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STRONG COMPLEMENTARY APPROXIMATE KARUSH-KUHN-TUCKER CONDITIONS FOR MULTIOBJECTIVE OPTIMIZATION PROBLEMS

Jitendra Kumar MAURYA, Shashi Kant MISHRA
Department of Mathematics, Institute of Science, Banaras Hindu university, Varanasi-221005, India
jitendramourya150@gmail.com; bhu.skmishra@gmail.com

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Abstract: In this paper, we establish strong complementary approximate Karush-Kuhn-Tucker (SCAKKT) sequential optimality conditions for multiobjective optimization problems with equality and inequality constraints without any constraint qualifications and introduce a weak constraint qualification which assures the equivalence between SCAKKT and the strong Karush-Kuhn-Tucker (J Optim Theory Appl 80 (3): 483–500, 1994) conditions for multiobjective optimization problems.

Keywords: Multiobjective Programming, Approximate Karush-Kuhn-Tucker Conditions, Nonlinear Programming, Sequential Optimality Conditions.

MSC: 90C26, 90C30, 90C46.

1. INTRODUCTION

Andreani *et al*. [1], observed that some scalar optimization problems do not satisfy Karush-Kuhn-Tucker optimality conditions at an optimal point. Therefore, to resolve the problem, Andreani *et al*.[1] developed sequential optimality conditions, so-called approximate Karush-Kuhn-Tucker (AKKT) optimality conditions using mixed penalty method of Fiacco and McCormick [2, Section 4.3]. Although AKKT conditions are satisfied at that point without any constraint qualifications, which is an additional benefit of this method, but Andreani *et al*. [3] present some examples satisfying AKKT conditions but the founded point is not an optimal solution. As a remedy to this situations Andreani *et al*. [3] proposed complementary approximate Karush-Kuhn-Tucker (CAKKT) optimality conditions using external penalty method [2, Section 4.2]. Birgin and Martínez [4, Theorem 6.1] noticed that constraint qualifications play an important role in implementation of sequential optimality conditions for stopping criteria of several practical optimization algorithms. In addition to that, Andreani *et al*. [5] introduced several constraint qualifications for various sequential optimality conditions in the support of stopping criteria.

Recently, Giorgi *et al*. [6] extended the concept of AKKT conditions of scalar optimization problems to multiobjective optimization problems and obtained necessary and sufficient optimality conditions where multipliers of gradient of objective functions are in small range and AKKT conditions coincide on KKT (see, Miettinen [7, Theorem 3.1.5]) conditions under Mangasarian Fromovitz constraint qualification (see, [8]). On the other hand, Feng and Li [9] established approximate strong Karush-Kuhn-Tucker (ASKKT) conditions using the techniques of Wendell and Lee [10], in which multipliers of gradient of objective functions are in wide range and also extended cone continuity property (CCP) for scalar optimization problems (see, Andreani *et al*. [11]) to cone continuity regularity (CCR) condition for multiobjective optimization problems, which guarantee that the ASKKT coincide with strong Karush-Kuhn-Tucker (SKKT [15]) conditions.

Motivated by the works of Andreani *et al*. [5], Feng and Li [9], and Wendell and Lee [10], we establish a new sequential optimality conditions, so-called strong complementary approximate Karush-Kuhn-Tucker (SCAKKT) conditions for multiobjective optimization problems using Hybrid approach from Chankong and Haimes [12, Section 4.6.3], in which domain of multipliers cover that of Giorgi *et al*. [6] as well as Feng and Li [9]. Further, we propose SCAKKT-regularity condition which is a constraint qualification weaker than CCR condition.

The outline of this paper is as follows: In Section 2, we recall some preliminaries. In Section 3, we define strong complementary approximate Karush-Kuhn-Tucker (SCAKKT) conditions and establish necessary and sufficient optimality conditions for multiobjective optimization problems. In Section 4, SCAKKT-regularity condition and related consequences are discussed.

