Yugoslav Journal of Operations Research
32 (2022), Number 1, 61–86
DOI: https://doi.org/10.2298/YJOR201115015J

MULTITIME MULTIOBJECTIVE VARIATIONAL PROBLEMS VIA η-APPROXIMATION METHOD

Shalini JHA*
Machine Intelligence Unit
(Indian Statistical Institute), 203, B.T. Road Kolkata 700 108, India
jhashalini.rash89@gmail.com
shalini.thakur@vitap.ac.in

Prasun DAS
SQC & OR Unit
(Indian Statistical Institute), 203, B.T. Road Kolkata 700 108, India
prasun@isical.ac.in

Sanghamitra BANDHYOPADHYAY
Machine Intelligence Unit
(Indian Statistical Institute), 203, B.T. Road Kolkata 700 108, India
sanghami@isical.ac.in

Received: October 2020 / Accepted: May 2021

Abstract: The present article is devoted to multitime multiobjective variational problems via η-approximation method. In this method, an η-approximation approach is applied to the considered problem, and a new problem is constructed, called as η-approximated multitime multiobjective variational problem that contains the change in objective and both constraints functions. The equivalence between an efficient (Pareto optimal) solution to the main multitime multiobjective variational problem is derived along with its associated η-approximated problem under invexity defined for a multitime functional. Furthermore, we have also discussed the saddle-point criteria for the problem considered and its associated η-approximated problems via generalized invexity assumptions.

Keywords: Multitime Multiobjective Variational Problem, η-approximation Method, Efficient Solution, Optimality Conditions, Generalized Invex Functions.

MSC: 49J20, 65K10, 93C35.

1. INTRODUCTION

The determination of optimal elements of a set, if it exist at all, is called a vector optimization problem/ multiobjective optimization problem/multi-criteria decision making problem. This type of problems exist not only in mathematical fields but also in economics [23] and engineering [6]. Bayes solution, multi-criteria decision making, Ekland's variational principle, n player differential game, and solution of boundary value problems, etc. are fields of mathematics that belong to multiobjective optimization problems. The standard optimality conditions for vector optimization problems were first introduced by Edgeworth [18], and Pareto [28]. Thereafter, several researchers had generalized these notions to different types of optimization problems.

The theory of convexity plays a vital role in optimization. However, there exist optimization problems of various type for which the concept of convexity cannot be used in proving the fundamental results. Therefore, its generalization is needed. Invexity is one of its generalization that was first explored by Hanson [20]. Later on, Hanson's results motivated other authors to generalize the role of invexity (see, [5, 8, 11, 15, 16, 24, 31, 35, 39]).

Calculus of variations is an important branch of pure and applied mathematics that deals with search of best possible objects like minimal surfaces or trajectory of the fastest travel, shortest distance between two points. Many researchers studied the necessary and sufficient conditions for calculus of variation in which Euler's condition is well known. The function that satisfies the Euler's condition is called the stationary point (or extremal point) of the problem but the converse is not necessarily true. Consequently, it was required to check whether the extremal point was an optimal solution. Hanson [19] discussed the relationship between optimization problems and variational problems. Later, several researchers have shown their interest to solve the variational control problems (see, for example, [2, 3, 4, 10, 12, 14, 22, 25, 27, 34, 36] and others). Craven [17], remarkably established KT-type necessary conditions for multiobjective variational problems under pseudo and quasi convexity assumptions. In [13], Arana-Jimenez et al. studied various duality results for multiobjective variational problems by using pseudoinvexity concept.

In 1932, the Physicist Dirac first introduced the term multitme, later used in mathematics [32, 37]. The multitime control theory is related to the partial derivatives of dynamical systems and their optimization over multitime, also known as the multidimensional control problems, which have wide applications theoretically as well as numerically [1]. Multitime is the extension of single-time dynamic programming that contains m-dimensional evolution and path independent curvillinear integral functional was explained by Udriste and Tevy [38]. Further, a curvilinear integral type multitime multiobjective variational problems is studied in [30] and the results on Mond-Weir type duality is established by using (ρ, b)-quasiinvexity.

Postolache [33] consider multitime multiobjective variational problems that minimizes curvilinear integral type quotients of multiobjective functionals and discusses duality theory. Very recently, in [29], author introduced another generalization, called univexity, and examined the sufficient conditions for efficiency and proper efficiency by using the newly introduced univexity concept.

On the continuation of the above research works, a new method has been introduced by Antczak to find the solution of mathematical programming problem and its duals from some associated optimization problem. In 2003, Antczak [7] first explored the modified objective function method for differentiable multiobjective programming problem. He used this approach to obtain optimality conditions for (weak) Pareto optimality for the considered nonconvex multiobjective programming problem by constructing for it an equivalent vector minimization problem.

The aim of our paper is to explore optimality conditions and an η-saddle-point criteria by using the η-approximation method for a new class of nonconvex optimization problems, that is, multitime multiobjective variational problems with invex functionals of curvilinear integral type. Hence, the modified objective function method, which was discussed by Jayswal et al. [21] for differentiable multitime variational problems, is extended to a new class of nonconvex multitime multiobjective variational problems. In other words, this method is used for the first time for characterization of solvability of multitime multiobjective variational problems. In this approach, for the original multitime multiobjective variational problem, we construct at a fixed feasible point its associated η-approximated multitime multiobjective variational problem. The equivalence between efficient solutions for the original problem and its associated η-approximated problem is established under invexity and generalized invexity hypotheses. Further, we have also discussed the η-saddle-point criteria.

The organization of this paper is as follows: some preliminary definitions, and a theorem, which we use in proving the main results in the paper have been mentioned in Section 2. In Section 3, using the η-approximation method, we construct a new η-approximated multitime multiobjective variational problem by modifying both the objective and constraint functions of the considered problem. Then, we establish the relationship between an efficient solution for the original considered problem and its associated η-approximated problem. Afterwards, in Section 4, we give a definition of the Lagrange function and its saddle-point in the associated η-approximated problem and establish the relationship between an efficient solution and an η-saddle-point in its associated η-approximated problem. Also, we have established the same results as in Section 4 for original considered problem in Section 5. Section 6 gives our conclusion.

2. NOTATIONS AND PRELIMINARIES

Let $\Lambda_n = \{1, \ldots, n\}$ be the indexed set. Suppose that τ and ζ are two Riemannian manifolds of dimension p and n, respectively, with the local co-ordinates of τ and ζ are $s = s^\alpha,\ \alpha \in \Lambda_p$, and $z = z^i,\ i \in \Lambda_n$. The symbol s is used for time. Let β_{s_0,s_1} denote the hyperparallelopiped in Euclidean space $\mathcal{R}^p$ with opposite points $s_0 = (s_0^1, \ldots, s_0^p)$ and $s_1 = (s_1^1, \ldots, s_1^p)$ can be written as in the interval $[s_0, s_1]$ by the help of product order relation on $\mathcal{R}^p$. Assume that Γ_{s_0,s_1} is the curve of

piecewise C^1-type connecting the points $s_0,\ s_1$ and $J^1(\tau,\zeta)$ is jet bundle together with $\tau,\ \zeta$ that contains first order derivatives.
For any vectors $\varsigma,\ \varkappa \in \zeta$, we use the following inequalities and equalities:
(i) $\varsigma = \varkappa \Leftrightarrow \varsigma^i = \varkappa^i,\ \forall\ i \in \Lambda_n$;
(ii) $\varsigma < \varkappa \Leftrightarrow \varsigma^i < \varkappa^i,\ \forall\ i \in \Lambda_n$;
(iii) $\varsigma \leqq \varkappa \Leftrightarrow \varsigma^i \leqq \varkappa^i,\ \forall\ i \in \Lambda_n$;
(iv) $\varsigma \leq \varkappa \Leftrightarrow \varsigma \leqq \varkappa$ and $\varsigma \neq \varkappa$.
Consider the problem:

$$\begin{aligned}
\textbf{(MMVP)}\quad & \text{minimize} \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))\,ds^\alpha \\
&= \Big(\int_{\Gamma_{s_0,s_1}} \varphi^1_\alpha(\pi_z(s))\,ds^\alpha,\ldots,\int_{\Gamma_{s_0,s_1}} \varphi^r_\alpha(\pi_z(s))\,ds^\alpha\Big) \\
&\text{subject to}\qquad g(\pi_z(s)) \leqq 0, \\
&\qquad z(s_0)=z_0,\ z(s_1)=z_1,\ s\in\beta_{s_0,s_1},
\end{aligned}$$

where $\varphi^i_\alpha : J^1(\tau,\zeta) \to \mathcal{R},\ i=1,\ldots,r$ is of C^∞-class and $g=(g^j_a) : J^1(\beta_{s_0,s_1},\zeta) \to \mathcal{R}^{ms};\ a=1,\ldots,s,\ j=1,\ldots,m;\ m<n$ is the Lagrange matrix density of C^∞-class. Let $z:\beta_{s_0,s_1} \to \zeta$. Assume that $\pi_z(s)=(s,z(s),z_\gamma(s)),\ z_\gamma(s)=\dfrac{\partial z(s)}{\partial s^\gamma};\gamma\in\Lambda_p$ are the partial velocities. We use the symbol z in place of $z(s)$ for our convenience.
Let Ω denote the feasible set of (MMVP), i.e.,

$$\Omega = \{z\in\zeta|\ z(s_0)=z_0,\ z(s_1)=z_1,\ g(\pi_z(s))\leqq 0,\ s\in\beta_{s_0,s_1}\}.$$

In multiobjective programming, all the objectives are in conflicting nature. Therefore, there is no singe solution that optimizes all the objectives simultaneously. Thus, minimization of multiobjective programming means determination of efficient solution in the following ways.

