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A STUDY OF NÖRLUND IDEAL CONVERGENT SEQUENCE SPACES

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Abstract: The Nörlund matrix N^t in the theory of sequence space was firstly used by Wang. In this paper, by using the Nörlund mean N^t and the notion of ideal convergence, we introduce some new sequence spaces $c_0^I(N^t)$, $c^I(N^t)$, and $\ell_\infty^I(N^t)$ as a domain of Nörlund mean. We study some topological and algebraic properties on these spaces. Further, some inclusion concerning these spaces are discussed.

Keywords: Nörlund matrix, matrix transformation, Nörlund I–convergence, Nörlund I–Cauchy, Nörlund I–bounded.

MSC: 40A35,46A45; 40G99.

1. INTRODUCTION

Through this paper, $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural and real numbers, respectively. Also, ω denotes the liner space of all real sequences. The sequence spaces c_0, c, and ℓ_∞ denote the spaces of all null, convergent, and bounded sequence, respectively, with the usual sup–norm defined by $\|x\|_\infty = \sup_k |x_k|$, for each $k \in \mathbb{N}$. Now, let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then A determines a *matrix transformation* from λ into μ denoted by $A : \lambda \longrightarrow \mu$ such that for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{A_n(x)\}$, the A–transform of x is in μ. Where the sequence $\{A_n(x)\}$ is defined as:

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k, \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

The pair (λ, μ) stand for the class of all matrices A such that A maps from λ into μ. Hence, $A \in (\lambda, \mu)$ if and only if the series on the right hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$, where $A_n = (a_{nk})_{k\in\mathbb{N}}$ express the sequence in the n–th row of A. The matrix domain of an infinite matrix A in a sequence space λ is a sequence space given by

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}. \tag{2}$$

Recall that in [1], let (t_k) denote a non–negative sequence of real numbers and $T_n = \sum_{k=0}^{n} t_k$, for all $n \in \mathbb{N}$ with $t_0 > 0$. Then, the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$ as follows:

$$a^t_{nk} = \begin{cases} \frac{t_{n-k}}{T_n}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n, \end{cases} \tag{3}$$

for all $n, k \in \mathbb{N}$. In [1], (Theorem 16, p. 64), the Nörlund matrix N^t is regular if and only if $t_n/T_n \to 0$ as $n \to \infty$, and it reduces to the Cesàro matrix of order one if $t = e = (1, 1, 1, \ldots)$. In case $t_n = A^{r-1}_n$ (for all $n \in \mathbb{N}$), the method N^t is reduced to the Cesàro method C_r of order $r > -1$, where

$$A^r_n = \begin{cases} \frac{(r+1)(r+2)\ldots(r+n)}{n!}, & n = 1, 2, 3, \ldots, \\ 1, & n = 0. \end{cases} \tag{4}$$

The inverse matrix $U^t = (u^t_{nk})$ of Nörlund matrix and some of its details can be found in [2, 3, 4].

Wang [5] used the Nörlund matrix to define the sequence space $\ell_\infty(N^t)$ as the set of all sequences whose N^t–transform are in the space ℓ_∞ as follows:

$$\ell_\infty(N^t) := \left\{ x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \left| \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k}x_k \right| < \infty \right\}.$$

Lately, Tug and Basar [6] introduced the sequence spaces $c_0(N^t)$ and $c(N^t)$ as the domain of Nörlund mean N^t in the spaces c_0 and c, respectively, defined by

$$c_0(N^t) := \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k}x_k = 0 \right\},$$

and

$$c(N^t) := \left\{ x = (x_k) \in \omega : \exists L \in \mathbb{C} \text{ such that } \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} x_k = L \right\},$$