2. PRELIMINARIES

In this section, we recall some notations and definitions which will be used throughout the paper. Let $\mathbb{R}^n$ denotes n-dimensional Euclidean space and open (closed) ball $B(x^\circ,\delta)=\{x\in\mathbb{R}^n;\|x-x^\circ\|<\delta\}(\bar{B}(x^\circ,\delta)=\{x\in\mathbb{R}^n;\|x-x^\circ\|\leqq\delta\})$, centered at $x^\circ\in\mathbb{R}^n$, radius $\delta>0$, notation $\|\cdot\|$ denotes Euclidean norm in $\mathbb{R}^n$ except otherwise specified. For $x=(x_1,...,x_n)\in\mathbb{R}^n,\ x_i(i=1,...,n)$ are its components. $\mathbb{R}^n_+$ denote non-negative orthant of $\mathbb{R}^n$ and for $c\in\mathbb{R},\ c_+=\max\{0,c\},\ c_+^2=(c_+)^2$. Following inequalities for $y,\ z\in\mathbb{R}^n$,

$$y\leqq z\Longleftrightarrow y_i\leqq z_i,\ i=1,...,n,$$

$$y\leq z\Longleftrightarrow y\leqq z\ \text{ and } y\neq z,$$

$$y < z \Longleftrightarrow y_i < z_i, \ i = 1, ..., n.$$

Consider the multiobjective optimization problem:

$$\begin{array}{rl} \text{(MOP)} & \min \big(f_1(x), f_2(x), \ldots, f_p(x)\big), \\ \text{subject to} & x \in \Omega = \{x \in \mathbb{R}^n : g(x) \leqq 0, \ h(x) = 0\}, \end{array} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p, g : \mathbb{R}^n \to \mathbb{R}^m, \ h : \mathbb{R}^n \to \mathbb{R}^r$ be continuously differentiable function. A point $x^\circ \in \Omega$ is an efficient (weak efficient, respectively) solution for (MOP) if there is no $x \in \Omega$ such that $f(x) \leq f(x^\circ)$ ($f(x) < f(x^\circ)$, respectively). A point $x^\circ \in \Omega$ is said to be a local efficient (local weak efficient, respectively) solution for (MOP) if there exists an open ball $B(x^\circ, \delta)$ around the point x° such that x° is an efficient (weak efficient, respectively) solution on $\Omega \cap B(x^\circ, \delta)$. We will use the following index sets further

$$I = \{1, ..., p\}, J(x^\circ) = \{j : g_j(x^\circ) = 0\}, L = \{1, ..., r\}.$$

We recall some already existing approximate optimality conditions for multiobjective optimization problems, where differences can be observed easily.

Definition 1. *(AKKT Conditions [6]) We say that AKKT conditions are satisfied for (MOP) at a feasible point x° if and only if there exist sequences $(x^k) \subset \mathbb{R}^n$ and $(\lambda^k, \mu^k, \tau^k) \subset \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that*

$(C1)$ $x^k \to x^\circ$,

$(C2)$ $\displaystyle\sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k) \to 0,$

$(C3)$ $\displaystyle\sum_{i=1}^{p} \lambda_i^k = 1,$

$(C4)$ $g_j(x^\circ) < 0 \implies \mu_j = 0$ *for sufficiently large* $k, \ j = 1, ..., m.$

Definition 2. *(ASKKT Conditions[9]) We say that ASKKT conditions are satisfied for $(MOP)'$*

$$\begin{array}{rl} \textit{(MOP)}' & \min \big(f_1(x), f_2(x), \ldots, f_p(x)\big), \\ \textit{subject to} & x \in \Omega = \{x \in \mathbb{R}^n : g(x) \leqq 0\}, \end{array}$$

at a feasible point x° if and only if there exist sequences $(x^k) \subset \mathbb{R}^n$ and $(\lambda^k, \mu^k) \subset \mathbb{R}^p_+ \times \mathbb{R}^m_+$ such that

$(C1)$ $x^k \to x^\circ$,

$(C2)$ $\displaystyle\sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) \to 0,$

$(C3)$ $\lambda_i^k \geqq 1, i = 1, ..., p,$

$(C4)$ $g_j(x^\circ) < 0 \implies \mu_j = 0$ *for sufficiently large* $k, \ j = 1, ..., m.$

3. SCAKKT CONDITIONS FOR MULTIOBJECTIVE OPTIMIZATION PROBLEMS

In this section, we extend the concept of complementary approximate Karush-Kuhn-Tucker (CAKKT) necessary conditions for single objective optimization problems given by Andreani *et al*. [3] to multiobjective optimization problems (MOP).