Definition 1. *A point $y(s)\in\Omega$ is called an efficient (Pareto optimal) solution to (MMVP) if no other $z(s)\in\Omega$ exist such that*

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))\,ds^\alpha \leq \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_y(s))\,ds^\alpha.$$

Definition 2. *A feasible solution $y(s)\in\Omega$ is called a weak efficient (weak Pareto optimal) solution to (MMVP) if no other $z(s)\in\Omega$ exist such that*

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))\,ds^\alpha < \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_y(s))\,ds^\alpha.$$

To establish the various results in subsequent part, firstly, we shall present the following definitions of invexity for a multitime multiobjective functional of

curvilinear integral type (see, [30]).
Let $f_i^\alpha : J^1(\beta_{s_0,s_1}, \zeta) \to \mathcal{R};\ i \in \Lambda_r$ be a path independent curvilinear vector valued functional of C^∞-class and $\eta : J^1(\beta_{t_0,t_1}, \zeta) \times J^1(\beta_{s_0,s_1}, \zeta) \to \mathcal{R}^n$ be such a vector valued function for which the condition $\eta(\pi_z(s), \pi_z(s)) = 0$ is satisfied for all $z(s) \in \zeta$.

Definition 3. *A functional $\int_{\Gamma_{s_0,s_1}} f_\alpha^i(\pi_z(s))ds^\alpha$ is called invex at $y(s) \in \zeta$ on ζ with respect to η if, for all $z(s) \in \zeta$,*

$$\int_{\Gamma_{s_0,s_1}} f_\alpha^i(\pi_z(s))ds^\alpha - \int_{\Gamma_{s_0,s_1}} f_\alpha^i(\pi_y(s))ds^\alpha \geqq \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_z(s), \pi_y(s))\frac{\partial f_\alpha^i}{\partial z}(\pi_y(s)) + D_\gamma \eta(\pi_z(s), \pi_y(s))\frac{\partial f_\alpha^i}{\partial z_\gamma}(\pi_y(s))\Big]ds^\alpha.$$

Now, we use the following necessary optimality conditions for (MMVP) established by Mititelu et al. [26].

Theorem 4. *Let $\bar{z} \in \Omega$ be a normal efficient solution to* (MMVP). *Then, there exist $\bar{\Upsilon} \in \mathcal{R}^r$ and $\bar{\upsilon} = \bar{\upsilon}_\alpha : J^1(\beta_{s_0,s_1}, \zeta) \to \mathcal{R}^{msp}$ which, for all $s \in \beta_{s_0,s_1}$, satisfy the following conditions:*

$$\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) = D_\gamma\Big(\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big), \tag{1}$$

$$\langle \bar{\upsilon}_\alpha(s), g(\pi_{\bar{z}}(s))\rangle = 0,\ \text{for each } \alpha = \{1, \dots, m\}, \tag{2}$$

$$\bar{\Upsilon} \geq 0,\ \langle \bar{\Upsilon}, e\rangle = 1,\ \bar{\upsilon}_\alpha(s) \geqq 0, e = (1, \dots, 1) \in \mathcal{R}^r. \tag{3}$$

Definition 5. *[26] An efficient solution $\bar{z} \in \Omega$ in the problem (MMVP) is called normal if $\bar{\Upsilon} \neq 0$.*
For proving the fundamental results, we can take $\bar{\Upsilon} = 1$.

Remark 6. *The following property:*

$$\int_{\Gamma_{s_0,s_1}} D_\gamma(\eta(\pi_z(s), \pi_{\bar{z}}(s)))\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(t))ds^\alpha = -\int_{\Gamma_{s_0,s_1}} \eta(\pi_z(s), \pi_{\bar{z}}(s))\Big(D_\gamma\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(t))\Big)ds^\alpha.$$

has been used to discuss the fundamental results in the onward section.

3. η-APPROXIMATED MULTITIME MULTIOBJECTIVE VARIATIONAL PROBLEM AND OPTIMALITY CONDITIONS

Let $\bar{z}$ be an arbitrary given feasible solution to (MMVP). Then, in the used η-approximation approach, the η-approximated multitime multiobjective variational problem (MMVP$_\eta(\bar{z})$) corresponding to (MMVP) is constructed as follows:

(MMVP$_\eta(\bar{z})$)

$$\text{minimize} \quad \int_{\Gamma_{s_0,s_1}} \varphi^i_\alpha(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi^i_\alpha}{\partial z}(\pi_{\bar{z}}(t)) + D_\gamma(\eta(\pi_z(s), \pi_{\bar{z}}(s))) \frac{\partial \varphi^i_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha$$

subject to

$$g^j_a(\pi_{\bar{z}}(s)) + \eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial g^j_a}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_z(s), \pi_{\bar{x}}(s)) \frac{\partial g^j_a}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0,$$

$$z(s_0) = z_0, \; z(s_1) = z_1, \; s \in \beta_{s_0,s_1},$$

where $\varphi, \; g$ are mentioned in the original problem (MMVP).

Let $\Omega(\bar{z}(s))$ denote the feasible set of (MMVP$_\eta(\bar{z})$), i.e.,

$$\Omega(\bar{z}(s)) = \{z(s) \in \zeta | \; z(s_0) = z_0, \; z(s_1) = z_1, \; g^j_a(\pi_{\bar{z}}(s)) + \eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial g^j_a}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_z(s), \pi_{\bar{z}}(s))) \frac{\partial g^j_a}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0, \; s \in \beta_{s_0,s_1}\}.$$

Remark 7. *As it follows directly from Definition 1, a point $\hat{y}(s) \in \Omega(\bar{z})$ is called an efficient solution to (MMVP$_\eta(\bar{z})$), if*

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_z(s), \pi_{\bar{z}}(s))) \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha$$
$$\leq \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\hat{y}}(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_{\hat{y}}(s), \pi_{\bar{z}}(s))) \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha$$

holds for no other $z(s) \in \Omega(\bar{z})$.

Now, we have to show that the relationship of feasible solution between original problem (MMVP) and (MMVP$_\eta(\bar{z})$).

Lemma 8. *Let* $\bar{z}(s)$ *be a feasible solution to* (MMVP) *at which* $g(\pi_z(s))$ *is invex with respect to* η *on* Ω. *Then any feasible solution of* (MMVP) *is also a feasible solution to* (MMVP$_\eta(\bar{z})$).

Proof. Let $\bar{z}(s)$ be a feasible solution to (MMVP). Since $g(\pi_z(s))$ is invex at $\bar{z}(s)$ with respect to η, therefore, by Definition 3, it follows that

$$g_a^j(\pi_z(s)) - g_a^j(\pi_{\bar{z}}(s)) \geqq \eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial g_a^j}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial g_a^j}{\partial z_\gamma}(\pi_{\bar{z}}(s))$$

hold for all $z(s) \in \Omega$.
Since $z(s) \in \Omega$, therefore, by the feasibility condition, the above inequality reduces to

$$g_a^j(\pi_{\bar{z}}(s)) + \eta(\pi_z(s), \pi_{\bar{z}}(s)) \frac{\partial g_a^j}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_z(s), \pi_{\bar{z}}(s))) \frac{\partial g_a^j}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0,$$

which shows that $z(s) \in \Omega(\bar{z}(s))$. This completes the proof. □

Now, we have to establish the equivalence between an efficient solution to (MMVP) and (MMVP$_\eta(\bar{z})$) under invexity assumptions.

