ON ANTI-FUZZY SUBGROUP

Sudipta GAYEN

Department of Mathematics, National Institute of Technology Jamshedpur, India
sudi23dipta@gmail.com

Sripati JHA

Department of Mathematics, National Institute of Technology Jamshedpur, India
sjha.math@nitjsr.ac.in

Manoranjan Kumar SINGH

Department of Mathematics, Magadh University, Bodh Gaya, India
drmksingh_gaya@yahoo.com

Ashish Kumar PRASAD*

School of Advanced Sciences, Vellore Institute of Technology, Vellore, India
ashishprasa@gmail.com

Abstract: In this paper, efforts have been taken to generalise the notion of anti-fuzzy subgroup. Proposed definition was supported by graphical comparison with our previous result and suitably constructed examples. Based on the introduced definition, we derive not only some results of anti-fuzzy subgroup but also we redefine lower level set and lower level subgroup, and derive some essential theorems to study some algebraic characteristics. Further, a modified notion of lower level subgroup is given.

Keywords: Fuzzy Subgroup, Anti-Fuzzy Subgroup, Lower Level Set, Lower level Subgroup.

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1. INTRODUCTION

The notion of fuzzy set was mathematically formulated by Zadeh [12]. From the very beginning, it has become an important area of research in various fields of analysis and applications. It is based on the assumption that classical set theory is inadequate in describing real-life problems because every object of our surrounding

carries some sort of degree of fuzziness. Initially, Rosenfeld [3] introduced notions of fuzzy subgroupoid, fuzzy subgroup, fuzzy ideal, fuzzy homomorphism. He also introduced the lattice of a fuzzy subgroup and a fuzzy ideal. The notion of level subgroup was introduced by Das [4] which plays a significant role in the characterization of a fuzzy subgroup. Mukherjee [8] defined fuzzy normal subgroup and fuzzy coset. He proved some interesting results on fuzzy normal subgroup related to conjugate class and level subgroup. He also established fuzzy version of Lagrange's theorem.

Singh [10] redefined the notion of fuzzy subgroup and showed that his proposed notion was more appealing. Based on his definition, he proved most of the important theorems and propositions related to fuzzy subgroup and also derived some other interesting results. He further redefined level subgroup, fuzzy ring, fuzzy field, fuzzy linear space, fuzzy module, fuzzy topological space, etc. Biswas [2] introduced the notions of anti-fuzzy subgroup, and lower level subgroup. But his definition of anti-fuzzy subgroup was based on the notion of fuzzy subgroup proposed in [3]. As the definition of fuzzy subgroup proposed in [10] is more general than the definition proposed in [3], the notion of anti-fuzzy subgroup can be redefined based on that. Depending on this logic, the notion of anti-fuzzy subgroup has been generalized in such a way that the domain of definition becomes more large.

Rasuli [9] gave an idea of anti fuzzy subrings with the help of t-conorm and established few algebraic properties including intersection, direct product, and homomorphisms for anti fuzzy subbrings. Gayen et al. [11] introduced a new notion of anti-fuzzy subgroup by considering general t-conorm, and based on this, they further defined infimum image of a fuzzy set and subgroup generated anti-fuzzy subgroup, and a function generated anti-fuzzy subgroup.

The organization of this article is as follows: in Section 2 we have mentioned some preliminary aspects of fuzzy set, fuzzy subgroup and anti-fuzzy subgroup. In Section 3, a redefined version of anti-fuzzy subgroup has been given. Also, we have given a graphical comparison between Biswas's notion and our redefined version. Furthermore, we have redefined lower level set and lower level subgroup based on our proposed definition. Finally, in Section 4 we have given the concluding segment, mentioning a scope of future researches.

2. PRELIMINARIES

Definition 1. *[12] A fuzzy subset α of a crisp set H is defined as a function $\alpha : H \to [0,1]$.*

Definition 2. *[3] A fuzzy subset α of a crisp group H is termed as a fuzzy subgroup of H if for all $p, q \in H$, the subsequent conditions are satisfied:*

  (i) $\alpha(pq) \geq min\{\alpha(p), \alpha(q)\}$

(ii) $\alpha(p^{-1}) \geq \alpha(p)$.

Here $\alpha(p^{-1}) = \alpha(p)$ and $\alpha(p) \leq \alpha(e)$, e represents the neutral element of H.