Additionally, they prove that these new spaces are isomorphic to the spaces c_0 and c, respectively. Furthermore, Tug and Basar [6] defined the sequence $N_n^t(x)$ to denote the N^t–transform of the sequence $(x_k) \in \omega$, where the sequence $N_n^t(x)$ is given by

$$N_n^t(x) := \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} x_k \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Recall in [7] that an ideal I is a non–empty family of subsets of $\mathbb{N}$ satisfying additive and heriditary properties, that is (i) $\emptyset \in I$, (ii) for each $A, B \in I$, we have $A \cup B \in I$, (iii) for each $A \in I$ with $B \subseteq A$,we have $B \in I$. An ideal I of $\mathbb{N}$ is said to be admissible in $\mathbb{N}$ if and only if $I \neq \mathbb{N}$ and $I \supset \{\{n\} : n \in \mathbb{N}\}$. A non–trivial ideal I is said to be maximal if there cannot exist any non-trivial ideal J containing I as a subset. A filter $\mathcal{F}$ in $\mathbb{N}$ is a non–empty collection of subsets of $\mathbb{N}$ satisfying (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$, (iii) for each $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$. A filter $\mathcal{F}(I)$ which is associated to each ideal I is given by $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$, where $K^c = X \backslash K$. In 1999, Kostyrko et. al [7] generalized the notion of statistical convergence to ideal convergence. Afterwards, the notion of ideal convergence was considered from the sequence space point of view and associated with the summability theory by many authors like Salát et. al. [8], Tripathy and Hazarika [9], Kolk [10], Savas [11], Khan et al. [12, 13]. For more details refer to [14, 15, 16, 17, 18].

Throughout this paper, c_0^I, c^I, and ℓ_∞^I serve as the I–null, I–convergent, and I–bounded sequence spaces, respectively. In this paper, by conjoining the definitions of the Nörlund mean N^t and ideal convergence, we introduce the Nörlund sequence spaces $c_0^I(N^t)$, $c^I(N^t)$, and $\ell_\infty^I(N^t)$ as the sets of all sequences whose N^t –transform are in the spaces c_0^I, c^I, and ℓ_∞^I, respectively. In addition , we study some inclusion relations concerning these spaces. Also, we study some topological and algebraic properties on these spaces.

Now, we recall some definitions and lemmas needed in the sequel.

Definition 1. *[7] A sequence $(x_k) \in \omega$ is said to be I–convergent to a number $L \in \mathbb{R}$ if, for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$. And we write $I\text{–}\lim x_k = L$. In case $L = 0$ then $(x_k) \in \omega$ is said to be I–null.*

Definition 2. *[8] A sequence $(x) = (x_k) \in \omega$ is said to be I–Cauchy if, for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I$.*

Definition 3. *[19] A sequence $(x_k) \in \omega$ is said to be I–bounded if there exists $K > 0$, such that, the set $\{k \in \mathbb{N} : |x_k| \geq K\} \in I$.*

Definition 4. *[19] Let $x = (x_k)$ and $z = (z_k)$ be two sequences. We say that $x_k = z_k$ for almost all k relative to I (in short a.a.k.r.I) if the set $\{k \in \mathbb{N} : x \neq z\} \in I$.*

Definition 5. *[8] A sequence space E is said to be solid or normal, if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence of scalars $(\alpha_k) \in \omega$ with $|\alpha_k| < 1$, for every $k \in \mathbb{N}$.*

Lemma 6. *[8] Every solid space is monotone.*

Lemma 7. *[8] If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$, we have either $A \in I$ or $\mathbb{N} \backslash A$*

Definition 8. *[8] Let $K = \{k_i \in \mathbb{N} : k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ and E be a sequence space. A K–step space of E is a sequence space*

$$\lambda_K^E = \{(x_{k_i}) \in \omega : (x_k) \in E\}.$$

A canonical pre–image of a sequence $(x_{k_i}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre–image of a step space λ_K^E is a set of canonical pre–images of all elements in λ_K^E., i.e., y is in canonical pre–image of λ_K^E iff y is canonical pre–image of some element $x \in \lambda_K^E$.