Theorem 9. *Let* $\bar{z}$ *be a normal efficient solution to* (MMVP) *at which the necessary optimality conditions* (1)-(3) *are satisfied with multipliers* $\bar{\Upsilon}, \bar{\upsilon}_\alpha(s)$. *Assume that* $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$. *If* $\bar{\Upsilon} > 0$, *then* $\bar{z}$ *is also an efficient solution* (MMVP$_\eta(\bar{z})$).

Proof. Suppose, efficiency of (MMVP$_\eta(\bar{z})$) fails at $\bar{z}$. Then the following inequalities hold for existence of $y(s) \in \Omega(\bar{z}(s))$:

$$\begin{aligned} &\int_{\Gamma_{s_0,s_1}} \varphi_\alpha^i(\pi_{\bar{z}}(s))\, ds^\alpha \\ &+ \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_y(s), \pi_{\bar{z}}(s))) \frac{\partial \varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha \\ &\leqq \int_{\Gamma_{s_0,s_1}} \varphi_\alpha^i(\pi_{\bar{z}}(s))\, ds^\alpha \\ &+ \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \frac{\partial \varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha, \\ &\forall\ i = 1, \ldots, r \end{aligned}$$

and

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha^{i^*}(\pi_{\bar z}(s))\, ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar z}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar z}(s)))\frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big] ds^\alpha$$
$$< \int_{\Gamma_{s_0,s_1}} \varphi_\alpha^{i^*}(\pi_{\bar z}(s))\, ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar z}(s),\pi_{\bar z}(s))\frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar z}(s)) + D_\gamma(\eta(\pi_{\bar z}(s),\pi_{\bar z}(s)))\frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big] ds^\alpha,$$
$$\text{for some } i^* = 1,\dots,r.$$

Since $\eta(\pi_{\bar z}(s),\pi_{\bar z}(s)) = 0$, therefore, the above inequalities reduces to

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial \varphi_\alpha^{i}}{\partial z}(\pi_{\bar z}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar z}(s)))\frac{\partial \varphi_\alpha^{i}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big] ds^\alpha$$
$$\leqq 0,\ \forall\ i = 1,\dots,r \tag{4}$$

and

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar z}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar z}(s)))\frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big] ds^\alpha$$
$$< 0,\ \text{for some } i^* = 1,\dots,r. \tag{5}$$

Since $\bar\Upsilon > 0$, therefore, from (4) and (5), we get

$$\int_{\Gamma_{s_0,s_1}} \Big[\langle \eta(\pi_y(s),\pi_{\bar z}(s)), \bar\Upsilon_i \frac{\partial \varphi_\alpha^{i}}{\partial z}(\pi_{\bar z}(s))\rangle$$
$$+\langle D_\gamma(\eta(\pi_y(s),\pi_{\bar z}(s))), \bar\Upsilon_i \frac{\partial \varphi_\alpha^{i}}{\partial z_\gamma}(\pi_{\bar z}(s))\rangle\Big] ds^\alpha \leqq 0,\ \forall\ i = 1,\dots,r \tag{6}$$

and

$$\int_{\Gamma_{s_0,s_1}} \Big[\langle \eta(\pi_y(s),\pi_{\bar z}(s)), \bar\Upsilon_i^* \frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar z}(s))\rangle$$
$$+\langle D_\gamma(\eta(\pi_y(s),\pi_{\bar z}(s))), \bar\Upsilon_i^* \frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big] ds^\alpha < 0,\ \text{for some } i^* = 1,\dots,r. \tag{7}$$

On combining (6) and (7), we get

$$\int_{\Gamma_{s_0,s_1}} \Big[\langle \eta(\pi_y(s),\pi_{\bar z}(s)), \bar\Upsilon \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar z}(s))\rangle$$
$$+\langle D_\gamma(\eta(\pi_y(s),\pi_{\bar z}(s))), \bar\Upsilon \frac{\partial \varphi_\alpha}{\partial x_\gamma}(\pi_{\bar z}(s))\rangle\Big] ds^\alpha < 0.$$

Using the Remark 6, the inequality above can be rewritten as

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\{\bar\Upsilon\frac{\partial\varphi_\alpha^i}{\partial z}(\pi_{\bar z}(s)) - D_\gamma(\bar\Upsilon\frac{\partial\varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar z}(s)))\}\Big]ds^\alpha < 0,$$

which by necessary optimality condition (1) implies that

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\{\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar z}(s)) - D_\gamma(\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar z}(s)))\}\Big]ds^\alpha > 0. \tag{8}$$

Now, $y \in \Omega(\bar z(s))$, therefore, from feasibility condition we get

$$g_a^j(\pi_{\bar z}(s)) + \eta(\pi_z(s),\pi_{\bar z}(s))\frac{\partial g_a^j}{\partial z}(\pi_{\bar z}(s)) + D_\gamma\eta(\pi_z(s),\pi_{\bar z}(s))\frac{\partial g_a^j}{\partial z_\gamma}(\pi_{\bar z}(s)) \leqq 0.$$

On multiplying by $\bar\upsilon_\alpha(s)$ in the above inequality and adding them, we get

$$\bar\upsilon_\alpha(s)g(\pi_{\bar z}(s)) + \eta(\pi_z(s),\pi_{\bar z}(s))\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar z}(s)) + D_\gamma(\eta(\pi_z(s),\pi_{\bar z}(s)))\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar z}(s)) \leqq 0,$$

which on using necessary condition (2) reduces to

$$\eta(\pi_z(s),\pi_{\bar z}(s))\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar z}(s)) + D_\gamma(\eta(\pi_z(s),\pi_{\bar z}(s)))\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar z}(s)) \leqq 0.$$

Integrating the above integral over Γ_{s_0,s_1} and using the Remark 6, we have

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\{\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar z}(s)) - D_\gamma(\bar\upsilon_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar z}(s)))\}\Big]ds^\alpha \leqq 0,$$

which contradicts (8). Hence, the proof is complete. □

Corollary 10. *Let $\bar z$ be a weakly efficient solution to* (MMVP) *at which the necessary optimality conditions* (1)-(3) *are satisfied with multipliers $\bar\Upsilon^\alpha(s) \in \mathcal{R}^r, \bar\upsilon_\alpha(s) \in \mathcal{R}^{msp}$. Assume that $\eta(\pi_{\bar z}(s),\pi_{\bar z}(s)) = 0$. Then $\bar z$ is also an efficient solution in its associated η-approximated* ($\text{MMVP}_\eta(\bar z)$).

Proof. Since every efficient solution is a weak efficient solution, therefore, proof follows the similar line as in Theorem 9, hence it is omitted. □

Now, we prove the converse of the above theorems under the assumptions of invexity on multitime multiobjective functional.

Theorem 11. *Let $\bar z \in \Omega$ be an efficient solution to* ($\text{MMVP}_\eta(\bar z)$). *Further, assume that $\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))ds^\alpha$ and $g(\pi_z(s))$ are invex at $\bar z$ on Ω with respect to η satisfying the condition $\eta(\pi_{\bar z}(s),\pi_{\bar z}(s)) = 0$. Then $\bar z$ is also an efficient solution to* (MMVP).