Proposition 3. *[3] α is a fuzzy subgroup of G iff for all p, $q \in G$, we have*

$$\alpha(pq^{-1}) \geq min\{\alpha(p), \alpha(q)\}$$

Definition 4. *[10] A fuzzy subset α of a group H is termed as a fuzzy subgroup of H if for all p, $q \in H$, the following conditions are held:*

(i) $\alpha(pq) \geq \alpha(p)\alpha(q)$ *and*

(ii) $\alpha(p^{-1}) \geq \alpha(p)$.

If α is a fuzzy subgroup of H according to Rosenfeld [3], then

$$\alpha(pq) \geq min\{\alpha(p), \alpha(q)\} \geq \alpha(p)\alpha(q), \forall \, p, \, q \in H. \tag{1}$$

From equation (1) it is evident that when α is a fuzzy subgroup under Definition 2, then it is a fuzzy subgroup under Definition 4. But the converse is not always true.

Definition 5. *[2] A fuzzy subset α of a group H is termed as an anti-fuzzy subgroup of H if for all $p, q \in H$, the subsequent conditions are fulfilled:*

(i) $\alpha(pq) \leq max\{\alpha(p), \alpha(q)\}$,

(ii) $\alpha(p^{-1}) \leq \alpha(p)$.

Here observe that $\alpha(p^{-1}) = \alpha(p)$, $\alpha(p) \geq \alpha(e)$, e represents the neutral element of H. Again α is an anti-fuzzy subgroup of H iff $\alpha(pq^{-1}) \leq max\{\alpha(p), \alpha(q)\}$ $\forall p, q \in H$.

In the next section, the notion of anti-fuzzy subgroup is generalized and graphical comparison with the previously defined one is given.

3. A MORE GENERAL NOTION OF ANTI-FUZZY SUBGROUP

Definition 6. *A fuzzy subset α of a group H is termed as an anti-fuzzy subgroup of H if for all $p, q \in H$, we have*

(i) $\alpha(pq) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$,

(ii) $\alpha(p^{-1}) \leq \alpha(p)$.

Here we observe that for all $p, q \in H$

$$\alpha(pq) \leq max\{\alpha(p), \alpha(q)\} \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q). \tag{2}$$

From equation (2), it is evident that whenever α is an anti-fuzzy subgroup according to Definition 5, it is an anti-fuzzy subgroup according to Definition 6. The opposite is not always true. So, the proposed Definition 6 of anti-fuzzy subgroup is more general than Definition 5. In the following section, we have given graphical comparison between Definition 5 and Definition 6.

3.1. Graphical representation and comparison between the definitions

Let α be an anti-fuzzy subgroup of a group H with $\alpha(p)$, $\alpha(q) \in \{0, 0.1, 0.2, 0.3,$ 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$\}$. Here $\forall p, q \in H$, $\alpha(pq)$ have been plotted on the basis of Definition 5 and Definition 6 of anti-fuzzy subgroup. In those definitions '$\leq$'operator has been replaced with '$=$'. Figure 1a is according to Definition 5 and Figure 1b is according to Definition 6.

(a) $\alpha(pq) = max\{\alpha(p), \alpha(q)\}$ (b) $\alpha(pq) = \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$
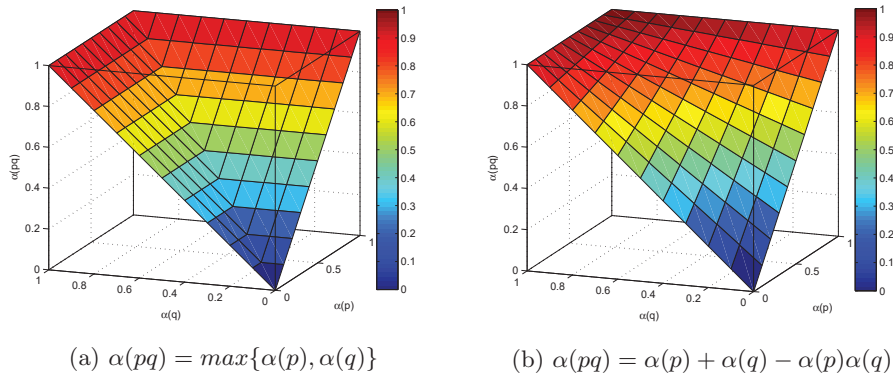
Figure 1: Comparison of Definition 5 and Definition 6

Let $\beta(pq) \leq \alpha(pq)$. Evidently figure 1b produces more possible $\beta(pq)$ than that of figure 1a. Clearly, the proposed Definition 6 of anti-fuzzy subgroup is followed by Definition 5 proposed in [2], i.e., the domain of definition has become more vast. So, Definition 6 is more general.

