(i\in I^-_h(\bar x))$, $-G_i(i\in I^+_{0+}(x)\cup I^+_{00}(\bar x)\cup I^{++}_{00}(\bar x))$, $-H_i(i\in \hat{I}^+_{+0}(\bar x)\cup\hat{I}^+_{00}(\bar x)\cup I^{++}_{00}(\bar x))$ are quasiconvex at $\bar x$.*

(i) If $f_i(i\in I)$ is pseudoconvex at $\bar x$, then $(\bar x,\bar\alpha,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)$ is a weakly efficient solution to $D_{MW}(\bar x)$.

(ii) If $f_i(i\in I)$ is strictly pseudoconvex at $\bar x$, then $(\bar x,\bar\alpha,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)$ is an efficient solution to $D_{MW}(\bar x)$.

Proof. By invoking Proposition 10, there exist $(\bar\alpha,\bar\lambda^g,\bar\lambda^h,\bar\lambda^G,\bar\lambda^H)\in\mathbb{R}^m_+\times\Lambda(\bar x)\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\sum_{i\in I}\bar\alpha_i = 1, \lambda^G_{I_{+0}(\bar x)} = 0, \bar\lambda^G_{I_{00}(\bar x)}\geq 0, \bar\lambda^H_{I_{00}(\bar x)}\geq 0$ and $\bar\lambda^H_{I_{0+}(\bar x)} = 0$ such that

$$\sum_{i\in I}\nabla f_i(\bar x)+\sum_{t\in T}\bar\lambda^g_t\nabla g_t(\bar x)+\sum_{i\in I_h}\bar\lambda^h_i\nabla h_i(\bar x)-\sum_{i\in I_l}\bar\lambda^G_i\nabla G_i(\bar x)-\sum_{i\in I_l}\bar\lambda^H_i\nabla H_i(\bar x)=0.$$

Since $\bar{x} \in \Omega$ and $\bar{\lambda}^g \in \Lambda(\bar{x})$, one has $\lambda_t g_t(\bar{x}) = 0$ and $g_t(\bar{x}) \leq 0$ for all $t \in T$. Hence, $g_t(\bar{x}) = 0$ for all $t \in I_g(\bar{x})$ and $g_t(\bar{x}) < 0$ for all $t \in T \setminus I_g(\bar{x})$, which in turn implies that $\bar{\lambda}^g_{T\setminus I_g(\bar{x})} = 0$. Again, the fact that $\bar{x} \in \Omega$ guarantees that $h_i(\bar{x}) = 0, \forall i \in I_h(\bar{x})$. In addition, we get from $G_i(\bar{x}) = 0$ for all $i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})$ that $G_i(\bar{x}) \geq 0$ for all $i \in I_{0+}(\bar{x}) \cup I_{00}(\bar{x})$. Similarly, we have $H_i(\bar{x}) \geq 0$ for all $i \in I_{+0}(\bar{x}) \cup I_{00}(\bar{x})$. Thus, $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \Omega_{MW}(\bar{x})$ and $f(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$.
(i) Arguing by contradiction, suppose that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is not a weakly efficient solution to $D_{MW}(\bar{x})$. By denotation, there exists $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$ such that

$$f(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \prec \widetilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

which contradicts with Proposition 20 (i), and thus, completes the proof.
(ii) Suppose to the contrary that $(\bar{x}, \bar{\alpha}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H)$ is not an efficient solution to $D_{MW}(\bar{x})$. In other words, there exists $(u, \alpha, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Omega_{MW}(\bar{x})$ such that

$$f(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \preceq \widetilde{L}(u, \lambda^g, \lambda^h, \lambda^G, \lambda^H),$$

which contradicts with Proposition 20 (ii). So, we arrive at the conclusion. □

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