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SUFFICIENCY AND DUALITY OF SET-VALUED FRACTIONAL PROGRAMMING PROBLEMS VIA SECOND-ORDER CONTINGENT EPIDERIVATIVE

Koushik DAS
Department of Mathematics, Taki Government College, Taki - 743429, West Bengal, India
koushikdas.maths@gmail.com

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Abstract: In this paper, we establish second-order sufficient KKT optimality conditions of a set-valued fractional programming problem under second-order generalized cone convexity assumptions. We also prove duality results between the primal problem and second-order dual problems of parametric, Mond-Weir, Wolfe, and mixed types via the notion of second-order contingent epiderivative.

Keywords: Convex Cone, Set-Calued Map, Contingent Epiderivative, Duality.

MSC: 26B25, 49N15.

1. INTRODUCTION

The theory of set-valued optimization problems, being an expanding branch of optimization theory, has attracted the attention of many researchers in the past few years. A set-valued optimization problem is one that involves set-valued maps as objective functions and constraints. Many optimization problems in mathematical economics, differential inclusions, image processing, optimal control, and viability theory are set-valued optimization problems, but there exist various types of notions of differentiability of set-valued maps. In 1997, Jahn and Rauh [22] introduced the notion of contingent epiderivative of set-valued maps. It has a vital role for establishment of optimality conditions of set-valued optimization problems. Borwein [6] introduced the notion of cone convexity of set-valued maps, which plays an important role in the theory of set-valued optimization problems. There are various classes of set-valued optimization problems. An important class is that of set-valued fractional programming problems. In 1997, Bhatia and Mehra [4] introduced the notion of cone preinvexity of set-valued maps and established the Lagrangian duality results for the set-valued fractional programming problems. They also [5] developed the duality results for Geoffrion efficient solutions of the set-valued fractional programming problems via cone convexity. Gadhi and Jawhar [20] established the necessary optimality conditions of the set-valued fractional programming problems without any convex separation approach in 2013. Many authors like Kaul and Lyall [23], Bhatia and Garg [3], Suneja and Gupta [27], Suneja and Lalitha [28], and Lee and Ho [24] established the optimality conditions and developed the duality theorems for vector-valued fractional programming problems under generalized convexity assumptions. Li et al. [25, 26] established the necessary and sufficient optimality conditions by using higher-order contingent derivative. They also formulated the higher-order Mond-Weir dual for set-valued optimization problems and studied the duality theorems under convexity assumptions. In 2015, Das and Nahak [11] established the sufficient Karush-Kuhn-Tucker (KKT) optimality conditions for set-valued fractional programming problems under contingent epiderivative and ρ-cone convexity assumptions. They also studied the duality results of parametric, Mond-Weir, Wolfe, and mixed types for the problem. Later, they [9] established the sufficient KKT optimality conditions of set-valued optimization problems under generalized convexity assumptions and higher-order contingent derivative assumptions. They also proved weak, strong, and converse duality theorems of Mond-Weir, Wolfe and mixed types.

In this paper, we establish the second-order sufficient KKT optimality conditions of a set-valued fractional programming problem by using second-order ρ-cone convexity assumptions. The duality results between the primal problem and second-order dual problems of parametric, Mond-Weir, Wolfe, and mixed types are also proved via the notion of second-order contingent epiderivative.

This paper is organized as follows. Section 2 deals with some definitions and preliminary concepts of set-valued maps. In Section 3, a set-valued fractional programming problem (FP) is considered and the second-order sufficient KKT conditions are established for the problem (FP). Various types of duality theorems are studied under second-order contingent epiderivative and second-order generalized cone convexity assumptions.

2. DEFINITIONS AND PRELIMINARIES

Let K be a nonempty subset of the m-dimensional Euclidean space $\mathbb{R}^m$. Then K is called a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Furthermore, K is called non-trivial if $K \neq \{\mathbf{0}_{\mathbb{R}^m}\}$, proper if $K \neq \mathbb{R}^m$, pointed if $K \cap (-K) = \{\mathbf{0}_{\mathbb{R}^m}\}$, solid if $\operatorname{int}(K) \neq \emptyset$, closed if $\overline{K} = K$, and convex if $\lambda K + (1-\lambda)K \subseteq K$, for all $\lambda \in [0, 1]$, where $\operatorname{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and $\mathbf{0}_{\mathbb{R}^m}$ is the zero element of $\mathbb{R}^m$.

Let us define the non-negative orthant $\mathbb{R}^m_+$ of $\mathbb{R}^m$ by

$$\mathbb{R}^m_+ = \{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, 2, ..., m\}.$$

Then $\mathbb{R}^m_+$ is a solid pointed closed convex cone and $\text{int}(\mathbb{R}^m_+) \cup \{\mathbf{0}_{\mathbb{R}^m}\}$ is a solid pointed convex cone in $\mathbb{R}^m$.

There are two types of cone-orderings in $\mathbb{R}^m$ with respect to the solid pointed convex cone $\mathbb{R}^m_+$ of $\mathbb{R}^m$. For any two elements $y_1, y_2 \in \mathbb{R}^m$, we have

$$y_1 \leq y_2 \text{ if } y_2 - y_1 \in \mathbb{R}^m_+$$

and

$$y_1 < y_2 \text{ if } y_2 - y_1 \in \text{int}(\mathbb{R}^m_+).$$

We say $y_2 \geq y_1$, if $y_1 \leq y_2$ and $y_2 > y_1$, if $y_1 < y_2$.

The following notions of minimality are mainly used with respect to the solid pointed convex cone $\mathbb{R}^m_+$ of $\mathbb{R}^m$.

Definition 2.1. *Let B be a nonempty subset of $\mathbb{R}^m$. Then minimal and weakly minimal points of B are defined as*

(i) *$y' \in B$ is a minimal point of B if there is no $y \in B \setminus \{y'\}$ such that $y \leq y'$.*

(ii) *$y' \in B$ is a weakly minimal point of B if there is no $y \in B$ such that $y < y'$.*

The sets of minimal points and weakly minimal points of B are denoted by $\min(B)$ and w-$\min(B)$, respectively and characterized as

$$\min(B) = \{y' \in B : (y' - \mathbb{R}^m_+) \cap B = \{y'\}\}$$

and

$$\text{w-}\min(B) = \{y' \in B : (y' - \text{int}(\mathbb{R}^m_+)) \cap B = \emptyset\}.$$

Similarly, the sets of maximal points and weakly maximal points of B can be defined and characterized.

We recall the notions of contingent cone and second-order contingent set in a real normed space.

Definition 2.2. *[1, 2] Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, and $y' \in \overline{B}$. The contingent cone to B at y' is denoted by $T(B, y')$ and defined as follows:*

An element $y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$ such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$ such that

$$t_n(y'_n - y') \to y.$$

The contingent cone $T(B, y')$ is actually a local approximation of the set $B - y'$. If $y' \in \text{int}(B)$, then $T(B, y') = Y$.

Proposition 2.1. *[2] The contingent cone $T(B, y')$ is a closed cone, but not necessarily convex and* $T(B, y') \subseteq \overline{\bigcup_{h>0} \frac{B - y'}{h}}$.