Definition 9. *[8] A sequence space E is said to be monotone if its contains the canonical pre–images of its step space. (i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(x_k) \in E$ the sequence $(\alpha_k x_k)$, where $\alpha_k = 1$ for $k \in K$ and $\alpha_k = 0$ otherwise, belongs to E).*

Definition 10. *[8] A sequence space E is said to be convergence free if $(x_k) \in E$, whenever $(y_k) \in E$ and $(y_k) = 0$ imply that $(x_k) = 0$ for all $k \in N$.*

Definition 11. *A map h defined on a domain $D \subset X$ i.e., $h : D \subset X \longrightarrow \mathbb{R}$ is said to satisfy Lipschitz condition if $|h(x) - h(y)| \leq K|x - y|$, where K is known as the Lipschitz constant.*

Remark 12. *[8] A convergence field of I–convergence is a set*

$$\mathcal{F}(I) = \{x = (x_k) \in \ell_\infty : \text{ there exists } I\text{–}\lim x \in \mathbb{R}\}$$

Definition 13. *[7] The convergence field $\mathcal{F}(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $\mathcal{F}(I) = \ell_\infty \cap c^I$.*

Lemma 14. *[20] Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Definition 15. *[8] The function $h : D \subset X \longrightarrow \mathbb{R}$ defined by $h(x) = I\text{–}\lim x$, for all $x \in \mathcal{F}(I)$ is a Lipschitz function.*

2. NÖRLUND I–CONVERGENT SEQUENCE SPACES

In this section, we proposed the sequence spaces $c_0^I(N^t)$, $c^I(N^t)$, and $\ell_\infty^I(N^t)$ as the sets of all sequences whose N^t–transforms are in the spaces c_0^I, c^I, and ℓ_∞^I respectively. Moreware, we study some inclusion relation topological and algebraic properties on these spaces. Throughout the article, we suppose that the sequences $x = (x_k) \in \omega$ and $N_n^t(x)$ are connected with the relation (5) and I is an admissible ideal of subset of $\mathbb{N}$. Define,

$$c_0^I(N^t) := \left\{x = (x_k) \in \omega : \{n \in \mathbb{N} : |N_n^t(x)| \geq \varepsilon\} \in I\right\}, \tag{6}$$

and

$$c^I(N^t) := \left\{x = (x_k) \in \omega : \{n \in \mathbb{N} : |N_n^t(x) - L| \geq \varepsilon \text{ for some } L \in \mathbb{R}\} \in I\right\}, \tag{7}$$

$$\ell_\infty^I(N^t) := \left\{x = (x_k) \in \omega : \exists K > 0 \text{ s.t } \left\{n \in \mathbb{N} : \left|N_n^t(x)\right| \geq K\right\} \in I\right\}. \tag{8}$$

We write

$$m_0^I(N^t) := c_0^I(N^t) \cap \ell_\infty(N^t), \tag{9}$$

and

$$m^I(N^t) := c^I(N^t) \cap \ell_\infty(N^t). \tag{10}$$

With the notation of (2), the spaces $c_0^I(N^t)$, $c^I(N^t)$, $\ell_\infty^I(N^t)$, $m^I(N^t)$, and $m_0^I(N^t)$ can be redefined as follows:

$$c_0^I(N^t) = (c_0^I)_{N^t}, \qquad c^I(N^t) = (c^I)_{N^t}, \qquad \ell_\infty^I(N^t) = (\ell_\infty^I)_{N^t},$$

$$m^I(N^t) = (m^I)_{N^t} \qquad \text{and} \qquad m_0^I(N^t) = (m_0^I)_{N^t}.$$

Definition 16. *Let I be an admissible ideal of subset of $\mathbb{N}$. If for each $\varepsilon > 0$ there exists a number $m = m(\varepsilon) \in \mathbb{N}$ such that $\{n \in \mathbb{N} : |N_n^t(x) - N_m^t(x)| \geq \varepsilon\} \in I$, then a sequence $(x) = (x_k) \in \omega$ is called Nörlund I–Cauchy.*