Proof. On the contrary, suppose $\bar{z}$ fails the condition of efficiency to (MMVP). Then, following inequalities hold for existence of $y \in \Omega$:

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_y(s))\, ds^\alpha \leq \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_{\bar{z}}(s))\, ds^\alpha.$$

Equivalently,

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha^i(\pi_y(s))ds^\alpha \leqq \int_{\Gamma_{s_0,s_1}} \varphi_\alpha^i \pi_{\bar{z}}(s))ds^\alpha,\ \forall\ i = 1,\dots,r, \tag{9}$$

and

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha^{i^*}(\pi_y(s))ds^\alpha < \int_{\Gamma_{s_0,s_1}} \varphi_\alpha^{i^*} \pi_{\bar{z}}(s))\, ds^\alpha,\ \text{for some } i^* = 1,\dots,r. \tag{10}$$

Since $g(\pi_z(s))$ is invex at $\bar{z} \in \Omega$, therefore, by Lemma 8, we say that $y \in \Omega(\bar{z}(s))$. Now, $\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))\, ds^\alpha$ is invex at $\bar{z}$ with respect to η, therefore, by Definition 3, we have

$$\begin{aligned} &\int_{\Gamma_{s_0,s_1}} \varphi_\alpha^i(\pi_y(s))ds^\alpha - \int_{\Gamma_{s_0,s_1}} \varphi_\alpha^i(\pi_{\bar{z}}(s))ds^\alpha \\ &\quad \geqq \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z}(\pi_{\bar{z}}(s)) \\ &\qquad + D_\gamma \eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha,\ \forall\ i = 1,\dots,r. \end{aligned} \tag{11}$$

On combining (9)-(11), we get

$$\begin{aligned} &\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z}(\pi_{\bar{z}}(s)) \\ &\qquad + D_\gamma \eta(\pi_z(s), \pi_y(s)) \frac{\partial \varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha \leqq 0,\ \forall\ i = 1,\dots,r, \end{aligned}$$

and

$$\begin{aligned} &\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha \\ &\qquad < 0,\ \text{for some } i^* = 1,\dots,r. \end{aligned}$$

Since $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$, therefore, the inequalities above can be rewritten as

$$\begin{aligned} &\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha \\ &\quad \leqq \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z}(\pi_{\bar{z}}(s)) \\ &\qquad + D_\gamma \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha^i}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha,\ \forall\ i = 1,\dots,r, \end{aligned}$$

and

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar z}(s)) + D_\gamma\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big]ds^\alpha$$
$$< \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar z}(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha^{i^*}}{\partial x}(\pi_{\bar z}(s)) + D_\gamma\eta(\pi_{\bar z}(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar z}(s))\Big]ds^\alpha,$$
$$\text{for some } i^* = 1,\dots,r.$$

That is, there exists some $y \in \Omega(\bar z(s))$ such that

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha}{\partial z}(\pi_{\bar z}(s)) + D_\gamma\eta(\pi_y(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha}{\partial z_\gamma}(\pi_{\bar z}(s))\Big]ds^\alpha$$
$$\leq \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar z}(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha}{\partial z}(\pi_{\bar z}(s)) + D_\gamma\eta(\pi_{\bar z}(s),\pi_{\bar z}(s))\frac{\partial\varphi_\alpha}{\partial z_\gamma}(\pi_{\bar z}(s))\Big]ds^\alpha,$$

which contradicts the fact that $\bar x$ is an efficient solution to (MMVP$_\eta(\bar z)$). □

Now, the numerical justification of Theorem 11 has been presented below:

Example 12. *Let* $\zeta = \mathcal{R},\ \alpha = \gamma = \{1,2\}, \Gamma_{s_0,s_1} = (s,s);\ 0 \leqq s \leqq 1$ *and* $\eta : J^1(\beta_{s_0,s_1},\zeta) \times J^1(\beta_{s_0,s_1},\zeta) \mapsto \mathcal{R}$ *is defined as*

$$\eta(\pi_z(s),\pi_{\bar x}(s)) = e^{z(s)} - e^{\bar z(s)}.$$

It is obvious that $\eta(\pi_{\bar z}(s),\pi_{\bar z}(s)) = 0$. *Consider the problem:*

$$\textbf{\textit{(MMVP1)}} \qquad \textit{minimize} \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))\,ds^\alpha$$
$$= \Big(\int_{\Gamma_{s_0,s_1}} ((\sin z(s))^2 + e^{\frac14(z(s))^2} + z(s) - e^{-(z(s))^2},$$
$$\frac12(\sin z(s))^2 + e^{\frac14(z(s))^2} + \frac12(z(s))^5 + z(s))\,ds^\alpha,$$
$$\int_{\Gamma_{s_0,s_1}} (\cos z(s) - \sin z(s) + \frac12 e^{-\frac12(z(s))^2} + \frac14(z(s))^2e^{-(z(s))^2},$$
$$1 + \cos z(s) - \sin z(s) + \frac12 e^{-\frac12(z(s))^2} + \frac14(z(s))^2e^{-(z(s))^2}\,ds^\alpha\Big)$$
$$\textit{subject to} \quad (z(s))^2 - z(s) \leqq 0.$$

The feasible set of (MMVP1) is $\Omega = \{z(s) \in \zeta : 0 \leqq z(s) \leqq 1\}$. *Consider* $\bar z(s) = 0$. *All the functions can be seen in following figures:*

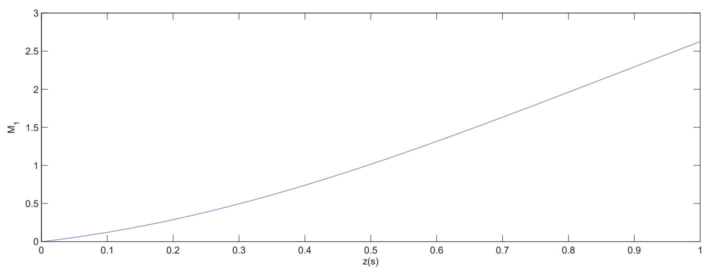

Figure 1: Graph of $[M_1 = (\sin z(s))^2 + e^{\frac14(z(s))^2} + z(s) - e^{-(z(s))^2}]$.

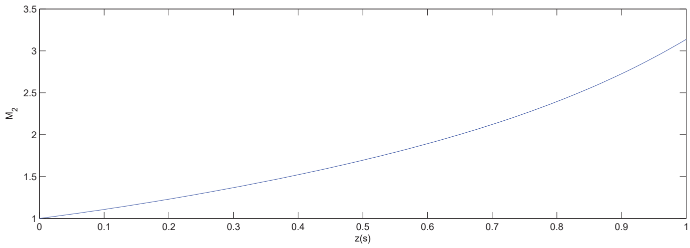

Figure 2: Graph of $[M_2 = \frac{1}{2}(\sin z(s))^2 + e^{\frac{1}{4}(z(s))^2} + \frac{1}{2}(z(s))^5 + z(s)]$.

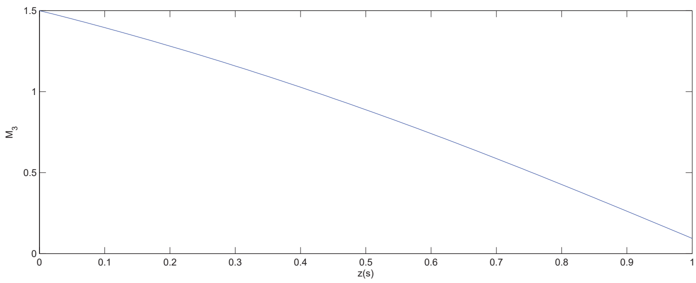

Figure 3: Graph of $[M_3 = \cos z(s) - \sin z(s) + \frac{1}{2}e^{-\frac{1}{2}(z(s))^2} + \frac{1}{4}(z(s))^2 e^{-(z(s))^2}]$.

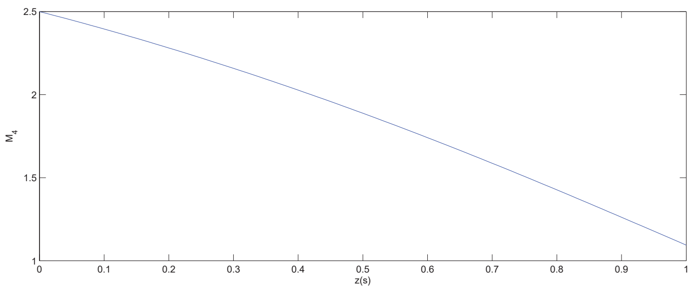

Figure 4: Graph of $[M_4 = 1 + \cos z(s) - \sin z(s) + \frac{1}{2}e^{-\frac{1}{2}(z(s))^2} + \frac{1}{4}(z(s))^2 e^{-(z(s))^2}]$.

Therefore, the multitime variational problem (MMVP1$_\eta(\bar{z})$) constructed in the η-approximation method is given as follows:

$$\textbf{(MMVP1}_\eta(\bar{z})\textbf{)} \quad \text{minimize} \quad \Big(\int_{\Gamma_{s_0,s_1}} (e^{z(s)}-1,\ e^{z(s)})\,ds^\alpha,$$
$$\int_{\Gamma_{t_0,t_1}} \Big(\frac{5}{2}-e^{z(s)},\ \frac{7}{2}-e^{z(s)}\Big)\,ds^\alpha\Big)$$
$$\text{subject to} \qquad 1-e^{z(s)} \leqq 0.$$

The efficiency of η-approximated problems can be seen in the following figures:

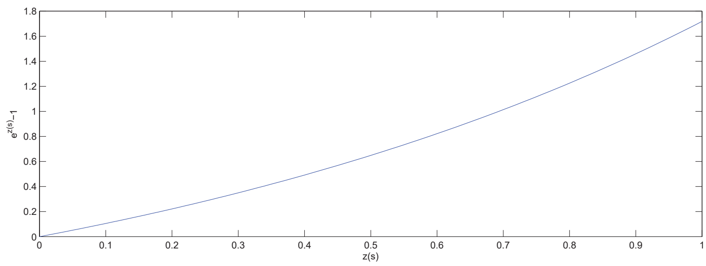

Figure 5: Graph of $[e^{z(s)}-1]$.