Definition 2.3. *[1, 2, 7] Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, $y' \in \overline{B}$, and $u \in Y$. The second-order contingent set to B at y' in the direction u is denoted by $T^2(B, y', u)$ and defined as*

An element $y \in T^2(B, y', u)$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$ such that

$$y' + \lambda_n u + \frac{1}{2}{\lambda_n}^2 y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$, $\{t'_n\}$ in $\mathbb{R}$, with $t_n, t'_n > 0$, $t_n \to \infty$, $t'_n \to \infty$, $\frac{t'_n}{t_n} \to 2$, and $\{y'_n\}$ in B, with $y'_n \to y'$ such that

$$t_n(y'_n - y') \to u \text{ and } t'_n(t_n(y'_n - y') - u) \to y.$$

Proposition 2.2. *[29] The second-order contingent set $T^2(B, y', u)$ is a closed set, but not necessarily a cone. Even, $T^2(B, y', u)$ may not be convex, though B is convex. Also, $T^2(B, y', \theta_Y) = T(T(B, y'), \theta_Y) = T(B, y')$.*

Let X, Y be real normed spaces, 2^Y be the set of all subsets of Y, and K be a solid pointed convex cone in Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y, i.e., $F(x) \subseteq Y$, for all $x \in X$. The effective domain, image, graph, and epigraph of F are defined respectively by

$$\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$F(A) = \bigcup_{x \in A} F(x), \text{ for any } A(\neq \emptyset) \subseteq X,$$

$$\text{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\},$$

and

$$\text{epi}(F) = \{(x, y) \in X \times Y : y \in F(x) + K\}.$$

Let A be a nonempty subset of X, $x' \in A$, $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$ and $y' \in F(x')$. Jahn and Rauh [22] introduced the notion of contingent epiderivative of set-valued maps, which plays an important role in set-valued optimization problems.

Definition 2.4. *[22] A single-valued map $D_\uparrow F(x', y') : X \to Y$ whose epigraph coincides with the contingent cone to the epigraph of F at (x', y'), i.e.,*

$$\text{epi}(D_\uparrow F(x', y')) = T(\text{epi}(F), (x', y')),$$

is said to be the contingent epiderivative of F at (x', y').

When $f : X \to \mathbb{R}$ is a real-valued map, being continuous at $x_0 \in X$ and convex,

$$D_{\uparrow} f(x_0, f(x_0))(u) = f'(x_0)(u), \forall u \in X,$$

where $f'(x_0)(u)$ is the directional derivative of f at x_0 in the direction u.

Jahn et al. [21] introduced the notion of second-order contingent epiderivative of set-valued maps which also has a vital role in set-valued optimization problems.

Definition 2.5. *[21] A single-valued map $D^2_{\uparrow} F(x', y', u, v) : X \to Y$ whose epigraph coincides with the second-order contingent set to the epigraph of F at $(x', y') \in \mathrm{gr}(F)$ in a direction $(u, v) \in X \times Y$, i.e.,*

$$\mathrm{epi}(D^2_{\uparrow} F(x', y', u, v)) = T^2(\mathrm{epi}(F), (x', y'), (u, v)),$$

is said to be the second-order contingent epiderivative of F at (x', y') in the direction (u, v).

Proposition 2.3. *[2] Let $\emptyset \neq A \subseteq X$, $x' \in A$, $u \in X$, and $f : X \to Y$ be a single-valued map that is twice continuously differentiable around x'. The second-order contingent epiderivative $D^2_{\uparrow} f(x', f(x'), u, f'(x')u)$ of f at $(x', f(x'))$ in the direction $(u, f'(x')u)$ is given by*

$$D^2_{\uparrow} f(x', f(x'), u, f'(x')u)(x) = f'(x')x + \frac{1}{2} f''(x')(u, u), x \in T^2(A, x', u).$$

Borwein [6] introduced the notion of cone convexity of set-valued maps.

Definition 2.6. *[6] Let A be a nonempty convex subset of a real normed space X. A set-valued map $F : X \to 2^Y$, with $A \subseteq \mathrm{dom}(F)$, is called K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,*

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda) x_2) + K.$$

It is clear that if a set-valued map $F : X \to 2^Y$ is K-convex on A, then $\mathrm{epi}(F)$ is a convex subset of $X \times Y$.

The following lemma represents cone convex set-valued maps in terms of contingent epiderivative.

Lemma 2.1. *[22] If $F : X \to 2^Y$ is K-convex on a nonempty convex subset A of a real normed space X, then for all $x, x' \in A$ and $y' \in F(x')$,*

$$F(x) - y' \subseteq D_{\uparrow} F(x', y')(x - x') + K.$$

Definition 2.7. *[29] Let A be a nonempty subset of a real normed space X and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \mathrm{dom}(F)$. Let $x', u \in A$, $y' \in F(x')$, and $v \in F(u) + K$. Assume that F is second-order contingent epiderivable at (x', y') in the direction $(u - x', v - y')$. Then F is said to be second-order K-convex at (x', y') in the direction $(u - x', v - y')$ on A if*

$$F(x) - y' \subseteq D^2_{\uparrow} F(x', y', u - x', v - y')(x - x') + K, \forall x \in A.$$

Let X be a real normed space and A a nonempty subset of X. Let $F : X \to 2^{\mathbb{R}^m}$, $G : X \to 2^{\mathbb{R}^m}$, and $H : X \to 2^{\mathbb{R}^k}$ be set-valued maps, with

$$A \subseteq \operatorname{dom}(F) \cap \operatorname{dom}(G) \cap \operatorname{dom}(H).$$

Let $F = (F_1, F_2, ..., F_m)$, $G = (G_1, G_2, ..., G_m)$, and $H = (H_1, H_2, ..., H_k)$, where the set-valued maps $F_i : X \to 2^{\mathbb{R}}$, $G_i : X \to 2^{\mathbb{R}}$; $i = 1, 2, ..., m$, and $H_j : X \to 2^{\mathbb{R}}$; $j = 1, 2, ..., k$, are defined by

$$\operatorname{dom}(F_i) = \operatorname{dom}(F), \operatorname{dom}(G_i) = \operatorname{dom}(G), \text{ and } \operatorname{dom}(H_j) = \operatorname{dom}(H),$$

$$x \in A, y = (y_1, y_2, ..., y_m) \in F(x) \Longrightarrow y_i \in F_i(x), \forall i = 1, 2, ..., m,$$
$$z = (z_1, z_2, ..., z_m) \in G(x) \Longrightarrow z_i \in G_i(x), \forall i = 1, 2, ..., m,$$

and

$$w = (w_1, w_2, ..., w_k) \in H(x) \Longrightarrow w_j \in H_j(x), \forall j = 1, 2, ..., k.$$