Example 17. *Define $I_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$. I_f is an admissible ideal in $\mathbb{N}$ and $c^{I_f}(N^t) = c(N^t)$.*

Example 18. *Consider a non–trivial ideal $I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ denotes the natural density of the set A. In this case, $c^{I_d}(N^t) = S(N^t)$, where $S(N^t)$ is the space of all Nörlund statistically convergent sequence defined as:*

$$S(N^t) := \left\{x = (x_k) \in \omega : d\big(\{n \in \mathbb{N} : |N_n^t(x) - L| \geq \varepsilon\}\big) = 0, \text{ for some } L \in \mathbb{R}\right\}. \tag{11}$$

Theorem 19. *The sequence spaces* $c^I(N^t)$, $c_0^I(N^t)$, $\ell_\infty^I(N^t)$, $m_0^I(N^t)$, *and* $m^I(N^t)$ *are real vector spaces.*

Proof. Let $x = (x_k)$, $y = (y_k)$ be two arbitrary elements of the space $c^I(N^t)$ and α, β are scalars. Now, since $x, y \in c^I(N^t)$, then for given $\varepsilon > 0$, there exist $L_1, L_2 \in \mathbb{R}$, such that

$$\left\{n \in \mathbb{N} : \left|N_n^t(x) - L_1\right| \geq \frac{\varepsilon}{2}\right\} \in I,$$

and

$$\left\{n \in \mathbb{N} : \left|N_n^t(y) - L_2\right| \geq \frac{\varepsilon}{2}\right\} \in I.$$

Now, let

$$A_1 = \left\{n \in \mathbb{N} : \left|N_n^t(x) - L_1\right| < \frac{\varepsilon}{2|\alpha|}\right\} \in \mathcal{F}(I),$$

$$A_2 = \left\{n \in \mathbb{N} : \left|N_n^t(y) - L_2\right| < \frac{\varepsilon}{2|\beta|}\right\} \in \mathcal{F}(I),$$

be such that $A_1^c, A_2^c \in I$. Then

$$A_3 = \left\{n \in \mathbb{N} : \left|\alpha N_n^t(x) + \beta N_n^t(y) - (\alpha L_1 + \beta L_2)\right| < \varepsilon\right\} \supseteq \{A_1 \cap A_2\}. \quad (12)$$

Thus, the sets on the right hand sides of (12) belong to $\mathcal{F}(I)$. By the definition of filter associated with an ideal I, the complement of the set on the left hand side of (12) belongs to I. This implies that $(\alpha x + \beta y) \in c^I(N^t)$. Hence, $c^I(N^t)$ is a linear space. The proof of the remaining results is similar. □

Theorem 20. *Let* $I \subseteq 2^{\mathbb{N}}$ *be a non–trivial ideal. Then the inclusion* $c(N^t) \subset c^I(N^t)$ *is strict.*

Proof. The inclusion $c \subseteq c^I$ is obvious and for any X and Y spaces, if $X \subseteq Y$, then $X(N^t) \subseteq Y(N^t)$. (see [6] Theorem 2.3). Therefore, we have $c(N^t) \subset c^I(N^t)$. For strict inclusion consider the following example.