Example 7. *Let $H = \{1, -1, i, -i\}$ be a group with regular multiplication. Let α be defined as $\alpha : H \to [0, 1]$ by*

$$\alpha(p) = \begin{cases} 0 & \text{if } p = 1 \\ 0.5 & \text{if } p = -1 \\ 1 & \text{if } p = i, -i \end{cases}$$

Evidently, α is an anti-fuzzy subgroup of H. Here, observe that $\forall p, q \in H$, $\alpha(pq) \leq max\{\alpha(p), \alpha(q)\} \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$. Here α is is an anti-fuzzy subgroup of H both according to Definition 5 and Definition 6.

Example 8. *Let $H = \{1, -1, i, -i\}$ be a group with regular multiplication. Let α be defined as $\alpha : H \to [0, 1]$ by*

$$\alpha(p) = \begin{cases} 0 & \text{if } p = 1 \\ 0.3 & \text{if } p = -1 \\ 0.6 & \text{if } p = i \\ 0.5 & \text{if } p = -i \end{cases}$$

Here $\alpha((-1) \times (-i)) = \alpha(i) = 0.6 \nleq max\{\alpha(-1), \alpha(-i)\}$. So, α is not an anti-fuzzy subgroup of H according to Definition 5. But $\alpha((-1) \times (-i)) = \alpha(i) = 0.6 \leq \alpha(-1) + \alpha(-i) - \alpha(-1)\alpha(-i)$. In fact $\forall p, q \in H$ $\alpha(pq) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$. So, it is an anti-fuzzy subgroup according to our proposed Definition 6.

Example 9. *Consider the infinite group of integers, $\mathbb{Z}$ with regular addition. Define $\alpha : \mathbb{Z} \to [0, 1]$ by*

$$\alpha(p) = \begin{cases} 0.3 & \text{if } p \in 2\mathbb{Z} \\ 0.5 & \text{if } p = 1 \\ 0.4 & \text{if } p \in 2\mathbb{Z} + 1 \smallsetminus \{1\} \end{cases}$$

Here $\alpha(-2 + 3) = \alpha(1) = 0.5 \nleq max\{\alpha(-2), \alpha(3)\}$. So, α is not an anti-fuzzy subgroup of H according to Definition 5. But $\alpha(-2+3) = \alpha(1) = 0.5 \leq \alpha(-2) + \alpha(3) - \alpha(-2)\alpha(3)$. In fact $\forall p, q \in H$ $\alpha(pq) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$. So, it is an anti-fuzzy subgroup according to our proposed Definition 6.

3.2. Some propositions and theorems

Proposition 10. *Let α be an anti-fuzzy subgroup of H, then*

(i) $\alpha(p^{-1}) = \alpha(p), \forall p \in H$.

(ii) $2\alpha(p) - \alpha^2(p) \geq \alpha(e)$, e represents the neutral element of H.

(iii) $\alpha(e) = 0$, if $\alpha(p) = 0$ for at-least one $p \in H$.

Proposition 11. *Let α be an anti-fuzzy subgroup of H, then for all $p, q \in H$, $\alpha(pq^{-1}) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$. If $\alpha(e) = 0$ with $\alpha(pq^{-1}) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$, $\forall p, q \in H$, then α is an anti fuzzy subgroup of H.*

Proof. Let α be an anti-fuzzy subgroup of H. Then $\forall p, q \in H$,

$$\alpha(pq^{-1}) \leq \alpha(p) + \alpha(q^{-1}) - \alpha(p)\alpha(q^{-1})$$
$$= \alpha(p) + \alpha(q) - \alpha(p)\alpha(q) \text{ [as } \alpha(q^{-1}) = \alpha(q), \ \forall \ q \in H].$$

Now let $\alpha(e) = 0$ with $\alpha(pq^{-1}) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)$, $\forall p, q \in H$. Replacing p with e, we have

$$\alpha(q^{-1}) \leq \alpha(e) + \alpha(q) - \alpha(e)\alpha(q)$$
$$= \alpha(q) \tag{3}$$

Again,

$$\alpha(q) = \alpha((q^{-1})^{-1})$$
$$\leq \alpha(q^{-1}) \tag{4}$$

By equations (3) and (4), $\alpha(q) = \alpha(q^{-1})$.
Again,

$$
\begin{aligned}
\alpha(pq) &= \alpha(p(q^{-1})^{-1}) \\
&\leq \alpha(p) + \alpha(q^{-1}) - \alpha(p)\alpha(q^{-1}) \\
&= \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)
\end{aligned} \tag{5}
$$