Assume that $F_i(x) \subseteq \mathbb{R}_+$ and $G_i(x) \subseteq \operatorname{int}(\mathbb{R}_+), \forall i = 1, 2, ..., m$ and $x \in A$. Let $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_m) \in \mathbb{R}^m_+$. Define elements $\frac{y}{z}, \lambda' z \in \mathbb{R}^m$ and a subset $\lambda' G(x)$ of $\mathbb{R}^m$ by

$$\frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, ..., \frac{y_m}{z_m}\right)$$
$$\lambda' z = (\lambda'_1 z_1, \lambda'_2 z_2, ..., \lambda'_m z_m),$$

and

$$\lambda' G(x) = \{\lambda' z : z \in G(x)\}.$$

For $x \in A$, define a subset $\frac{F(x)}{G(x)}$ of $\mathbb{R}^m$ by

$$\frac{F(x)}{G(x)} = \left\{ \frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, ..., \frac{y_m}{z_m}\right) : y = (y_1, y_2, ..., y_m) \in F(x), \right.$$
$$\left. z = (z_1, z_2, ..., z_m) \in G(x) \right\}.$$

We consider a set-valued fractional programming problem (FP).

$$\begin{aligned} &\underset{x \in A}{\text{minimize}} && \frac{F(x)}{G(x)} \\ &\text{subject to} && H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset. \end{aligned} \tag{FP}$$

The feasible set of the problem (FP) is given by

$$S = \{x \in A : H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset\}.$$

Definition 2.8. *A point* $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$*, with* $x' \in S$*,* $y' \in F(x')$*, and* $z' \in G(x')$*, is called a minimizer of the problem* (FP) *if for all* $(x, \frac{y}{z}) \in X \times \mathbb{R}^m$*, with* $x \in S$*,* $y \in F(x)$*, and* $z \in G(x)$*,*

$$\tfrac{y}{z} - \tfrac{y'}{z'} \notin (-\mathbb{R}^m_+) \setminus \{\boldsymbol{0}_{\mathbb{R}^m}\}.$$

Definition 2.9. *A point* $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$*, with* $x' \in S$*,* $y' \in F(x')$*, and* $z' \in G(x')$*, is called a weak minimizer of the problem* (FP) *if for all* $(x, \frac{y}{z}) \in X \times \mathbb{R}^m$*, with* $x \in S$*,* $y \in F(x)$*, and* $z \in G(x)$*,*

$$\frac{y}{z} - \frac{y'}{z'} \notin (-\mathrm{int}(\mathbb{R}^m_+)).$$

Let $\lambda' \in \mathbb{R}^m_+$. We consider a parametric problem $(FP_{\lambda'})$ associated with the set-valued fractional programming problem (FP).

$$\begin{aligned} &\underset{x\in A}{\text{minimize}} && F(x) - \lambda' G(x) \\ &\text{subject to} && H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset. \end{aligned} \qquad (FP_{\lambda'})$$

Definition 2.10. *A point* $(x', y' - \lambda' z') \in X \times \mathbb{R}^m$*, with* $x' \in S$*,* $y' \in F(x')$*, and* $z' \in G(x')$*, is called a minimizer of the problem* $(FP_{\lambda'})$ *if for all* $(x, y - \lambda' z) \in X \times \mathbb{R}^m$*, with* $x \in S$*,* $y \in F(x)$*, and* $z \in G(x)$*,*

$$(y - \lambda' z) - (y' - \lambda' z') \notin (-\mathbb{R}^m_+) \setminus \{\boldsymbol{0}_{\mathbb{R}^m}\}.$$

Definition 2.11. *A point* $(x', y' - \lambda' z') \in X \times \mathbb{R}^m$*, with* $x' \in S$*,* $y' \in F(x')$*, and* $z' \in G(x')$*, is called a weak minimizer of the problem* $(FP_{\lambda'})$ *if for all* $(x, y - \lambda' z) \in X \times \mathbb{R}^m$*, with* $x \in S$*,* $y \in F(x)$*, and* $z \in G(x)$*,*

$$(y - \lambda' z) - (y' - \lambda' z') \notin (-\mathrm{int}(\mathbb{R}^m_+)).$$

Gadhi and Jawhar [20] proved the relationship between the solutions of the problems (FP) and $(FP_{\lambda'})$.

Lemma 2.2. *[20] A point* $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$ *is a weak minimizer of the problem* (FP) *if and only if* $(x', \boldsymbol{0}_{\mathbb{R}^m})$ *is a weak minimizer of the problem* $(FP_{\lambda'})$*, where* $\lambda' = \frac{y'}{z'}$*.*

For the special case, when $f : X \to \mathbb{R}^m$, $g : X \to \mathbb{R}^m$, and $h : X \to \mathbb{R}^k$ are single-valued maps, we have a multiobjective fractional programming problem as

$$\begin{aligned} &\underset{x\in A}{\text{minimize}} && \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_m(x)}{g_m(x)}\right) \\ &\text{subject to} && h(x) \in (-\mathbb{R}^k_+), \end{aligned}$$

where $f = (f_1, f_2, ..., f_m)$ and $g = (g_1, g_2, ..., g_m)$, by considering $F(x) = \{f(x)\}$, $G(x) = \{g(x)\}$, and $H(x) = \{h(x)\}$ in the problem (FP).

3. MAIN RESULTS

Das and Nahak [8]-[19] introduced the notion of ρ-cone convex set-valued maps. They established the sufficient KKT conditions and studied the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions. For $\rho = 0$, we have the usual notion of cone convexity of set-valued maps introduced by Borwein [6].

Definition 3.1. *[8, 11] Let A be a nonempty convex subset of $\mathbb{R}^n$, $e \in \text{int}(\mathbb{R}^m_+)$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then F is said to be ρ-$\mathbb{R}^m_+$-convex with respect to e on A if there exists $\rho \in \mathbb{R}$ such that*

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + \rho\lambda(1-\lambda)\|x_1 - x_2\|^2 e + \mathbb{R}^m_+, \quad \forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0,1].$$

Das and Nahak [11] constructed an example of ρ-cone convex set-valued map, which is not cone convex. They also characterized ρ-cone convexity of set-valued maps in terms of contingent epiderivative.

Theorem 3.1. *[11] Let A be a nonempty convex subset of $\mathbb{R}^n$, $e \in \text{int}(\mathbb{R}^m_+)$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be ρ-$\mathbb{R}^m_+$-convex with respect to e on A. Let $x' \in A$ and $y' \in F(x')$. Then,*

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + \mathbb{R}^m_+, \forall x \in A.$$

Das and Nahak [9] introduced second-order ρ-cone convexity of set-valued maps via second-order contingent epiderivative.

Definition 3.2. *[9] Let A be a nonempty subset of $\mathbb{R}^n$, $e \in \text{int}(\mathbb{R}^m_+)$, and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Let $x', u \in A$, $y' \in F(x')$, and $v \in F(u) + \mathbb{R}^m_+$. Assume that F is second-order contingent epiderivable at (x', y') in the direction $(u - x', v - y')$. Then F is said to be second-order ρ-$\mathbb{R}^m_+$-convex with respect to e at (x', y') in the direction $(u - x', v - y')$ on A if there exists $\rho \in \mathbb{R}$ such that*

$$F(x) - y' \subseteq D^2_{\uparrow}F(x', y', u - x', v - y')(x - x') + \rho\|x - x'\|^2 e + \mathbb{R}^m_+, \forall x \in A.$$

Remark 3.1. *For $u = x'$ and $v = y'$, we have*

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + \mathbb{R}^m_+, \forall x \in A.$$

In this case, we have the first order ρ-$\mathbb{R}^m_+$-convexity via contingent epiderivative.