Example 21. *Define the sequence* $x = (x_k) \in \omega$ *such that*

$$N_n^t(x) = \begin{cases} \sqrt{n}, & \text{if } n = i^2, \text{ for } i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in c^{I_d}(N^t)$ *but* $x \notin c(N^t)$.
□

Theorem 22. *A sequence* $x = (x_k) \in \omega$ *is Nörlund* I*–convergent if and only if for every* $\varepsilon > 0$*, there exists* $m = m(\varepsilon) \in \mathbb{N}$*, such that*

$$\left\{n \in \mathbb{N} : \left|N_n^t(x) - N_m^t(x)\right| < \varepsilon\right\} \in \mathcal{F}(I). \quad (13)$$

Proof. Let the sequence $x = (x_n) \in \omega$ be Nörlund I–convergent to some number $L \in \mathbb{R}$. Then, for a given $\varepsilon > 0$, the set

$$A_\varepsilon = \left\{ n \in \mathbb{N} : \left| N_n^t(x) - L \right| < \frac{\varepsilon}{2} \right\} \in \mathcal{F}(I).$$

Fix an integer $m = m(\varepsilon) \in A_\varepsilon$. Then, we have

$$\left| N_n^t(x) - N_m^t(x) \right| \leq \left| N_n^t(x) - L \right| + \left| L - N_m^t(x) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \in A_\varepsilon$. Hence (13) holds.

Conversely, suppose that (13) holds for all $\varepsilon > 0$. Then

$$E_\varepsilon = \left\{ n \in \mathbb{N} : N_n^t(x) \in \left[N_n^t(x) - \varepsilon, N_n^t(x) + \varepsilon \right] \right\} \in \mathcal{F}(I), \text{ for all } \varepsilon > 0.$$

Let $J_\varepsilon = [N_n^t(x) - \varepsilon, N_n^t(x) + \varepsilon]$. Fixing $\varepsilon > 0$, we have $E_\varepsilon \in \mathcal{F}(I)$ as well as $E_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. Hence $E_\varepsilon \cap E_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. This implies that $J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset$. That is, $\{n \in \mathbb{N} : N_n^t(x) \in J\} \in \mathcal{F}(I)$ and thus, diam $(J) \leq \frac{1}{2}$ diam (J_ε), where, the diam of J denotes the length of interval J. Proceeding in this way, by induction, we get a sequence of closed intervals

$$J_\varepsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that

$$\text{diam } (I_n) \leq \frac{1}{2} \text{ diam } (I_{n-1}), \qquad \text{for } n = (2, 3, \dots)$$

and

$$\left\{ n \in \mathbb{N} : N_n^t(x) \in I_n \right\} \in \mathcal{F}(I).$$

Then there exists a number $L \in \bigcap_{n \in \mathbb{N}} I_n$ and it is a routine work to verify that $L = I\text{–}\lim N_n^t(x)$ showing that $x = (x_k) \in \omega$ is Nörlund I–converges. Hence the result. $\square$

Theorem 23. *The inclusions $c_0^I(N^t) \subset c^I(N^t) \subset L_\infty^I(N^t)$ are strict.*

Proof. The inclusion $c_0^I(N^t) \subset c^I(N^t)$ is obvious. Now, to show the strictness of the inclusion, consider the sequence $x = (x_k) \in \omega$ such that $N_n^t(x) = 1$. It is easy to see that the sequence $N_n^t(x) \in c^I$ but $N_n^t(x) \notin c_0^I$. That is, $x \in c^I(N^t) \backslash c_0^I(N^t)$. Next, let $x = (x_k) \in c^I(N^t)$. Then there exists a number $L \in \mathbb{R}$ such that $I\text{–}\lim |N_n^t(x) - L| = 0$. That is,

$$\left\{ n \in \mathbb{N} : \left| N_n^t(x) - L \right| \geq \varepsilon \right\} \in I.$$

We have

$$\left| N_n^t(x) \right| = \left| N_n^t(x) - L + L \right| \leq \left| N_n^t(x) - L \right| + |L| \,.$$

From this it easily follows that the sequence (x_k) must belong to $\ell_\infty^I(N^t)$. Further, we show the strictness of the inclusion $c^I(N^t) \subset \ell_\infty^I(N^t)$ by constructing the following example.