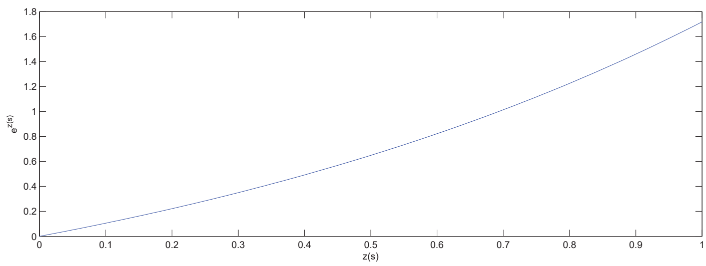

Figure 6: Graph of $[e^{z(s)}]$.

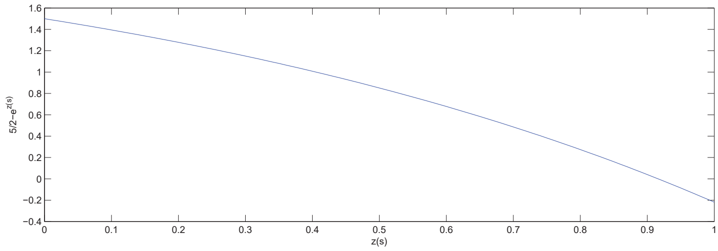

Figure 7: Graph of $[\frac{5}{2} - e^{z(s)}]$.

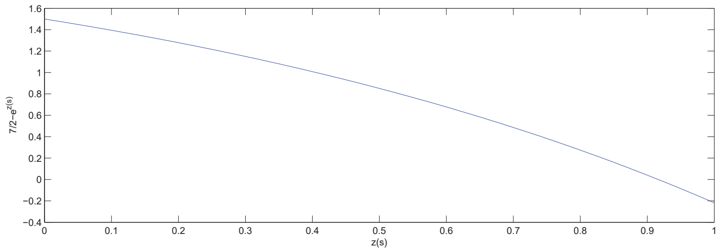

Figure 8: Graph of $[\frac{7}{2} - e^{z(s)}]$.

From Figure 5 to Figure 8, we can see that $\bar{z}(s) = 0$ is an efficient solution to (MMVP1$_\eta(\bar{z})$). Note that (MMVP1$_\eta(\bar{z})$) is reduced to a simpler form in comparison to (MMVP1). Also, we have checked that the functionals $\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))ds^\alpha$ and $g(\pi_z(s))$ are invex at $\bar{z}(s) = 0$ on Ω with respect to η given above. Since all assumptions of Theorem 11 are fulfilled, $\bar{z}(s) = 0$ is, therefore, $\bar{z}(s) = 0$ is also an efficient solution to (MMVP1). Also, the efficiency of (MMVP1) can be easily verified from Figures 1 to 4.

4. SADDLE POINT CRITERIA FOR η-APPROXIMATED MULTITIME MULTIOBJECTIVE VARIATIONAL PROBLEM (MMVP$_\eta(\bar{z})$)

In this section, we establish the relationship between an efficient solution and an η-saddle-point for η-Lagrange function in the η-approximated multitime variational problem (MMVP$_\eta(\bar{z})$) without invexity assumption on functional.

Motivated by Antczak [9], we define the η-Lagrange function and its saddle-point for the multitime multiobjective variational problem (MMVP$_\eta(\bar{z})$) constructed in the η-approximation method.

Definition 13. *The Lagrange function in (MMVP$_\eta(\bar{z})$) is denoted by L^η_α and defined as*

$$L^\eta_\alpha(z,\Upsilon,\upsilon_\alpha)=\int_{\Gamma_{s_0,s_1}}\Big[\Upsilon\varphi_\alpha(\pi_{\bar{z}}(s))+\upsilon_\alpha g(\pi_{\bar{z}}(s))\Big]ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}}\Big[\eta(\pi_z(s),\pi_{\bar{z}}(s))\{\Upsilon\frac{\partial\varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s))+\upsilon_\alpha\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\}$$
$$+D_\gamma(\eta(\pi_z(s),\pi_{\bar{z}}(s)))\{\Upsilon\frac{\partial\varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))+\upsilon_\alpha\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big]ds^\alpha$$

Remark 14. *It is obvious that*

$$L^\eta_\alpha(z,\Upsilon,\upsilon_\alpha)=L_\alpha(z,\Upsilon,\upsilon_\alpha),\ \textit{if}\ \eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))=0.$$

Definition 15. *A point $(\bar{z},\bar{\Upsilon},\bar{\upsilon}_\alpha)\in\Omega(\bar{z}(s))\times\mathcal{R}^r\times\mathcal{R}^{msp}$ is called an η-saddle-point for the η-Lagrange function in (MMVP$_\eta(\bar{z})$) if the following inequalities hold:*
(i) $L^\eta_\alpha(\bar{z},\bar{\Upsilon},\upsilon_\alpha)\leqq L^\eta_\alpha(\bar{z},\bar{\Upsilon},\bar{\upsilon}_\alpha),\ \forall\ \upsilon_\alpha(s)\in\mathcal{R}^{msp}$,
(ii) $L^\eta_\alpha(\bar{z},\bar{\Upsilon},\bar{\upsilon}_\alpha)\leqq L^\eta_\alpha(z,\bar{\Upsilon},\bar{\upsilon}_\alpha),\ \forall\ z(s)\in\Omega(\bar{z}(s))$.

Theorem 16. *Let $(\bar{z},\bar{\Upsilon},\bar{\upsilon}_\alpha)\in\Omega(\bar{z}(s))\times\mathcal{R}^r\times\mathcal{R}^{msp}$ be an η-saddle-point for the η-Lagrange function in* (MMVP$_\eta(\bar{z})$) *and $\bar{\Upsilon}>0$. Then $\bar{z}$ is an efficient solution to* (MMVP$_\eta(\bar{z})$).

Proof. Suppose, efficiency of (MMVP$_\eta(\bar{z})$) fails at $\bar{z}$. Then the following inequalities hold for existence of $y(s)\in\Omega(\bar{z}(s))$:

$$\int_{\Gamma_{s_0,s_1}}\varphi^i_\alpha(\pi_{\bar{z}}(s))\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}}\Big[\eta(\pi_y(s),\pi_{\bar{z}}(s))\frac{\partial\varphi^i_\alpha}{\partial z}(\pi_{\bar{z}}(s))+D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\frac{\partial\varphi^i_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big]ds^\alpha$$
$$\leqq\int_{\Gamma_{s_0,s_1}}\varphi^i_\alpha(\pi_{\bar{z}}(s))\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}}\Big[\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))\frac{\partial\varphi^i_\alpha}{\partial z}(\pi_{\bar{z}}(s))+D_\gamma(\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)))\frac{\partial\varphi^i_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big]ds^\alpha,$$
$$\forall\ i=1,\ldots,r$$

and

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha^{i^*}(\pi_{\bar{z}}(s))\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar{z}}(s))\frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar{z}}(t))\Big]ds^\alpha$$
$$< \int_{\Gamma_{s_0,s_1}} \varphi_\alpha^{i^*}(\pi_{\bar{z}}(s))\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))\frac{\partial \varphi_\alpha^{i^*}}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)))\frac{\partial \varphi_\alpha^{i^*}}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big]ds^\alpha,$$
$$\text{for some } i^* = 1,\ldots,r.$$

Since $\bar{\Upsilon} > 0$, therefore, multiplying the above inequalities by $\bar{\Upsilon}$ and adding them, we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}(s))}\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar{z}}(s))\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big]ds^\alpha$$
$$< \int_{\Gamma_{s_0,s_1}} \bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s))\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)))\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big]ds^\alpha. \tag{12}$$

Now, $y \in \Omega(\bar{z}(s))$, therefore, from feasibility condition we get

$$g_a^j(\pi_{\bar{z}}(s)) + \eta(\pi_y(s),\pi_{\bar{z}}(s))\frac{\partial g_a^j}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma\eta(\pi_y(s),\pi_{\bar{z}}(s))\frac{\partial g_a^j}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0.$$