If $\rho > 0$, then F is said to be strongly second-order ρ-$\mathbb{R}^m_+$-convex, if $\rho = 0$, we have the usual notion of second-order $\mathbb{R}^m_+$-convexity, and if $\rho < 0$, then F is said to be weakly second-order ρ-$\mathbb{R}^m_+$-convex.

Obviously, strongly second-order ρ-$\mathbb{R}^m_+$-convexity $\Rightarrow$ second-order $\mathbb{R}^m_+$-convexity $\Rightarrow$ weakly second-order ρ-$\mathbb{R}^m_+$-convexity.

Das and Nahak [9] constructed a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$, which is second-order ρ-$\mathbb{R}^2_+$-convex for some ρ, but is not second-order $\mathbb{R}^m_+$-convex.

Remark 3.2. *For the case of single-valued map, Definition 3.2 coincides with the existing one. Let X, Y be real normed spaces, K be a solid pointed convex cone in Y, $e \in \mathrm{int}(K)$, $u \in X$, and $v \in Y$. Let $f : X \to Y$ be a second-order continuously differentiable function at $x' \in X$. By considering $F(x) = \{f(x)\}$, from Definition 3.2 and Proposition 2.3, we can conclude that f is called second-order ρ-K-convex with respect to e at $(x', f(x'))$ in the direction $(u - x', v - f(x'))$ if there exists $\rho \in \mathbb{R}$ such that*

$$f(x)-f(x') \in f'(x')(x-x')+\frac{1}{2}f''(x')(u-x', u-x')+\rho\|x-x'\|^2 e+K, \forall x \in X,$$

where $v - f(x') = f'(x')(u - x')$.
The followings are some special cases.
When $Y = \mathbb{R}^m$, $K = \mathbb{R}^m_+$, $f = (f_1, f_2, ..., f_m)$, and $e = (1, 1, ..., 1) = \mathbf{1}_{\mathbb{R}^m}$, we have

$$f_i(x) - f_i(x') \geq f_i'(x')(x - x') + \frac{1}{2}f_i''(x')(u - x', u - x') + \rho\|x - x'\|^2,$$
$$\forall x \in X \text{ and } i = 1, 2, ..., m.$$

When $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x) - f(x') \geq f'(x')(x - x') + \frac{1}{2}f''(x')(u - x', u - x') + \rho\|x - x'\|^2, \forall x \in X.$$

When $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x)-f(x') \geq (x-x')^T\nabla f(x')+\frac{1}{2}(u-x')^T H(x')(u-x')+\rho\|x-x')\|^2, \forall x \in X,$$

where $\nabla f(x')$ and $H(x')$ are the gradient and Hessain matrix of f at x', respectively.

3.1. Second-order optimality conditions

Let $\lambda' \in \mathbb{R}^m_+$ and $G : X \to 2^{\mathbb{R}^m}$ be a set-valued map. Define a set-valued map $(-\lambda' G) : X \to 2^{\mathbb{R}^m}$ by

$$(-\lambda' G)(x) = -\lambda' G(x), \forall x \in \mathrm{dom}(G).$$

We establish the second-order sufficient optimality conditions of the problem (FP) under second-order contingent epiderivative and second-order ρ-cone convexity assumptions.

Theorem 3.2. *(Second-order sufficient optimality conditions) Let A be a nonempty convex subset of a real normed space X, x' be an element of the feasible set S of the problem* (FP)*, $y' \in F(x')$, $z' \in G(x')$, $\lambda' \in \frac{F(x')}{G(x')}$, $w' \in H(x') \cap (-L)$,* *and* $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$. *Let* $u \in A$, $v \in F(u) + \mathbb{R}^m_+$, $r \in (-\lambda' G)(u) + \mathbb{R}^m_+$, *and* $s \in H(u) + \mathbb{R}^k_+$. *Assume that* F *is second-order* ρ_1-$\mathbb{R}^m_+$*-convex at* (x', y') *in the direction* $(u - x', v - y')$, $-\lambda' G$ *is second-order* ρ_2-$\mathbb{R}^m_+$*-convex at* $(x', -\lambda' z')$ *in the direction* $(u - x', r + \lambda' z')$, *respectively, with respect to* $\mathbf{1}_{\mathbb{R}^m}$ *and* H *is second-order* ρ_3-$\mathbb{R}^k_+$*-convex at* (x', w') *in the direction* $(u - x', s - w')$, *with respect to* $\mathbf{1}_{\mathbb{R}^k}$, *on* A. *Suppose that there exists* $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, *with* $y^* \neq \mathbf{0}_{\mathbb{R}^m}$, *and*

$$(\rho_1 + \rho_2)\langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle + \rho_3\langle z^*, \mathbf{1}_{\mathbb{R}^k}\rangle \geq 0, \tag{3.1}$$

such that

$$\begin{aligned} &\Big\langle y^*, \Big(D^2_{\uparrow}F(x', y', u - x', v - y') \\ &+ D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u - x', r + \lambda' z')\Big)(x - x')\Big\rangle \\ &+ \langle z^*, D^2_{\uparrow}H(x', w', u - x', s - w')(x - x')\rangle \geq 0, \forall x \in A, \end{aligned} \tag{3.2}$$

$$y' - \lambda' z' = \mathbf{0}_{\mathbb{R}^m}, \tag{3.3}$$

and

$$\langle z^*, w'\rangle = 0. \tag{3.4}$$

Then $(x', \frac{y'}{z'})$ *is a weak minimizer of the problem* (FP).

Proof. We prove the theorem by the method of contradiction. Let $(x', \frac{y'}{z'})$ not be a weak minimzer of the problem (FP). Then there exist $x \in S$, $y \in F(x)$, and $z \in G(x)$ such that

$$\frac{y}{z} < \frac{y'}{z'}.$$

As $y' - \lambda' z' = \mathbf{0}_{\mathbb{R}^m}$, we have

$$\frac{y}{z} < \lambda'.$$

So,

$$y - \lambda' z < \mathbf{0}_{\mathbb{R}^m}.$$

Hence,

$$\langle y^*, y - \lambda' z\rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, as $y' - \lambda' z' = \mathbf{0}_{\mathbb{R}^m}$, we have

$$\langle y^*, y' - \lambda' z'\rangle = 0.$$

Since $x \in S$, there exists an element $w \in H(x) \cap (-\mathbb{R}^k_+)$. Therefore,

$$\langle z^*, w\rangle \leq 0.$$

So,

$$\langle z^*, w - w' \rangle \leq 0, \text{ as } \langle z^*, w' \rangle = 0.$$

Hence,

$$\langle y^*, y - \lambda' z - (y' - \lambda' z') \rangle + \langle z^*, w - w' \rangle < 0. \tag{3.5}$$