Example 24. *Consider the sequence $x = (x_k) \in \omega$ to be such that*

$$N_n^t(x) = \begin{cases} \sqrt{n}, & \textit{if } n \textit{ is square} \\ 1, & \textit{if } n \textit{ is odd non–square} \\ 0, & \textit{if } n \textit{ is even non–square.} \end{cases}$$

Then the sequence $N_n^t(x) \in \ell_\infty^I$ but $N_n^t(x) \notin c^I$, which implies that the sequence $x \in \ell_\infty^I(N^t)\backslash c^I(N^t)$.

Thus, the inclusions $c_0^I(N^t) \subset c^I(N^t) \subset \ell_\infty^I(N^t)$ are strict. □

Remark 25. *A Nörlund bounded sequence is obviously Nörlund I–bounded as the empty set belongs to the ideal I. However, the converse is not true. For example, we consider the sequence*

$$N_n^t(x) = \begin{cases} n, & \textit{if } n \textit{ is square} \\ 0, & \textit{if } n \textit{ is not square.} \end{cases}$$

Clearly $N_n^t(x)$ is not a bounded sequence. However, $\{n \in \mathbb{N} : |N_n^t(x)| \geq \frac{1}{2}\} \in I$. Hence (x_k) is Nörlund I–bounded.

Theorem 26. *The spaces $m^I(N^t)$ and $m_0^I(N^t)$ are Banach spaces normed by*

$$\|x\|_{X(N^t)} = \sup_n |N_n^t(x)| \textit{ where } X \in \{m^I, m_0^I\}.$$

Proof. Let (x_k^i) be a Cauchy sequence in $m^I(N^t) \subset \ell_\infty(N^t)$. Then we have (x_k^i) converges in $\ell_\infty(N^t)$ and $\lim_{i\to\infty} N_n^t(x^i) = N_n^t(x)$. Let $I\text{–}\lim N_n^t(x^i) = \ell_i$ for each $i \in \mathbb{N}$. Then we have to show that

(i) (L_i) is convergent say to L,

(ii) $I\text{–}\lim N_n^t(x) = L$.

(i) Since (x_k^i) is a Cauchy, so for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left|N_n^t(x^i) - N_n^t(x^j)\right| < \frac{\varepsilon}{3}, \text{ for all } i, j \geq n_0. \tag{14}$$

Now, let A and B be the following sets in I:

$$A = \left\{n \in \mathbb{N} : \left|N_n^t(x^i) - L_i\right| \geq \frac{\varepsilon}{3}\right\} \tag{15}$$

and

$$B = \left\{n \in \mathbb{N} : \left|N_n^t(x^j) - L_j\right| \geq \frac{\varepsilon}{3}\right\}. \tag{16}$$

Consider $i, j \geq n_0$ and $n \notin A \cap B$. Then we have

$$|L_i - L_j| \leq \left|N_n^t(x^i) - L_i\right| + \left|N_n^t(x^j) - L_j\right| + \left|N_n^t(x^i) - N_n^t(x^j)\right| < \varepsilon \text{ by (14), (15) and (16).}$$

Thus (L_i) is a Cauchy sequence of $\mathbb{R}$ and thus convergent, say to L, that is, $\lim_{i\to\infty} L_i = L$.

(ii) Let $\delta > 0$ be given, then we can fined m_0 such that

$$|L_i - L| < \frac{\delta}{3}, \text{ for each } i > m_0. \tag{17}$$

We have $(x_k^i) \to x_k$ as $i \to \infty$. Thus

$$\left|N_n^t(x^i) - N_n^t(x)\right| < \frac{\delta}{3}, \text{ for each } i > m_0. \tag{18}$$

Since (x_k^j) is I–convergent to L_j, there exists $D \in I$ such that, for each $n \notin D$, we have

$$\left|N_n^t(x^j) - L_j\right| < \frac{\delta}{3}. \tag{19}$$

Without loss of generality, let $j > m_0$ then for all $n \notin D$, we have by (17), (18) and (19) that

$$\left|N_n^t(x) - L\right| \leq \left|N_n^t(x) - N_n^{(t)}(x^j)\right| + \left|N_n^{(t)}(x^j) - L_j\right| + |L_j - L| < \delta.$$

Hence (x_k) is Nörlund I–convergent to L. Thus $m^I(N^t)$ is a Banach space. The other cases can be Similarly established.