On multiplying by $\bar{\upsilon}_\alpha(s)$ in the above inequality and adding them, we get

$$\bar{\upsilon}_\alpha(s)g(\pi_{\bar{z}}(s)) + \eta(\pi_y(s),\pi_{\bar{z}}(s))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))$$
$$+ D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0.$$

Integrating the above inequality over Γ_{s_0,s_1}, we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s)g(\pi_{\bar{z}}(s))\,ds^\alpha$$
$$+\int_{\Gamma_{s_0,s_1}} \eta(\pi_y(s),\pi_{\bar{z}}(s))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))$$
$$+D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\,ds^\alpha \leqq 0. \tag{13}$$

Similarly, for $\bar{z} \in \Omega(\bar{z}(s))$, we have

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\, ds^\alpha \leqq 0. \tag{14}$$

Now, the point $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha) \in \Omega(\bar{z}(s)) \times \mathcal{R}^r \times \mathcal{R}^{msp}$ is an η-saddle-point for the η-Lagrange function in multitime multiobjective variational problem (MMVP$_\eta(\bar{z})$), therefore from condition (i) of η-saddle-point, we get

$$L^\eta_\alpha(\bar{z}, \bar{\Upsilon}, \upsilon_\alpha) \leqq L^\eta_\alpha(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha), \ \forall\ \upsilon_\alpha(s) \in \mathcal{R}^{msp},$$

which can be rewritten as

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \upsilon_\alpha g(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \{\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \upsilon_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \{\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \upsilon_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha \\
\leqq\ &\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \{\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \{\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha.
\end{aligned}$$

Taking $\upsilon_\alpha = 0$ in the above inequality, we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \{\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\, ds^\alpha \geqq 0. \tag{15}$$

Combining (14) and (15), we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha$$
$$+ \int_{\Gamma_{s_0,s_1}} \{\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))$$
$$+ D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\, ds^\alpha = 0. \quad (16)$$

Again, combining (13) and (16), we have

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \{\eta(\pi_y(s), \pi_{\bar{z}}(s)) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))$$
$$+ D_\gamma(\eta(\pi_y(s), \pi_{\bar{z}}(s))) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\, ds^\alpha$$
$$\leqq \int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \{\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))$$
$$+ D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\, ds^\alpha. \quad (17)$$

Now, adding (12) and (17), we have

$$\int_{\Gamma_{s_0,s_1}} \left[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\right] ds^\alpha$$
$$+ \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\}$$
$$+ D_\gamma(\eta(\pi_y(s), \pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha$$
$$< \int_{\Gamma_{s_0,s_1}} \left[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\right] ds^\alpha$$
$$+ \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\}$$
$$+ D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha,$$

which by the definition of Lagrange function, yields

$$L^\eta_\alpha(y, \bar{\Upsilon}, \bar{\upsilon}_\alpha) < L^\eta_\alpha(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha), \text{ for } y(s) \in \Omega(\bar{z}(s)).$$

This is the contradiction to the second inequality in the definition of η-saddle-point. Hence, the proof is complete. □

Corollary 17. *Let* $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha) \in \Omega(\bar{z}(s)) \times \mathcal{R}^r \times \mathcal{R}^{msp}$ *be an* η*-saddle-point for the* η*-Lagrange function in* $(\text{MMVP}_\eta(\bar{z}))$ *and* $\bar{\Upsilon} > 0$*. Then* $\bar{z}$ *is a weak efficient solution to* $(\text{MMVP}_\eta(\bar{z}))$.

5. SADDLE-POINT CRITERIA FOR MULTITIME MULTIOBJECTIVE VARIATIONAL PROBLEM (MMVP)

In this section, we establish the relationship between an (weak) efficient solution of (MMVP) and an η-saddle-point for the Lagrange function in (MMVP$_\eta(\bar{z})$).

Theorem 18. *Let $\bar{z} \in \Omega$ at which the $\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))ds^\alpha$ and $g(\pi_z(s))$ are invex at $\bar{z}$ on Ω with respect to η satisfying the condition $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$. If $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha)$ is an η-saddle-point for the η-Lagrange function in* (MMVP$_\eta(\bar{z})$) *and $\bar{\Upsilon} \neq 0$, then $\bar{z}$ is an efficient solution to* (MMVP).

Proof. Since $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha)$ is an η-saddle-point for the η-Lagrange function in (MMVP$_\eta(\bar{z})$), therefore, by the inequality (i), we have

$$L^\eta_\alpha(\bar{z}, \bar{\Upsilon}, \upsilon_\alpha) \leqq L^\eta_\alpha(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha), \ \forall \ \upsilon_\alpha(s) \in \mathcal{R}^{msp},$$

which can be rewritten as

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \upsilon_\alpha g(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \upsilon_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \upsilon_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha \\
\leqq\ &\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha.
\end{aligned}$$

Taking $\upsilon_\alpha = 0$ and using the hypothesis $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$ in the above inequality, we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha \geqq 0. \tag{18}$$

Since $\bar{z} \in \Omega(\bar{z}(s))$, multiplying constraints of (MMVP$_\eta(\bar{z})$) by $\bar{\upsilon}_\alpha(s)$ and integrating over Γ_{s_0,s_1}, we have

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha + \int_{\Gamma_{s_0,s_1}} \{\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\, ds^\alpha \leqq 0,
\end{aligned}$$

which on using the hypothesis $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$ reduces to

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha \leqq 0. \tag{19}$$

On combining (18) and (19), we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha = 0. \tag{20}$$

On the contrary, suppose $\bar{z}$ fails the condition of efficiency to (MMVP). Then, following inequalities hold for existence of $y \in \Omega$:

$$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_y(s))\, ds^\alpha \leq \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_{\bar{z}}(s))\, ds^\alpha.$$

Equivalently,

$$\int_{\Gamma_{s_0,s_1}} \varphi^i_\alpha(\pi_y(s)) ds^\alpha \leqq \int_{\Gamma_{s_0,s_1}} \varphi^i_\alpha \pi_{\bar{z}}(s)) ds^\alpha,\ \forall\ i = 1, \ldots, r,$$

and

$$\int_{\Gamma_{s_0,s_1}} \varphi^{i^*}_\alpha(\pi_y(s)) ds^\alpha < \int_{\Gamma_{t_0,t_1}} \varphi^{i^*}_\alpha \pi_{\bar{z}}(s))\, ds^\alpha,\ \text{for some } i^* = 1, \ldots, r.$$

Since $\bar{\Upsilon} > 0$, therefore multiplying the inequalities above by $\bar{\Upsilon}$ and taking summation throughout, we have

$$\int_{\Gamma_{s_0,s_1}} \bar{\Upsilon}\varphi_\alpha(\pi_y(s)) ds^\alpha < \int_{\Gamma_{s_0,s_1}} \bar{\Upsilon}\varphi_\alpha \pi_{\bar{z}}(s)) ds^\alpha. \tag{21}$$

$\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))\, ds^\alpha$ is invex at $\bar{z}$ with respect to η, therefore, by Definition 3, we have

$$\begin{aligned} &\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_y(s)) ds^\alpha - \int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_{\bar{z}}(s)) ds^\alpha \\ \geqq &\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha. \end{aligned}$$

Multiplying the above inequality by $\bar{\Upsilon}$ and combining with (21), we get

$$\int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s), \pi_{\bar{z}}(s)) \bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_y(s), \pi_{\bar{z}}(s)) \bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big] ds^\alpha < 0. \tag{22}$$

Now, by using Lemma 8, we can say that $y \in \Omega(\bar{z}(s))$, therefore, by the feasibility condition

$$g^j_a(\pi_{\bar{z}}(s)) + \eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial g^j_a}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma \eta(\pi_y(s), \pi_{\bar{z}}(s)) \frac{\partial g^j_a}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0.$$

On multiplying by $\bar{\upsilon}_\alpha(t)$ in the above inequality and adding them, we get

$$\bar{\upsilon}_\alpha(s)g(\pi_{\bar{z}}(s)) + \eta(\pi_y(s),\pi_{\bar{z}}(s))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \leqq 0.$$

Integrating the above inequality over Γ_{s_0,s_1}, we get

$$\int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s)g(\pi_{\bar{z}}(s))\,ds^\alpha + \int_{\Gamma_{s_0,s_1}} \{\eta(\pi_y(s),\pi_{\bar{z}}(s))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) + D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\,ds^\alpha \leqq 0. \quad (23)$$