As F is second-order ρ_1-$\mathbb{R}^m_+$-convex at (x', y') in the direction $(u - x', v - y')$, $-\lambda' G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -\lambda' z')$ in the direction $(u - x', r + \lambda' z')$, respectively, with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$F(x) - y' \subseteq D^2_{\uparrow} F(x', y', u - x', v - y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

$$(-\lambda' G)(x) + \lambda' z' \subseteq D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u - x', r + \lambda' z')(x - x') + \rho_2 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

and

$$H(x) - w' \subseteq D^2_{\uparrow} H(x', w', u - x', s - w')(x - x') + \rho_3 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$y - y' \in D^2_{\uparrow} F(x', y', u - x', v - y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

$$-\lambda' z + \lambda' z' \in D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u - x', r + \lambda' z')(x - x') + \rho_2 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

and

$$w - w' \in D^2_{\uparrow} H(x', w', u - x', s - w')(x - x') + \rho_3 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from (3.1) and (3.2), we have

$$\langle y^*, y - \lambda' z - (y' - \lambda' z') \rangle + \langle z^*, w - w' \rangle \geq 0,$$

which contradicts (3.5).
Consequently, (x', y') is a weak minimizer of the problem (FP). $\square$

Let $x \in A$, $x' \in S$, $y' = (y'_1, y'_2, ..., y'_m) \in F(x')$, and $z' = (z'_1, z'_2, ..., z'_m) \in G(x')$. Define subsets $z'F(x)$ and $-y'G(x)$ of $\mathbb{R}^m$ by

$$z'F(x) = \{z'y = (z'_1 y_1, z'_2 y_2, ..., z'_m y_m) : y = (y_1, y_2, ..., y_m) \in F(x)\}$$

and

$$-y'G(x) = \{-y'z = (-y'_1 z_1, -y'_2 z_2, ..., -y'_m z_m) : z = (z_1, z_2, ..., z_m) \in G(x)\}.$$

Define set-valued maps $z'F, (-y'G) : X \to 2^{\mathbb{R}^m}$ by

$$(z'F)(x) = z'F(x), \forall x \in \mathrm{dom}(F)$$

and

$$(-y'G)(x) = -y'G(x), \forall x \in \mathrm{dom}(G).$$

We can also prove the following second-order sufficient optimality conditions through the same approach of Theorem 3.2.

Theorem 3.3. *(Second-order sufficient optimality conditions) Let A be a nonempty convex subset of a real normed space X, x' be an element of the feasible set S of the problem* (FP)*, $y' \in F(x')$, $z' \in G(x')$, $w' \in H(x') \cap (-L)$, and $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$. Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in the direction $(u - x', v - z'y')$, $-y'G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -y'z')$ in the direction $(u - x', r + y'z')$, respectively, with respect to $\boldsymbol{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\boldsymbol{1}_{\mathbb{R}^k}$, on A. Suppose that there exists $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^* \neq \boldsymbol{0}_{\mathbb{R}^m}$, and* (3.1) *and* (3.4) *are satisfied, with*

$$
\begin{aligned}
&\Big\langle y^*, \Big(D^2_\uparrow(z'F)(x', z'y', u - x', v - z'y') \\
&+ D^2_\uparrow(-y'G)(x', -y'z', u - x', r + y'z')\Big)(x - x')\Big\rangle \\
&+ \langle z^*, D^2_\uparrow H(x', w, u - x', s - w')(x - x')\rangle \geq 0, \forall x \in A,
\end{aligned}
\tag{3.6}
$$

Then $(x', \frac{y'}{z'})$ is a weak minimizer of the problem (FP).

We formulate second-order parametric (PD), Mond-Weir (MWD), Wolfe (WD), and mixed types (MD) duals of the problem (FP) and prove the corresponding duality results.

3.2. Second-order parametric type dual

We consider a second-order parametric type dual (PD) of the problem (FP), where F, $-\lambda'G$, and H are second-order contingent epiderivable set-valued maps for all $\lambda' \in \mathbb{R}^m_+$. Let $u \in A, v \in F(u) + \mathbb{R}^m_+,$ and $r \in (-\lambda'G)(u) + \mathbb{R}^m_+$.

$$
\begin{aligned}
&\text{maximize} \quad \lambda' \qquad\qquad (PD)\\
&\text{subject to}\\
&\Big\langle y^*, \Big(D^2_\uparrow F(x', y', u - x', v - y') \\
&+ D^2_\uparrow(-\lambda'G)(x', -\lambda'z', u - x', r + \lambda'z')\Big)(x - x')\Big\rangle \\
&+ \langle z^*, D^2_\uparrow H(x', w', u - x', s - w')(x - x')\rangle \geq 0, \forall x \in A,\\
&s \in H(u) + \mathbb{R}^k_+,\\
&y' - \lambda'z' \geq 0,\\
&x' \in A, y' \in F(x'), z' \in G(x'), \lambda' \in \frac{F(x')}{G(x')}, w' \in H(x'),\\
&y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+, \langle z^*, w'\rangle \geq 0, \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1.
\end{aligned}
$$

Definition 3.3. *A feasible point $(x', y', z', \lambda', w', y^*, z^*)$ of the problem (PD) is called a weak maximizer of (PD) if for all feasible points $(x, y, z, \lambda, w, y_1^*, z_1^*)$ of (PD),*

$$\lambda - \lambda' \notin \operatorname{int}(\mathbb{R}^m_+).$$

Theorem 3.4. *(Second-order weak duality) Let A be a nonempty convex subset of a real normed space X and $\overline{x}$ be an element of the feasible set S of the problem* (FP). *Let $(x', y', z', \lambda', w', y^*, z^*)$ be a feasible point of the problem (PD). Let $u \in A$, $v \in F(u) + \mathbb{R}^m_+$, $r \in (-\lambda' G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that F is second-order ρ_1-$\mathbb{R}^m_+$-convex at (x', y') in the direction $(u - x', v - y')$, $-\lambda' G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -\lambda' z')$ in the direction $(u - x', r + \lambda' z')$, respectively, with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A such that*

$$(\rho_1 + \rho_2) + \rho_3 \langle z^*, \mathbf{1}_{\mathbb{R}^k} \rangle \geq 0. \tag{3.7}$$

Then,

$$\frac{F(\overline{x})}{G(\overline{x})} - \lambda' \subseteq \mathbb{R}^m \setminus (-\mathrm{int}(\mathbb{R}^m_+)).$$

Proof. We prove the theorem by the method of contradiction. Suppose that for some $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$,

$$\frac{\overline{y}}{\overline{z}} - \lambda' \in (-\mathrm{int}(\mathbb{R}^m_+)).$$

Therefore,

$$\frac{\overline{y}}{\overline{z}} < \lambda'.$$

So,

$$\overline{y} - \lambda' \overline{z} < \mathbf{0}_{\mathbb{R}^m}.$$

Therefore,

$$\langle y^*, \overline{y} - \lambda' \overline{z} \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, from the constraints of (PD), we have

$$y' - \lambda' z' \geq 0.$$

Therefore,

$$\langle y^*, y' - \lambda' z' \rangle \geq 0.$$

Since $\overline{x} \in S$, we have

$$H(\overline{x}) \cap (-\mathbb{R}^k_+) \neq \emptyset.$$

Choose $\overline{w} \in H(\overline{x}) \cap (-\mathbb{R}^k_+)$. So, we have

$$\langle z^*, \overline{w} \rangle \leq 0.$$

From the constraints of (PD), we have

$$\langle z^*, w' \rangle \geq 0.$$

So,

$$\langle z^*, \overline{w} - w' \rangle = \langle z^*, \overline{w} \rangle - \langle z^*, w' \rangle \leq 0.$$