□

Theorem 27. *If I is not maximal ideal then the space $c^I(N^t)$ is neither solid nor monotone.*

Proof. Consider the sequence $x_k = 1$ for all $k \in \mathbb{N}$, then $(x_k) \in c^I(N^t)$. Since, I is not maximal, by Lemma 7 there exists a subset K of $\mathbb{N}$ such that, $K \notin I$ and $K^c \notin I$. Let us define $y = (y_k)$ by

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

Then, (y_k) belongs to the canonical pre–image of the K–step space of $c^I(N^t)$. But $(y_k) \notin c^I(N^t)$. Hence, $c^I(N^t)$ is not monotone. Therefore, by Lemma 6 is not solid.

□

Theorem 28. *The spaces $c_0^I(N^t)$ and $m_0^I(N^t)$ are solid and monotone.*

Proof. We shall prove the result for $c_0^I(N^t)$. Other follows similarity. Let $x = (x_k) \in c_0^I(N^t)$ for $\varepsilon > 0$, the set

$$\left\{n \in \mathbb{N} : \left|N_n^t(x)\right| \geq \varepsilon\right\} \in I. \tag{20}$$

Let $\alpha = (\alpha_k)$ be a sequence of scalars with $|\alpha| \leq 1$ for all $k \in \mathbb{N}$. Then

$$\left|N_n^t(\alpha x)\right| = \left|\alpha N_n^t(x)\right| \leq |\alpha| \left|N_n^t(x)\right| \leq \left|N_n^t(x)\right|, \text{ for all } n \in \mathbb{N}.$$

Thus, from the above inequality and (20):

$$\left\{n \in \mathbb{N} : \left|N_n^t(\alpha x)\right| \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \left|N_n^t(x)\right| \geq \varepsilon\right\} \in I$$

implies

$$\left\{n \in \mathbb{N} : \left|N_n^t(\alpha x)\right| \geq \varepsilon\right\} \in I$$

Therefore, $(\alpha x_n) \in c_0^I(N^t)$. Hence the space $c_0^I(N^t)$ is solid, and hence by Lemma, 6 the space $c_0^I(N^t)$ is monotone. □

Theorem 29. *The spaces $c^I(N^t)$ and $c_0^I(N^t)$ are not convergence free.*

Proof. The proof of this result follows from the following example.

Example 30. *Let $I = I_d$. Consider $x_k = \frac{1}{k}$ and $y_k = k$ for all k. Then (x_k) belongs to $c^I(N^t)$ and $c_0^I(N^t)$, but (y_k) does not belongs to $c^I(N^t)$ and $c_0^I(N^t)$. Hence, the spaces are not convergence free.*

□

Theorem 31. *The function $g : m^I(N^t) \to \mathbb{R}$ defined by $g(x) = |I\text{–}\lim N_n^t(x)|$, where $m^I(N^t) = \ell_\infty(N^t) \cap c^I(N^t)$, is a Lipschitz function and hence uniformly continuous.*

Proof. First of all, we show that the function is well defined. Let $x, y \in m^I(N^t)$, such that

$$\begin{aligned} x = y &\Rightarrow I\text{–}\lim N_n^t(x) = I\text{–}\lim N_n^t(y) \\ &\Rightarrow |I\text{–}\lim N_n^t(x)| = |I\text{–}\lim N_n^t(y)| \Rightarrow g(x) = g(y). \end{aligned}$$