Definition 3. *(SCAKKT Conditions) We say that SCAKKT conditions are satisfied for (MOP) at a feasible point x° if and only if there exist sequences $(x^k) \subset \mathbb{R}^n$ and $(\lambda^k, \mu^k, \tau^k) \subset \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that*

$(C1)$ $x^k \to x^\circ$,

$(C2)$ $\displaystyle\sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k) \to 0,$

$(C3)$ $\lambda_i^k > 0, i = 1, ..., p,$

$(C4)$ $\displaystyle\lim_{k\to\infty} \lambda_i^k (f_i(x^k) - f_i(x^\circ)) = 0 \ \forall i, \ \lim_{k\to\infty} \mu_j^k g_j(x^k) = 0 \ \forall j, \ \lim_{k\to\infty} \tau_l^k h_l(x^k) = 0 \ \forall l.$

Theorem 4. *(Necessary conditions) If $x^\circ \in \Omega$ is a local efficient solution of (MOP), then x° satisfies the SCAKKT conditions.*

Proof. Since x° is a local efficient solution to (MOP). Then, from Miettinen [7, Theorem 3.3.1] and Chankong and Haimes [12, Section 4.6.3], x° is a local solution to the problem,

$$\begin{gathered}\min \sum_{i=1}^{p} w_i f_i(x) + \frac{1}{2}\|x - x^\circ\|^2, \\ \text{subject to } f_i(x) \leqq f_i(x^\circ), \ i = 1, ..., p, \ x \in \Omega \cap \bar{B}(x^\circ, \delta), \ w \in \mathbb{R}^p, w > 0.\end{gathered} \tag{2}$$

We can suppose x° is a unique solution of (2). Now, we define an unconstrained optimization problem corresponding to constrained optimization problem (2), with the help of penalty function [13, p. 255] as follows:

$$\begin{aligned}\varphi_k(x) = \sum_{i=1}^{p} w_i f_i(x) + \frac{1}{2}\|x - x^\circ\|^2 + \frac{\rho_k}{2}\Big\{ \sum_{i=1}^{p} (f_i(x) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j(x)_+^2 \\ + \sum_{l=1}^{r} h_l(x)^2 \Big\},\end{aligned} \tag{3}$$

for all $k \in \mathbb{N}$. Let x^k be a global solution to the problem

$$\min \ \varphi_k(x), \ \text{subject to } \|x - x^\circ\| \leqq \delta. \tag{4}$$

In (4), x^k exists corresponding to large enough ρ_k, because $\varphi_k(x)$ is continuous and $\bar{B}(x^\circ, \delta)$ is compact. Let z be a limit point of sequence x^k. We may suppose that $x^k \to z$. From (3), we have

$$\sum_{i=1}^{p} w_i f_i(x^k) \leqq \varphi_k(x^k),$$

because

$$\varphi_k(x^k) - \sum_{i=1}^{p} w_i f_i(x^k) = \frac{1}{2}\|x^k - x^\circ\|^2 + \frac{\rho_k}{2}\Big\{ \sum_{i=1}^{p}(f_i(x^k) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j(x^k)_+^2 + \sum_{l=1}^{r} h_l(x^k)^2 \Big\} \geqq 0.$$

Since x^k is the solution of (4) and x° is a feasible point, then we have

$$\varphi_k(x^k) \leqq \varphi_k(x^\circ) = \sum_{i=1}^{p} w_i f_i(x^\circ). \tag{5}$$

We claim that z is a feasible point of the problem (4), for this suppose if possible

$$\sum_{i=1}^{p}(f_i(z) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j^2(z)_+ + \sum_{l=1}^{r} h_l^2(z) > 0,$$

for sufficiently large k, then there exists $c > 0$, such that

$$\sum_{i=1}^{p}(f_i(x^k) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j^2(x^k)_+ + \sum_{l=1}^{r} h_l^2(x^k) > c.$$

Therefore, from continuity of all functions and $x^k \to z$, we have

$$\sum_{i=1}^{p} w_i f_i(x^k) + \frac{1}{2}\|x^k - x^\circ\|^2 + \frac{\rho_k}{2}\Big\{ \sum_{i=1}^{p}(f_i(x^k) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j(x^k)_+^2 + \sum_{l=1}^{r} h_l(x^k)^2 \Big\} > \sum_{i=1}^{p} w_i f_i(x^k) + \frac{1}{2}\rho_k c.$$

Taking the limit $\rho_k \longrightarrow \infty$, we obtain $\varphi_k(x^k) \longrightarrow \infty$, which contradicts (5). Consequently, $\sum_{i=1}^{p}(f_i(z) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j^2(z)_+ + \sum_{l=1}^{r} h_l^2(z) = 0$, which implies z is a feasible point. From (4), we have

$$\varphi_k(x^k) = \sum_{i=1}^{p} w_i f_i(x^k) + \frac{1}{2}\|x^k - x^\circ\|^2 + \frac{\rho_k}{2}\Big\{ \sum_{i=1}^{p}(f_i(x^k) - f_i(x^\circ))_+^2 + \sum_{j=1}^{m} g_j(x^k)_+^2 + \sum_{l=1}^{r} h_l(x^k)^2 \Big\} \leqq \sum_{i=1}^{p} w_i f_i(x^\circ). \tag{6}$$