Hence,

$$\langle y^*, \overline{y} - \lambda' \overline{z} - (y' - \lambda' z') \rangle + \langle z^*, \overline{w} - w' \rangle < 0. \tag{3.8}$$

As F is second-order ρ_1-$\mathbb{R}^m_+$-convex at (x', y') in the direction $(u - x', v - y')$, $-\lambda' G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -\lambda' z')$ in the direction $(u - x', r + \lambda' z')$, respectively, with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$F(\overline{x}) - y' \subseteq D^2_{\uparrow} F(x', y', u - x', v - y')(\overline{x} - x') + \rho_1 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

$$(-\lambda' G)(\overline{x}) + \lambda' z' \subseteq D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u - x', r + \lambda' z')(\overline{x} - x') + \rho_2 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

and

$$H(\overline{x}) - w' \subseteq D^2_{\uparrow} H(x', w', u - x', s - w')(\overline{x} - x') + \rho_3 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$\overline{y} - y' \in D^2_{\uparrow} F(x', y', u - x', v - y')(\overline{x} - x') + \rho_1 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

$$-\lambda' \overline{z} + \lambda' z' \in D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u - x', r + \lambda' z')(\overline{x} - x') + \rho_2 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

and

$$\overline{w} - w' \in D^2_{\uparrow} H(x', w', u - x', s - w')(\overline{x} - x') + \rho_3 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from the constraints of (PD) and (3.7), we have

$$\langle y^*, \overline{y} - \lambda' \overline{z} - (y' - \lambda' z') \rangle + \langle z^*, \overline{w} - w' \rangle \geq 0,$$

which contradicts (3.8). Therefore,

$$\frac{\overline{y}}{\overline{z}} - \lambda' \notin (-\mathrm{int}(\mathbb{R}^m_+)).$$

Since $\overline{y} \in F(\overline{x})$ is arbitrary, we have

$$\frac{F(\overline{x})}{G(\overline{x})} - \lambda' \subseteq \mathbb{R}^m \setminus (-\mathrm{int}(\mathbb{R}^m_+)),$$

which completes the proof of the theorem. $\square$

Theorem 3.5. *(Second-order strong duality) Let* $(x', \frac{y'}{z'})$ *be a weak minimizer of the problem* (FP) *and* $w' \in H(x') \cap (-\mathbb{R}^k_+)$. *Assume that for some* $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, *with* $\langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1$ *and* $\lambda' \in \frac{F(x')}{G(x')}$, *(3.2), (3.3), and (3.4) are satisfied at the point* $(x', y', z', \lambda', w', y^*, z^*)$. *Then* $(x', y', z', \lambda', w', y^*, z^*)$ *is a feasible solution of the problem* (PD). *Furthermore, if the second-order weak duality Theorem 3.4 holds between the problems* (FP) *and* (PD), *then* $(x', y', z', \lambda', w', y^*, z^*)$ *is a weak maximizer of* (PD).

Proof. As the (3.2), (3.3), and (3.4) are satisfied at $(x', y', z', \lambda', w', y^*, z^*)$, we have

$$\Big\langle y^*, \Big(D^2_{\uparrow}F(x', y', u - x', v - y') + D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u - x', r + \lambda' z')\Big)(x - x')\Big\rangle$$
$$+ \langle z^*, D^2_{\uparrow}H(x', w', u - x', s - w')(x - x')\rangle \geq 0, \forall x \in A,$$

$$y' - \lambda' z' = \mathbf{0}_{\mathbb{R}^m},$$

and

$$\langle z^*, w'\rangle = 0.$$

Hence $(x', y', z', \lambda', w', y^*, z^*)$ is a feasible solution of (PD).
Suppose that the second-order weak duality Theorem 3.4 holds between (FP) and (PD) and $(x', y', z', \lambda', w', y^*, z^*)$ is not a weak maximizer of (PD).
Then there exists a feasible point $(x, y, z, \lambda, w, y_1^*, z_1^*)$ of (PD) such that

$$\lambda - \lambda' \in \operatorname{int}(\mathbb{R}^m_+).$$

As $y' - \lambda' z' = \mathbf{0}_{\mathbb{R}^m}$, we have

$$\lambda - \frac{y'}{z'} \in \operatorname{int}(\mathbb{R}^m_+).$$

which contradicts the second-order weak duality Theorem 3.4 between (FP) and (PD).
Consequently, $(x', y', z', \lambda', w', y^*, z^*)$ is a weak maximizer of (PD). □

Theorem 3.6. *(Second-order converse duality) Let* A *be a nonempty convex subset of a real normed space* X *and* $(x', y', z', \lambda', w', y^*, z^*)$ *be a feasible point of the problem* (PD), *where* $\lambda' = \frac{y'}{z'}$. *Let* $(x', y', z', \lambda', w', y^*, z^*)$ *be a feasible point of the problem* (PD). *Let* $u \in A$, $v \in F(u) + \mathbb{R}^m_+$, $r \in (-\lambda' G)(u) + \mathbb{R}^m_+$, *and* $s \in H(u) + \mathbb{R}^k_+$. *Assume that* F *is second-order* ρ_1-$\mathbb{R}^m_+$*-convex at* (x', y') *in the direction* $(u - x', v - y')$, $-\lambda' G$ *is second-order* ρ_2-$\mathbb{R}^m_+$*-convex at* $(x', -\lambda' z')$ *in the direction* $(u - x', r + \lambda' z')$, *respectively, with respect to* $\mathbf{1}_{\mathbb{R}^m}$ *and* H *is second-order* ρ_3-$\mathbb{R}^k_+$*-convex at* (x', w') *in the direction* $(u - x', s - w')$, *with respect to* $\mathbf{1}_{\mathbb{R}^k}$, *on* A, *satisfying* (3.7). *If* x' *is an element of the feasible set* S *of the problem* (FP), *then* $(x', \frac{y'}{z'})$ *is a weak minimizer of the problem* (FP).