Evidently, from equations (3) and (5), α is an anti-fuzzy subgroup. $\square$

Proposition 12. *α is an anti-fuzzy subgroup of H iff its complement α^c is a fuzzy subgroup of H.*

Proof. Let α be an anti-fuzzy subgroup of H. So, $\forall p, q \in H$,

$$
\begin{aligned}
&\alpha(pq) \leq \alpha(p) + \alpha(q) - \alpha(p)\alpha(q) \\
\Rightarrow &1 - \alpha^c(pq) \leq 1 - \alpha^c(p) + 1 - \alpha^c(q) - (1 - \alpha^c(p))(1 - \alpha^c(q)) \\
\Rightarrow &1 - \alpha^c(pq) \leq 1 - \alpha^c(p) + 1 - \alpha^c(q) - 1 + \alpha^c(q) + \alpha^c(p) - \alpha^c(p)\alpha^c(q) \\
\Rightarrow &-\alpha^c(pq) \leq -\alpha^c(p)\alpha^c(q) \\
\Rightarrow &\alpha^c(pq) \geq \alpha^c(p)\alpha^c(q)
\end{aligned} \tag{6}
$$

Again,

$$
\begin{aligned}
&\alpha(p^{-1}) \leq \alpha(p) \\
\Rightarrow &1 - \alpha^c(p^{-1}) \leq 1 - \alpha^c(p) \\
\Rightarrow &\alpha^c(p^{-1}) \geq \alpha^c(p)
\end{aligned} \tag{7}
$$

Thus by equations (6) and (7), α^c is a fuzzy subgroup according to Definition 4. The converse part can also be proved similarly. $\square$

In the next segment, based on the proposed Definition 6, we have given redefined versions of lower level subset and lower level subgroup.

3.3. Lower level set and lower level subgroup

Let α_t and $\bar{\alpha}_t$ be respectively level and lower level subset of the fuzzy subset α. So, $\alpha_t = \{p \in H : \alpha(p) \geq t\}$ and $\bar{\alpha}_t = \{p \in H : \alpha(p) \leq t\}$. Evidently, α_t and $\bar{\alpha}_t$ are subsets of H. Here we notice that if H is a crisp group and α is an anti-fuzzy subgroup of H, then according to Definition 6 $\forall p \in H$, we have

$$
\begin{aligned}
&\alpha(e) = \alpha(pp^{-1}) \leq \alpha(p) + \alpha(p^{-1}) - \alpha(p)\alpha(p^{-1}) \\
\Rightarrow &\alpha(e) \leq \alpha(p) + \alpha(p) - \alpha(p)\alpha(p) \\
\Rightarrow &\alpha(e) \leq 2\alpha(p) - \alpha^2(p)
\end{aligned}
$$

Under the light of above, lower level subset can be redefined as

Definition 13. *Let α be a fuzzy subset of a set H and let $t \in [0, 1]$. Then the set $\bar{\alpha}_{2t-t^2} = \left\{ p \in H : \alpha(p) \leq 2t - t^2 \right\}$ is termed as a lower level subset of α.*

Here observe that $\bar{\alpha}_1 = H$ and $\alpha_{2t-t^2} \cup \bar{\alpha}_{2t-t^2} = H$. Again if $t_1 < t_2$, then $\bar{\alpha}_{2t_1-t_1{}^2} \subseteq \bar{\alpha}_{2t_2-t_2{}^2}$.

Proposition 14. *Let α be an anti-fuzzy subgroup of a group H such that $\forall p \in H$ $\alpha(p) \notin (t, 2t - t^2]$ with $\alpha(e) \leq 2t - t^2$ for all $t \in [0, 1]$, then $\bar{\alpha}_{2t-t^2}$ is a crisp subgroup of H.*