Since, $\frac{\rho_k}{2}\{\sum_{i=1}^{p}(f_i(x^k)-f_i(x^\circ))_+^2+\sum_{j=1}^{m}g_j(x)_+^2+\sum_{l=1}^{r}h_l(x)^2\} \geqq 0$, then from (6), we have

$$\sum_{i=1}^{p} w_i f_i(x^k)+\frac{1}{2}\|x^k-x^\circ\|^2 \leqq \sum_{i=1}^{p} w_i f_i(x^\circ),$$

taking the limit, we get

$$\sum_{i=1}^{p} w_i f_i(z)+\frac{1}{2}\|z-x^\circ\|^2 \leqq \sum_{i=1}^{p} w_i f_i(x^\circ).$$

As x° is the unique solution to the problem (2), we conclude that $z = x^\circ$. Then, $x^k \longrightarrow x^\circ$ and $\|x^k - x^\circ\| < \delta$ for all k sufficiently large. Since x^k is a solution to problem (3) and it is an interior point of the feasible set for sufficiently large k, then from optimality conditions $\nabla\varphi_k(x^k) = 0$, that is

$$\begin{aligned}\sum_{i=1}^{p} w_i\nabla f_i(x^k)+(x^k-x^\circ)+\sum_{i=1}^{p}\rho_k(f_i(x^k)-f_i(x^\circ))_+\nabla f_i(x^k)\\ +\sum_{j=1}^{m}\rho_k g_j(x^k)_+\nabla g_j(x^k)+\sum_{l=1}^{r}\rho_k h_l(x^k)\nabla h_l(x^k)=0.\end{aligned}\tag{7}$$

If we suppose $\lambda_i^k = w_i + \rho_k(f_i(x^k) - f_i(x^\circ))_+ > 0,\ i = 1, 2, ..., p$, and as $\mu_j^k = \rho_k g_j(x^k)_+$, $\tau_l^k = \rho_k h_l(x^k)$, then from (7), we get

$$\sum_{i=1}^{p}\lambda_i^k\nabla f_i(x^k)+\sum_{j=1}^{m}\mu_j^k\nabla g_j(x^k)+\sum_{l=1}^{r}\tau_l^k\nabla h_l(x^k)=x^\circ-x^k\to 0,$$

as $x^k \longrightarrow x^\circ$. Now, from [13, p. 257, Theorem 2.1], we have

$$\begin{aligned}\sum_{i=1}^{p} w_i f_i(x^k)+\frac{1}{2}\|x^k-x^\circ\|^2+\frac{\rho_k}{2}\Big\{\sum_{i=1}^{p}(f_i(x^k)-f_i(x^\circ))_+^2+\sum_{j=1}^{m}g_j(x^k)_+^2\\ +\sum_{l=1}^{r}h_l(x^k)^2\Big\}\leqq\sum_{i=1}^{p}w_i f_i(x^\circ),\end{aligned}\tag{8}$$

taking limit, we get

$$\begin{aligned}\lim_{x^k\to x^\circ}\Bigg[\frac{1}{2}\|x^k-x^\circ\|^2+\frac{\rho_k}{2}\Big\{\sum_{i=1}^{p}(f_i(x^k)-f_i(x^\circ))_+^2+\sum_{j=1}^{m}g_j(x^k)_+^2\\ +\sum_{l=1}^{r}h_l(x^k)^2\Big\}\Bigg]\leqq 0.\end{aligned}\tag{9}$$

Since $\lambda_i^k - w_i = \rho_k(f_i(x^k) - f_i(x^\circ))_+,\ \mu_j^k = \rho_k g_j(x^k)_+ \geq 0$ and $\tau_l^k = \rho_k h_l(x^k)$, then

$$\lim_{x^k \to x^\circ} \left[\sum_{i=1}^{p} |(\lambda_i^k - w_i)(f_i(x^k) - f_i(x^\circ))_+| + \sum_{j=1}^{m} |\mu_j^k g_j(x^k)_+| + \sum_{l=1}^{r} |\tau_l^k h_l(x^k)| \right] = 0. \tag{10}$$