Thus, g is well defined. Next, let $x = (x_k), y = (y_k) \in m^I(N_n^t)$, $x \neq y$. Then

$$A_1 = \left\{n \in \mathbb{N} : \left|N_n^t(x) - g(x)\right| \geq |x - y|_*\right\} \in I,$$

$$A_2 = \left\{n \in \mathbb{N} : \left|N_n^t(y) - g(y)\right| \geq |x - y|_*\right\} \in I,$$

where $|x - y|_* = \sup_n \left|N_n^t(x) - N_n^t(y)\right|$. Thus

$$B_1 = \left\{n \in \mathbb{N} : \left|N_n^t(x) - g(x)\right| < |x - y|_*\right\} \in \mathcal{F}(I)$$

and

$$B_2 = \left\{n \in \mathbb{N} : \left|N_n^t(y) - g(y)\right| < |x - y|_*\right\} \in \mathcal{F}(I).$$

Hence $B = B_1 \cap B_2 \in \mathcal{F}(I)$, so that B is non–empty set. Therefore choosing $n \in B$, we have

$$\begin{aligned} |g(x) - g(y)| &\leq \left|g(x) - N_n^t(x)\right| + \left|N_n^t(x) - N_n^t(y)\right| + \left|N_n^t(y) - g(y)\right| \\ &\leq 3|x - y|_*. \end{aligned}$$

Thus, g is Lipschitz function and hence uniformly continuous. □

Theorem 32. *If* $x = (x_k), y = (y_k) \in m^I(N^t)$ *with* $N_n^t(x \cdot y) = N_n^t(x) \cdot N_n^t(y)$, *then* $(x \cdot y) \in m^I(N^t)$ *and* $(x \cdot y) = g(x) \cdot g(y)$, *where* $g : m^I(N^t) \to \mathbb{R}$ *is defined by* $g(x) = |I\text{–}\lim N_n^t(x)|$.

Proof. For $\varepsilon > 0$,

$$A = \left\{ n \in \mathbb{N} : \left| N_n^t(x) - g(x) \right| < \varepsilon \right\} \in \mathcal{F}(I), \tag{21}$$

and

$$B = \left\{ n \in \mathbb{N} : \left| N_n^t(y) - g(y) \right| < \varepsilon \right\} \in \mathcal{F}(I) \tag{22}$$

where $\varepsilon = |x - y|_* = \sup_n \left| N_n^t(x) - N_n^t(y) \right|$. Now, we have

$$\begin{aligned} \left| N_n^t(x \cdot y) - g(x)g(y) \right| &= \left| N_n^t(x) N_n^t(y) - N_n^t(x) g(y) + N_n^t(x) g(y) - g(x)g(y) \right| \\ &\leq \left| N_n^t(x) \right| \left| N_n^t(y) - g(y) \right| + |g(y)| \left| N_n^t(x) - g(x) \right|. \end{aligned} \tag{23}$$

As $m^I(N^t) \subseteq \ell_\infty(N^t)$, there exists an $M \in \mathbb{R}$ such that $|N_n^t(x)| < M$. Therefore, from the equations (21), (22) and (23), we have

$$\begin{aligned} \left| N_n^t(xy) - g(x)g(y) \right| &= \left| N_n^t(x) \cdot N_n^t(y) - g(x)g(y) \right| \\ &\leq M\varepsilon + |g(y)|\varepsilon = \varepsilon_1, \ \text{(say)} \end{aligned}$$

for all $n \in A \cap B \in \mathcal{F}(I)$. Hence $(x \cdot y) \in m^I(N^t)$ and $g(x \cdot y) = g(x) \cdot g(y)$. □

3. CONCLUSIONS AND SUGGESTIONS

In this present paper we have defined and investigated some sequence spaces, $c_0^I(N^t)$, $c^I(N^t)$, and $\ell_\infty^I(N^t)$ that is derived by Nörlund mean. Also, we presented some inclusions relation concerning these spaces. Finally, some properties on these spaces were studied. These new spaces and results related to them furnish new approach to deal with the convergence problems of sequences occurring in many branches of science and engineering.

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