Combining (22) and (23), we get

$$\int_{\Gamma_{s_0,s_1}} \Big[\bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\Big]\,ds^\alpha + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial\varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} + D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial\varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big]\,ds^\alpha < 0.$$

Adding the term $\int_{\Gamma_{s_0,s_1}} \bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s))\,ds^\alpha$ on both side, using the hypothesis $\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)) = 0$ and using (20), the above inequality can be rewritten as

$$\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\Big]\,ds^\alpha + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_y(s),\pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial\varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} + D_\gamma(\eta(\pi_y(s),\pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial\varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big]\,ds^\alpha$$

$$< \int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\Big]\,ds^\alpha + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial\varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} + D_\gamma(\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial\varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big]\,ds^\alpha,$$

which by the definition of Lagrange function, yields

$$L^\eta_\alpha(y,\bar{\Upsilon},\bar{\upsilon}_\alpha) < L^\eta_\alpha(z,\bar{\Upsilon},\bar{\upsilon}_\alpha), \text{ for } y(s)\in\Omega(\bar{z}(s)).$$

This is the contradiction to the second inequality in the definition of η-saddle-point. Hence, the proof is complete. □

Corollary 19. *Let $\bar{z} \in \Omega$ at which the $\int_{\Gamma_{s_0,s_1}} \varphi_\alpha(\pi_z(s))ds^\alpha$ and $g(\pi_z(s))$ are invex at $\bar{z}$ on Ω with respect to η satisfying the condition $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$. If $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha)$ is an η-saddle-point for the η-Lagrange function in* (MMVP$_\eta(\bar{z})$) *and $\bar{\Upsilon} \neq 0$, then $\bar{z}$ is a weak efficient solution to* (MMVP).

Theorem 20. *Let $\bar{z}$ be a normal optimal solution of* (MMVP) *at which the necessary optimality conditions are satisfied with the multipliers $\bar{\Upsilon}$, $\bar{\upsilon}_\alpha(s)$. If $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$, then $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha(s))$ is an η-saddle-point for the η-Lagrange function in* (MMVP$_\eta(\bar{z})$).

Proof. Since $\bar{z}$ is an efficient solution of (MMVP) therefore, conditions (1)-(3) are satisfied for $\bar{\Upsilon}$ and $\bar{\upsilon}_\alpha(s)$. Now, by the feasibility condition

$$g(\pi_{\bar{z}}(s)) \leqq 0.$$

On multiplying by $\upsilon_\alpha(s)$ in the above inequality and adding them, we get

$$\upsilon_\alpha(s) g(\pi_{\bar{z}}(s)) \leqq 0.$$

Integrating the above inequality over Γ_{s_0,s_1}, we get

$$\int_{\Gamma_{s_0,s_1}} \upsilon_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha \leqq 0.$$

Using the necessary optimality condition (2) with integration sign, we have

$$\int_{\Gamma_{s_0,s_1}} \upsilon_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha \leqq \int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha.$$

Since $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$, therefore, the above inequality yields

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \upsilon_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha \\
&+ \int_{\Gamma_{s_0,s_1}} \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \Big[\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \upsilon_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \Big] ds^\alpha \\
&\qquad + \int_{\Gamma_{s_0,s_1}} D_\gamma \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \Big(\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \upsilon_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \Big) ds^\alpha \\
&\quad \leqq \int_{\Gamma_{s_0,s_1}} \bar{\upsilon}_\alpha(s) g(\pi_{\bar{z}}(s))\, ds^\alpha \\
&+ \int_{\Gamma_{s_0,s_1}} \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \Big[\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \Big] ds^\alpha \\
&\qquad + \int_{\Gamma_{s_0,s_1}} D_\gamma \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \Big(\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \Big) ds^\alpha
\end{aligned}$$

Adding the term $\int_{\Gamma_{s_0,s_1}} \bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s))\, ds^\alpha$ on both sides, we have

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \upsilon_\alpha g(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \upsilon_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \upsilon_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha \\
&\leqq \int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon}\varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\} \\
&\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s),\pi_{\bar{z}}(s)))\{\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\}\Big] ds^\alpha,
\end{aligned}$$

which by the definition of Lagrange function, yields

$$L^\eta_\alpha(\bar{z},\bar{\Upsilon},\upsilon_\alpha) \leqq L^\eta_\alpha(\bar{z},\bar{\Upsilon},\bar{\upsilon}_\alpha), \text{ for all } \upsilon_\alpha(s) \in \mathcal{R}^{msp}. \tag{24}$$

From the necessary optimality condition (1), we have

$$\begin{aligned}
&\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \\
&\qquad - D_\gamma\Big(\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big) = 0.
\end{aligned}$$

Let $x \in \Omega(\bar{z}(s))$ be any arbitrary feasible point. Multiplying by $\eta(\pi_z(s),\pi_{\bar{z}}(s))$ on both sides of above equation and integrating over Γ_{s_0,s_1} throughout, we get

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \eta(\pi_z(s),\pi_{\bar{z}}(s))\Big[\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad - \int_{\Gamma_{s_0,s_1}} \eta(\pi_z(s),\pi_{\bar{z}}(s))\Big[D_\gamma\Big(\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big)\Big]\, ds^\alpha = 0.
\end{aligned}$$

Using the Remark 6, the above equation can also be rewritten as

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \eta(\pi_z(s),\pi_{\bar{z}}(s))\Big[\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z}(\pi_{\bar{z}}(s))\Big]\, ds^\alpha \\
&\quad + \int_{\Gamma_{s_0,s_1}} D_\gamma\eta(\pi_z(s),\pi_{\bar{z}}(s))\Big(\bar{\Upsilon}\frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s)\frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s))\Big)\, ds^\alpha = 0.
\end{aligned}$$

Since $\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) = 0$, therefore, the above equation yields

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \eta(\pi_z(s), \pi_{\bar{z}}(s)) \Big[\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \Big] \, ds^\alpha \\
&\qquad + \int_{\Gamma_{s_0,s_1}} D_\gamma \eta(\pi_z(s), \pi_{\bar{z}}(s)) \Big(\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \Big) \, ds^\alpha \\
&= \int_{\Gamma_{s_0,s_1}} \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \Big[\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \Big] \, ds^\alpha \\
&\qquad\quad + \int_{\Gamma_{s_0,s_1}} D_\gamma \eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \Big(\bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha(s) \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \Big) \, ds^\alpha.
\end{aligned}$$

Using the necessary optimality condition (2) with integration sign and adding the term $\int_{\Gamma_{s_0,s_1}} \bar{\Upsilon} \varphi_\alpha(\pi_{\bar{z}}(s)) \, ds^\alpha$ on both sides, we get

$$\begin{aligned}
&\int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon} \varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s)) \Big] \, ds^\alpha \\
&\qquad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_z(s), \pi_{\bar{z}}(s)) \{ \bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \} \\
&\qquad\qquad + D_\gamma(\eta(\pi_z(s), \pi_{\bar{z}}(s))) \{ \bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \} \Big] ds^\alpha \\
&= \int_{\Gamma_{s_0,s_1}} \Big[\bar{\Upsilon} \varphi_\alpha(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha g(\pi_{\bar{z}}(s)) \Big] \, ds^\alpha \\
&\qquad + \int_{\Gamma_{s_0,s_1}} \Big[\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s)) \{ \bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z}(\pi_{\bar{z}}(s)) \} \\
&\qquad\qquad + D_\gamma(\eta(\pi_{\bar{z}}(s), \pi_{\bar{z}}(s))) \{ \bar{\Upsilon} \frac{\partial \varphi_\alpha}{\partial z_\gamma}(\pi_{\bar{z}}(s)) + \bar{\upsilon}_\alpha \frac{\partial g}{\partial z_\gamma}(\pi_{\bar{z}}(s)) \} \Big] ds^\alpha,
\end{aligned}$$

which by the definition of Lagrange function, yields

$$L_\alpha^\eta(z, \bar{\Upsilon}, \bar{\upsilon}_\alpha) = L_\alpha^\eta(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha), \text{ for } z(s) \in \Omega(\bar{z}(s)). \tag{25}$$

Thus, from (24) and (25), we conclude that $(\bar{z}, \bar{\Upsilon}, \bar{\upsilon}_\alpha(s))$ is an η-saddle point for the η-Lagrange function in (MMVP$_\eta(\bar{z})$). Hence, the proof is complete. □

6. CONCLUSIONS

In the present article, we have used the η-approximation approach to obtain characterization of efficiency of the main problem. Using this approach, efficient solutions of the main problem are characterized by minimizers of η-approximated problem. Then, we can characterize solvability of the given problem, in general, by the help of a less complex η-approximated multitime multiobjective variational problem. In some cases, an η-approximated is linear and/or convex (and such a case was illustrated in the paper). This is an important property of the analyzed method since efficient solutions of nonconvex multitime multiobjective variational problems with complex objective functions can be characterized by the help of minimizers of linear and/or convex multitime variational problems. Further, we have presented the characterization of an η-saddle-point of the Lagrange function defined for the approximated problem. Further, we shall extend this idea to interval program.