Thus, we get

$$\lim_{k\to\infty} \lambda_i^k (f_i(x^k) - f_i(x^\circ)) = 0,\ \lim_{k\to\infty} \mu_j^k g_j(x^k) = 0,\ \text{and}\ \lim_{k\to\infty} \tau_l^k h_l(x^k) = 0.$$

Hence the conditions are satisfied. □

Remark 5. *If we consider a scalar optimization problem, then Theorem 3.1 reduces to Theorem 3.3 of Andreani et al. [3]*

Remark 6. *ASKKT implies SCAKKT, but converse implication may not be true,*

Example 7. *Consider the following problem*

$$min\ (f_1(x_1, x_2), f_2(x_1, x_2)),\ subject\ to\ g(x_1, x_2) = -x_1 \leqq 0,$$

where $f_1(x_1, x_2) = x_1, f_2(x_1, x_2) = x_2$. *For* $x^\circ = (0, 0)$, *choose* $\lambda_1^k = 1, \lambda_2^k = \frac{1}{k}, \mu^k = 1, x^k = (-\frac{1}{k}, \frac{1}{k})$

$$\lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \mu^k \nabla g(x^k) \to 0,\ as\ x^k \to x^\circ = (0, 0).$$

$$\lambda_i^k (f_i(x^k) - f_i(x^\circ)) \to 0,\ i = 1, 2,\ \mu^k g(x^k) \to 0.$$

Hence, x° *is SCAKKT point but can not be ASKKT point. Moreover,* x° *also AKKT point.*

Theorem 8. *(Sufficient conditions) Assume that* $f_i (i = 1, ..., p)$, $g_j (j = 1, ..., m)$ *are convex functions and* $h_l (l = 1, ..., r)$ *are affine. If* $x^\circ \in \Omega$ *satisfies the SCAKKT conditions with the sequences* $(x^k) \subset \mathbb{R}^n$, $(\lambda^k, \mu^k, \tau^k) \subset \mathbb{R}_+^p \times \mathbb{R}_+^m \times \mathbb{R}^r$ *and* $\lambda^k \to \lambda^\circ > 0$. *Then,* x° *is an efficient solution of (MOP).*

Proof. Suppose that x° is not an efficient solution then, there exists $\bar{x} \in S$ such that

$$f(\bar{x}) \leq f(x^\circ). \tag{11}$$

Since f_i, g_j are convex and h_l are affine, then for all k we have

$$f_i(\bar{x}) \geqq f_i(x^k) + \langle \nabla f_i(x^k), \bar{x} - x^k \rangle, \forall\ i = 1, ..., p. \tag{12}$$

$$g_j(\bar{x}) \geqq g_j(x^k) + \langle \nabla g_j(x^k), \bar{x} - x^k \rangle, \forall j = 1, ..., m. \tag{13}$$

$$h_l(\bar{x}) = h_l(x^k) + \langle \nabla h_l(x^k),\ \bar{x} - x^k \rangle, \forall l = 1, ..., r. \tag{14}$$

For $\bar{x}$ we can write

$$\sum_{i=1}^{p} \lambda_i^k f_i(\bar{x}) \geqq \sum_{l=1}^{p} \lambda_i^k f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j^k g_j(\bar{x}) + \sum_{l=1}^{r} \tau_l^k h_l(\bar{x}), \tag{15}$$

then from (12) to (15), we get

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i^k f_i(\bar{x}) \geqq \sum_{i=1}^{p} \lambda_i^k f_i(x^k) + \sum_{j=1}^{m} \mu_j^k g_j(x^k) + \sum_{l=1}^{r} \tau_l^k h_l(x^k) \\ + \left\langle \sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k),\ \bar{x} - x^k \right\rangle. \end{aligned} \tag{16}$$

From $(C1) - (C4)$ and (16), we get

$$\sum_{i=1}^{p} \lambda_i^\circ f_i(\bar{x}) \geqq \sum_{i=1}^{p} \lambda_i^\circ f_i(x^\circ),$$

as $x^k \to x^\circ$, which contradicts (11). $\square$

Remark 9. *If we take scalar optimization problem, then Theorem 3.2 reduces to Theorem 4.2 of Andreani et al. [3]*