Proof. Here α is an anti-fuzzy subgroup of a group H such that $\forall p \in H$ $\alpha(p) \notin (t, 2t - t^2]$ with $\alpha(e) \leq 2t - t^2$ for all $t \in [0, 1]$.
Evidently $\bar{\alpha}_{2t-t^2}$ is nonempty as $e \in \bar{\alpha}_{2t-t^2}$.
Let $p, q \in \bar{\alpha}_{2t-t^2}$. Then $\alpha(p), \alpha(q) \leq 2t - t^2$. Again, by supposition we have $\alpha(p), \alpha(q) \leq t$.
Now,

$$\begin{aligned}
\alpha(pq^{-1}) &\leq \alpha(p) + \alpha(q^{-1}) - \alpha(p)\alpha(q^{-1}) \\
&\leq t + t - t \cdot t \\
&= 2t - t^2
\end{aligned} \tag{8}$$

Hence, by equation (8), $pq^{-1} \in \bar{\alpha}_{2t-t^2}$, i.e., $\bar{\alpha}_{2t-t^2}$ is a subgroup of H. $\square$

Here observe that $\bar{\alpha}_{2t-t^2} = \bar{\alpha}_t \cup \{p \in H : t < \alpha(p) \leq 2t - t^2\}$. So, if $\forall p \in H$ $\alpha(p) \notin (t, 2t - t^2]$ for $t \in [0, 1]$, then $\bar{\alpha}_{2t-t^2} = \bar{\alpha}_t$.

Proposition 15. *Let us consider α as an anti-fuzzy subgroup of a group H such that $\forall p \in H$ $\alpha(p) \notin (t, 2t - t^2]$ with $\alpha(e) \leq t$ for all $t \in [0, 1]$, then $\bar{\alpha}_t$ is a subgroup of H.*

Proposition 16. *Let α be a fuzzy subset of a group H and $\bar{\alpha}_{2t-t^2}$ are subgroups of H for all $t \in [0, 1]$ with $\alpha(e) \leq 2t - t^2$, then α is an anti-fuzzy subgroup of H.*

Proof. Let us consider $p, q \in H$ such that $\alpha(p) = 2t_1 - t_1{}^2$ and $\alpha(q) = 2t_2 - t_2{}^2$. Then $p \in \bar{\alpha}_{2t_1-t_1{}^2}$ and $q \in \bar{\alpha}_{2t_2-t_2{}^2}$. Let us suppose that $t_1 < t_2$. Then $p, q \in \bar{\alpha}_{2t_2-t_2{}^2}$. As $\bar{\alpha}_{2t_2-t_2{}^2}$ is a subgroup of H, we have $pq \in \bar{\alpha}_{2t_2-t_2{}^2}$. So,

$$\begin{aligned}
\alpha(pq) &\leq 2t_2 - t_2{}^2 \\
&\leq 2t_1 - t_1{}^2 + 2t_2 - t_2{}^2 - (2t_1 - t_1{}^2) \cdot (2t_2 - t_2{}^2) \\
&= \alpha(p) + \alpha(q) - \alpha(p)\alpha(q)
\end{aligned}$$

Next, let $p \in H$ such that $\alpha(p) = 2t - t^2$. Then $p \in \bar{\alpha}_{2t-t^2}$. As $\bar{\alpha}_{2t-t^2}$ is a subgroup of H, we have $p^{-1} \in \bar{\alpha}_{2t-t^2}$.
$\therefore \alpha(p^{-1}) \leq 2t - t^2 = \alpha(p)$ i.e. $\alpha(p^{-1}) \leq \alpha(p)$
$\therefore \alpha$ is an anti-fuzzy subgroup of H. $\square$

So, it can be concluded that $\bar{\alpha}_{2t-t^2}$ are subgroups of an anti-fuzzy subgroup α for $\alpha(p) \notin (t, 2t - t^2]$ and $t \in [0, 1]$ satisfying $\alpha(e) \leq 2t - t^2$. Hence, lower level subgroup can be redefined as given below.

Definition 17. *Let α be an anti-fuzzy subgroup of a group H such that $\forall p \in H$ $\alpha(p) \notin (t, 2t - t^2]$ with $\alpha(e) \leq 2t - t^2$ for all $t \in [0, 1]$. Then the subgroups $\bar{\alpha}_{2t-t^2}$ for all $t \in [0, 1]$ are termed as lower level subgroups of α.*

4. CONCLUSION

This paper proposed a more general notion of anti-fuzzy subgroup and derived some important propositions and results. In order to support our claim, a graphical comparison between Biswas's version and our proposed version of anti-fuzzy subgroup is given and some suitable examples are constructed. Based on our proposed definition, we further redefined the notions of lower level set and lower level subgroup and derived theorems needed to study the algebraic characteristics posed by the introduced definitions. Further, we can redefine and generalize the notion of intuitionistic anti-fuzzy subgroup.

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