Acknowledgement: The authors would like to express their sincere gratitude to the J. C. Bose Fellowship (SB/SJ/JCB-033/201.6) of Department of Science and Technology, GoI, New Delhi for providing financial support.

REFERENCES

[1] Ahmed, H. F., "A numerical technique for solving multi-dimensional fractional optimal control problems", *Journal of Taibah University for Science*, 12 (5) (2018) 1–12.

[2] Ahmad, I., Gualti, T. R., "Mixed type duality for multiobjective variational problems with generalized (F, ρ)-convexity", *Journal of Mathematical Analysis and Applications*, 306 (2) (2005) 669–683.

[3] Ahmad, I., Husain, Z., Al-Homidan, S., "Mixed type symmetric duality for multiobjective variational problems with cone constraints", *International Transactions in Operational Research*, 21 (2) (2014) 291–310.

[4] Ahmad, I., Sharma, S., "Sufficiency and duality for multiobjective variational control problems with generalized $(F, \alpha, \rho, \theta) - V$-convexity", *Nonlinear Analysis*, 72 (2010) 2564–2579.

[5] Ahmad, I., Singh, D., Dar, B. A., "Optimality conditions for invex interval valued nonlinear programming problems involving generalized H-derivative", *FILOMAT*, 30 (8) (2016) 2121–2138.

[6] Andersson, J., *A Survey of Multiobjective Optimization in Engineering Design*, Linköping University 581 83 Linköping, Sweden, 2000.

[7] Antczak, T., "A new approach to multiobjective programming with a modified objective function", *Journal of Global Optimization*, 27 (4) (2003) 485–495.

[8] Antczak, T., "G-pre-invex functions in mathematical programming", *Journal of Computational and Applied Mathematics*, 217 (1) (2008) 212–226.

[9] Antczak, T., "Saddle points criteria in nondifferentiable multiobjective programming with V-invex functions via an η-approximation method", *Computers & Mathematics with Applications*, 60 (9) (2010) 2689–2700.

[10] Antczak, T., "On efficiency and mixed duality for a new class of nonconvex multiobjective variational control problems", *Journal of Global Optimization*, 59 (4) (2014) 757–785.

[11] Antczak, T., "Optimality conditions and duality results for nonsmooth vector optimization problems with the multiple interval-valued objective function", *Acta Mathematica Scientia*, 37 (4) (2017) 1133–1150.

[12] Antczak, T., Jayswal, A., Jha, S., "The modified objective function method for univex multiobjective variational problems", *Bulletin of the Iranian Mathematical Society*, 45 (1) (2019) 267–282.

[13] Arana-Jiménez, M., Ruiz-Garzón, G., Rufián-Lizana, A., Osuna-Gómez, R., "A necessary and sufficient condition for duality in multiobjective variational problems", *European Journal of Operational Research*, 201 (3) (2010) 672–681.

[14] Arana-Jiménez, M., Ruiz-Garzón, G., Rufián-Lizana, A., Osuna-Gómez, R., "Weak efficiency in multiobjective variational problems under generalized convexity", *Journal of Global Optimization*, 52 (2012) 109–121.

[15] Ben-Israel, A., Mond, B., "What is invexity?", *Journal of the Australian Mathematical Society Series B*, 28 (1) (1986) 1–9.

[16] Craven, B. D., "Invex functions and constrained local minima", *Bulletin of the Iranian Mathematical Society*, 24 (3) (1981) 357–366.

[17] Craven, B. D., “On continuous programming with generalized convexity”, *Asia-Pacific Journal of Operational Research*, 10 (1993) 219–232.

[18] Edgeworth, F.Y., *Mathematical Psychics*, Kegan Paul, London, 1881.

[19] Hanson, M. A., “Bounds for functionally convex optimal control problems”, *Journal of Mathematical Analysis and Applications*, 8 (1964) 84–89.

[20] Hanson, M. A., “On sufficiency of the Kuhn-Tucker conditions”, *Journal of Mathematical Analysis and Applications*, 80 (2) (1981) 545–550.

[21] Jayswal, A., Antczak, T., Jha, S., “Modified objective function approach for multitime variational problems”, *Turkish Journal of Mathematics*, 42 (2018) 1111–1129.

[22] Khazafi, K., Rueda, N., Enflo, P., “Sufficiency and duality for multiobjective control problems under generalized (B, ρ)-type I functions”, *Journal of Global Optimization*, 46 (1) (2010) 111–132.

[23] Martins, F., Costa, C. A. V., “Multiobjective optimization with economic and environmental objective functions using Modified Simulated Annealing”, *Computer Aided Chemical Engineering*, 28 (2010) 919–924.

[24] Martin, D. H., “The essence of invexity”, *Journal of Optimization Theory and Applications*, 47 (1985) 65–76.

[25] Mititelu, S., Postolache, M., “Mond-Weir dualities with Lagrangians for multiobjective fractional and nonfractional variational problems”, *Journal of Advanced Mathematical Studies*, 3 (1) (2010) 41–58.

[26] Mititelu, S., Postolache, M., “Efficiency and duality for multitime vector fractional variational problems on manifolds”, *Balkan Journal of Geometry and Its Applications*, 16 (2011) 90–101.

[27] Mititelu, S., Stancu-Minasian, I. M., “Efficiency and duality for multiobjective fractional variational problems with (ρ, b)-quasiinvexity”, *Yugoslav Journal of Operations Research*, 19 (1) (2009) 85–99.

[28] Pareto, V., “Manual of political economy”, *Translated by Schwier, A.S.. Augustus M., Kelley Publishers, New York*, 1971.

[29] Pitea, A., Antczak, T., “Proper efficiency and duality for a new class of nonconvex multitime multiobjective variational problems”, *Journal of Inequalities and Applications*, 333 (2014) 1-20.

[30] Pitea, A., Postolache, M., “Duality theorems for a new class of multitime multiobjective variational problems”, *Journal of Global Optimization*, 53 (1) (2012) 47–58.

[31] Pitea, A., Postolache, M., “Minimization of vectors of curvilinear functionals on the second order jet bundle. Sufficient efficiency conditions”, *Optimization Letters*, 6 (8) (2012) 1657–1669.

[32] Pitea, A., Udriste, C., Mititelu, S., “PDI & PDE-constrained optimization problems with curvilinear functional quotients as objective vectors”, *Balkan Journal of Geometry and Its Applications*, 14 (2009) 75–88.

[33] Postolache, M., “Duality for multitime multiobjective ratio variational problems on first order jet bundle”, *Abstract and Applied Analysis*, 2 (2012) 1-18.

[34] Ruiz-Garzon, G., Osuna-Gomez, R., Rufian-Lizana, A., Hernandez-Jimenez, B., “Optimality in continuous-time multiobjective optimization and vector variational-like inequalities”, *TOP*, 23 (2015) 198–219.

[35] Stancu, A. M., *Mathematical Programming with Type-I Functions*, Matrix Rom, Bucharest, 2013.

[36] Stancu-Minasian, I.M., Mititelu, S., “Multiobjective fractional variational problems with b-quasiinvexity”, *Proceedings of the Romanian Academy Series A - Mathematics Physics Technical Sciences Information Science* , 9 (1) (2008) 5–11.

[37] Udriste, C., “Multitime controllability, observability and bang-bang principle”, *Journal of Optimization Theory and Applications*, 139 (1) (2008) 141–157.

[38] Udriste, C., Tevy, I., “Multitime dynamic programming for curvilinear integral actions”, *Journal of Optimization Theory and Applications*, 146 (1) (2010) 189–207.

[39] Yan, C., Feng, B., “Sufficiency and Duality for Nonsmooth Multiobjective Programming Problems Involving Generalized $(\Phi, \rho) - V$-Type I Functions”, *Journal of Mathematical Modelling and Algorithms in Operations Research*, 14 (2015) 159–172.