Proof. We prove the theorem by the method of contradiction.
Suppose $(x', \frac{y'}{z'})$ is not a weak minimzer of the problem (FP).
Therefore there exist $x \in S$, $y \in F(x)$, and $z \in G(x)$ such that

$$\frac{y}{z} < \frac{y'}{z'}.$$

As $\lambda' = \frac{y'}{z'}$, we have

$$\frac{y}{z} < \lambda'.$$

So,

$$y - \lambda' z < \mathbf{0}_{\mathbb{R}^m}.$$

Hence,

$$\langle y^*, y - \lambda' z \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, from the constraints of (PD),

$$y' - \lambda' z' \geq 0.$$

Therefore,

$$\langle y^*, y' - \lambda' z' \rangle \geq 0.$$

Since $x \in S$, there exists an element

$$w \in H(x) \cap (-\mathbb{R}^k_+).$$

Therefore,

$$\langle z^*, w \rangle \leq 0.$$

We have

$$\langle z^*, w - w' \rangle \leq 0, \text{ as } \langle z^*, w' \rangle \geq 0.$$

Hence,

$$\langle y^*, y - \lambda' z - (y' - \lambda' z') \rangle + \langle z^*, w - w' \rangle < 0. \tag{3.9}$$

As F is second-order ρ_1-$\mathbb{R}^m_+$-convex at (x', y') in the direction $(u - x', v - y')$, $-\lambda' G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -\lambda' z')$ in the direction $(u - x', r + \lambda' z')$, respectively, with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$F(x) - y' \subseteq D^2_{\uparrow} F(x', y', u - x', v - y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

$(-\lambda' G)(x)+\lambda' z' \subseteq D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u-x', r+\lambda' z')(x-x')+\rho_2\|x-x'\|^2\mathbf{1}_{\mathbb{R}^m}+\mathbb{R}^m_+,$

and

$$H(x) - w' \subseteq D^2_{\uparrow}H(x', w', u - x', s - w')(x - x') + \rho_3\|x - x'\|^2\mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$y - y' \in D^2_{\uparrow}F(x', y', u - x', v - y')(x - x') + \rho_1\|x - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$

$$-\lambda' z+\lambda' z' \in D^2_{\uparrow}(-\lambda' G)(x', -\lambda' z', u-x', r+\lambda' z')(x-x')+\rho_2\|x-x'\|^2\mathbf{1}_{\mathbb{R}^m}+\mathbb{R}^m_+,$$

and

$$w - w' \in D^2_{\uparrow}H(x', w', u - x', s - w')(x - x') + \rho_3\|x - x'\|^2\mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from the constraints of (PD) and (3.7), we have

$$\langle y^*, y - \lambda' z - (y' - \lambda' z')\rangle + \langle z^*, w - w'\rangle \geq 0,$$

which contradicts (3.9).
Consequently, $(x', \frac{y'}{z'})$ is a weak minimizer of the problem (FP). $\square$

3.3. Second-order Mond-Weir type dual

We consider a second-order Mond-Weir type dual (MWD) of the problem (FP), where $z'F$, $-y'G$, and H are second-order contingent epiderivable set-valued maps for all $y', z' \in \mathbb{R}^m_+$. Let $u \in A, v \in (z'F)(u) + \mathbb{R}^m_+,$ and $r \in (-y'G)(u) + \mathbb{R}^m_+$.

$$\text{maximize} \quad \frac{y'}{z'} \tag{MWD}$$

subject to

$$\begin{aligned}
&\Big\langle y^*, \Big(D^2_{\uparrow}(z'F)(x', z'y', u - x', v - z'y') \\
&+ D^2_{\uparrow}(-y'G)(x', -y'z', u - x', r + y'z')\Big)(x - x')\Big\rangle \\
&+ \langle z^*, D^2_{\uparrow}H(x', w, u - x', s - w')(x - x')\rangle \geq 0, \forall x \in A, \\
&s \in H(u) + \mathbb{R}^k_+, \\
&\langle z^*, w'\rangle \geq 0, \\
&x' \in A, y' \in F(x'), z' \in G(x'), y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+, \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1.
\end{aligned}$$

Definition 3.4. *A feasible point $(x', y', z', w', y^*, z^*)$ of the problem (MWD) is called a weak maximizer of (MWD) if for all feasible points $(x, y, z, w, y_1^*, z_1^*)$ of (MWD),*

$$\frac{y}{z} - \frac{y'}{z'} \notin \text{int}(\mathbb{R}^m_+).$$

We prove the duality results of Mond-Weir type of the problem (FP). The proofs are very similar to Theorems 3.4 - 3.6, and hence omitted.

Theorem 3.7. *(Second-order weak duality) Let A be a nonempty convex subset of a real normed space X and $\overline{x}$ be an element of the feasible set S of* (FP). *Let $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MWD). Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in the direction $(u - x', v - z'y')$, $-y'G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -y'z')$ in the direction $(u - x', r + y'z')$, respectively, with respect to $\boldsymbol{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\boldsymbol{1}_{\mathbb{R}^k}$, on A, satisfying* (3.7). *Then,*

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y'}{z'} \subseteq \mathbb{R}^m \setminus (-\mathrm{int}(\mathbb{R}^m_+)).$$

Theorem 3.8. *(Second-order strong duality) Let $(x', \frac{y'}{z'})$ be a weak minimizer of the problem* (FP) *and $w' \in H(x') \cap (-\mathbb{R}^k_+)$. Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $\langle y^*, \boldsymbol{1}_{\mathbb{R}^m} \rangle = 1$, (3.4) and (3.6) are satisfied at $(x', y', z', w', y^*, z^*)$. Then $(x', y', z', w', y^*, z^*)$ is a feasible solution of the problem (MWD). Furthermore, if the second-order weak duality Theorem 3.7 holds between* (FP) *and (MWD), then the point $(x', y', z', w', y^*, z^*)$ is a weak maximizer of (MWD).*

Theorem 3.9. *(Second-order converse duality) Let A be a nonempty convex subset of a real normed space X and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MWD). Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in the direction $(u - x', v - z'y')$, $-y'G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -y'z')$ in the direction $(u - x', r + y'z')$, respectively, with respect to $\boldsymbol{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\boldsymbol{1}_{\mathbb{R}^k}$, on A, satisfying* (3.7). *If x' is an element of the feasible set S of the problem* (FP), *then $(x', \frac{y'}{z'})$ is a weak minimizer of the problem* (FP).

3.4. Second-order Wolfe type dual

We consider a second-order Wolfe type dual (WD) of the problem (FP), where $z'F$, $-y'G$, and H are second-order contingent epiderivable set-valued maps for all $y', z' \in \mathbb{R}^m_+$. Let $u \in A, v \in (z'F)(u) + \mathbb{R}^m_+$, and $r \in (-y'G)(u) + \mathbb{R}^m_+$.

$$\text{maximize} \quad \frac{y' + \langle z^*, w' \rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \qquad (WD)$$

subject to

$$\begin{aligned}
&\Big\langle y^*, \Big(D^2_{\uparrow}(z'F)(x', z'y', u - x', v - z'y') \\
&+ D^2_{\uparrow}(-y'G)(x', -y'z', u - x', r + y'z')\Big)(x - x')\Big\rangle \\
&+ \langle z^*, D^2_{\uparrow} H(x', w, u - x', s - w')(x - x') \rangle \geq 0, \forall x \in A, \\
&s \in H(u) + \mathbb{R}^k_+, \\
&x' \in A, y' \in F(x'), z' \in G(x'), y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+, \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m} \rangle = 1.
\end{aligned}$$

Definition 3.5. *A feasible point $(x', y', z', w', y^*, z^*)$ of the problem (WD) is called a weak maximizer of (WD) if for all feasible points $(x, y, z, w, y_1^*, z_1^*)$ of (WD),*

$$\frac{y + \langle z_1^*, w \rangle \boldsymbol{1}_{\mathbb{R}^m}}{z} - \frac{y' + \langle z^*, w' \rangle \boldsymbol{1}_{\mathbb{R}^m}}{z'} \notin \operatorname{int}(\mathbb{R}^m_+).$$

We prove the duality results of Wolfe type of the problem (FP). The proofs are very similar to Theorems 3.4 - 3.6, and hence omitted.

Theorem 3.10. *(Second-order weak duality) Let A be a nonempty convex subset of a real normed space X and $\overline{x}$ be an element of the feasible set S of* (FP)*. Let the point $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (WD). Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in the direction $(u - x', v - z'y')$, $-y'G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -y'z')$ in the direction $(u - x', r + y'z')$, respectively, with respect to $\boldsymbol{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\boldsymbol{1}_{\mathbb{R}^k}$, on A, satisfying* (3.7)*. Then,*

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y' + \langle z^*, w' \rangle \boldsymbol{1}_{\mathbb{R}^m}}{z'} \subseteq \mathbb{R}^m \setminus (-\operatorname{int}(\mathbb{R}^m_+)).$$

Theorem 3.11. *(Second-order strong duality) Let $(x', \frac{y'}{z'})$ be a weak minimizer of the problem* (FP) *and $w' \in H(x') \cap (-\mathbb{R}^k_+)$. Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $\langle y^*, \boldsymbol{1}_{\mathbb{R}^m} \rangle = 1$, (3.4) and (3.6) are satisfied at $(x', y', z', w', y^*, z^*)$. Then $(x', y', z', w', y^*, z^*)$ is a feasible solution of the problem (WD). Furthermore, if the second-order weak duality Theorem 3.10 holds between the problems* (FP) *and (WD), then $(x', y', z', w', y^*, z^*)$ is a weak maximizer of (WD).*

Theorem 3.12. *(Second-order converse duality) Let A be a nonempty convex subset of a real normed space X and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (WD), with $\langle z^*, w' \rangle \geq 0$. Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in the direction $(u - x', v - z'y')$, $-y'G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -y'z')$ in the direction $(u - x', r + y'z')$, respectively, with respect to $\boldsymbol{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\boldsymbol{1}_{\mathbb{R}^k}$, on A, satisfying* (3.7)*. If x' is an element of the feasible set S of the problem* (FP)*, then $(x', \frac{y'}{z'})$ is a weak minimizer of the problem* (FP)*.*

3.5. Second-order mixed type dual

We consider a second-order mixed type dual (MD) of the problem (FP), where $z'F$, $-y'G$, and H are second-order contingent epiderivable set-valued maps for all $y', z' \in \mathbb{R}^m_+$. Let $u \in A, v \in (z'F)(u) + \mathbb{R}^m_+,$ and $r \in (-y'G)(u) + \mathbb{R}^m_+$.

$$\text{maximize} \quad \frac{y' + \langle z^*, w' \rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \tag{MD}$$

subject to

$$\begin{aligned}
&\Big\langle y^*, \Big(D^2_{\uparrow}(z'F)(x', z'y', u - x', v - z'y') \\
&+ D^2_{\uparrow}(-y'G)(x', -y'z', u - x', r + y'z') \Big)(x - x') \Big\rangle \\
&+ \langle z^*, D^2_{\uparrow} H(x', w, u - x', s - w')(x - x') \rangle \geq 0, \forall x \in A, \\
&s \in H(u) + \mathbb{R}^k_+, \\
&\langle z^*, w' \rangle \geq 0, \\
&x' \in A, y' \in F(x'), z' \in G(x'), y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+, \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m} \rangle = 1.
\end{aligned}$$

Definition 3.6. *A feasible point $(x', y', z', w', y^*, z^*)$ of the problem (MD) is called a weak maximizer of (MD) if for all feasible points $(x, y, z, w, y_1^*, z_1^*)$ of (MD),*

$$\frac{y + \langle z_1^*, w \rangle \mathbf{1}_{\mathbb{R}^m}}{z} - \frac{y' + \langle z^*, w' \rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \notin \operatorname{int}(\mathbb{R}^m_+).$$

We prove the duality results of mixed type of the problem (FP). The proofs are very similar to Theorems 3.4 - 3.6, and hence omitted.

Theorem 3.13. *(Second-order weak duality) Let A be a nonempty convex subset of a real normed space X and $\overline{x}$ be an element of the feasible set S of* (FP). *Let the point $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MD). Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in the direction $(u - x', v - z'y')$, $-y'G$ is second-order ρ_2-$\mathbb{R}^m_+$-convex at $(x', -y'z')$ in the direction $(u - x', r + y'z')$, respectively, with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is second-order ρ_3-$\mathbb{R}^k_+$-convex at (x', w') in the direction $(u - x', s - w')$, with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, satisfying* (3.7). *Then,*

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y' + \langle z^*, w' \rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \subseteq \mathbb{R}^m \setminus (-\operatorname{int}(\mathbb{R}^m_+)).$$

Theorem 3.14. *(Second-order strong duality) Let $(x', \frac{y'}{z'})$ be a weak minimizer of the problem* (FP) *and $w' \in H(x') \cap (-\mathbb{R}^k_+)$. Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $\langle y^*, \mathbf{1}_{\mathbb{R}^m} \rangle = 1$, (3.4) and (3.6) are satisfied at $(x', y', z', w', y^*, z^*)$. Then $(x', y', z', w', y^*, z^*)$ is a feasible solution of the problem (MD). Furthermore, if the second-order weak duality Theorem 3.13 holds between the problems* (FP) *and (MD), then $(x', y', z', w', y^*, z^*)$ is a weak maximizer of (MD).*

Theorem 3.15. *(Second-order converse duality) Let A be a nonempty convex subset of a real normed space X and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MD). Let $u \in A$, $v \in (z'F)(u) + \mathbb{R}^m_+$, $r \in (-y'G)(u) + \mathbb{R}^m_+$, and $s \in H(u) + \mathbb{R}^k_+$. Assume that $(z'F)$ is second-order ρ_1-$\mathbb{R}^m_+$-convex at $(x', z'y')$ in* *the direction* $(u-x', v-z'y')$, $-y'G$ *is second-order* ρ_2-$\mathbb{R}^m_+$*-convex at* $(x', -y'z')$ *in the direction* $(u-x', r+y'z')$, *respectively, with respect to* $\mathbf{1}_{\mathbb{R}^m}$ *and* H *is second-order* ρ_3-$\mathbb{R}^k_+$*-convex at* (x', w') *in the direction* $(u-x', s-w')$, *with respect to* $\mathbf{1}_{\mathbb{R}^k}$, *on* A, *satisfying* (3.7). *If* x' *is an element of the feasible set* S *of the problem* (FP), *then* $(x', \frac{y'}{z'})$ *is a weak minimizer of the problem* (FP).

4. CONCLUSIONS

In this paper, we establish the sufficient KKT conditions of the set-valued fractional programming problem (FP) via contingent epiderivative and ρ-cone convexity assumptions. We also develop the weak, strong, and converse duality theorems between the primal problem (FP) and the second-order dual problems of parametric (PD), Mond-Weir (MWD), Wolfe (WD), and mixed (MD) types, respectively.

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