4. SCAKKT-REGULARITY CONDITION

We propose some sets to define SCAKKT-regularity condition, which are given as: For $x^\circ \in \Omega$, linearized cone at x° is

$$\begin{aligned} \mathcal{P}(x^\circ) = \Big\{ d \in \mathbb{R}^n : \langle \nabla f_i(x^\circ), d \rangle \leqq 0, \forall i \in I, \langle \nabla g_j(x^\circ), d \rangle \leqq 0, \forall j \in J(x^\circ), \\ \langle \nabla h_l(x^\circ), d \rangle = 0, \forall l \in L \Big\}, \end{aligned} \tag{17}$$

$$\begin{aligned} \mathcal{Q}(x, r) = \Big\{ \sum_{i=1}^{p} \lambda_i \nabla f_i(x) + \sum_{j=1}^{m} \mu_j \nabla g_i(x) + \sum_{l=1}^{r} \tau_l \nabla h_l(x) : \\ \sum_{i=1}^{p} |\lambda_i (f_i(x) - f_i(x^\circ))_+| + \sum_{j=1}^{m} |\mu_j g_i(x)_+| + \sum_{l=1}^{r} |\tau_l h_l(x)| \leqq r,\ \lambda_i \geqq 0, \mu_j \geqq 0, \tau_l \in \mathbb{R} \Big\}. \end{aligned} \tag{18}$$

and

$$\mathcal{D}(x) = \Big\{ \sum_{i=1}^{p} \lambda_i \nabla f_i(x) + \sum_{j=1}^{m} \mu_j \nabla g_i(x) + \sum_{l=1}^{r} \tau_l \nabla h_l(x) : \lambda_i \geqq 0, \mu_j \geqq 0, \tau_l \in \mathbb{R} \Big\}. \tag{19}$$

From (17) and (19), we have

$$\mathcal{Q}(x^\circ, 0) = \mathcal{D}(x^\circ).$$

For basic properties of set-valued mapping, tangent cone $T_\Omega(x^\circ)$, normal cone $N_\Omega(x^\circ)$ at $x^\circ \in \Omega$, and dual cone $\mathcal{K}^\circ$ of $\mathcal{K} \subset \mathbb{R}^n$ see, Rockafellar and Wets [14].

Definition 10. *Feasible point x° satisfies the SCAKKT-regularity condition if the set-valued mapping*

$$(x, r) \in \mathbb{R}^n \times \mathbb{R}_+ \rightrightarrows \mathcal{Q}(x, r),$$

is outer semicontinuous at $(x^\circ, 0)$, in other words, the following inclusion holds:

$$\limsup_{(x,r)\to(x^\circ,0)} \mathcal{Q}(x, r) \subset \mathcal{Q}(x^\circ, 0).$$

An extended form of CCR-condition and Abadie's constraint qualification [9] are defined as follows:

Definition 11. *(CCR condition) Feasible point x° satisfies the cone continuity regularity (CCR) condition if the set-valued mapping*

$$x \in \mathbb{R}^n \rightrightarrows \mathcal{D}(x)$$

is outer semicontinuous at x°. In other words, the following inclusion holds,

$$\limsup_{x\to x^\circ} \mathcal{D}(x) \subset \mathcal{D}(x^\circ).$$

Definition 12. *Abadie's constraint qualification holds at a feasible point x^0, if*

$$\mathcal{P}(x^\circ) \subset T_\Omega(x^\circ).$$

Proposition 13. *CCR implies SCAKKT-regularity, but converse may not be true*

Proof. The first implication is a direct consequence from Definition 10 and Definition 11, and for converse part, we consider the following problem.

$$\text{Min } (f_1(x), f_2(x)), \text{ subject to } g(x) = x_2 e^{x_1} \leqq 0, \; h(x) = x_2 = 0,$$
$$\text{where } f_1(x) = 2x_2, f_2(x) = -x_2, \; x \in \mathbb{R}^2. \text{ Consider at point } x^\circ = (0, 0).$$

$$\begin{aligned}\mathcal{Q}(x^\circ, 0) =& \{\lambda_1 \nabla f_1(x^\circ) + \lambda_2 \nabla f_2(x^\circ) + \mu \nabla g(x^\circ) + \tau \nabla h(x^\circ) : \lambda_i \geqq 0, \mu \geqq 0, \tau \in \mathbb{R}\},\\ =& \{\lambda_1(0, 2) + \lambda_1(0, -1) + \mu(0, 1) + \tau(0, 1) : \lambda_i \geqq 0, \mu \geqq 0, \tau \in \mathbb{R}\},\\ =& \{0\} \times \mathbb{R}.\end{aligned}$$

Let $\omega^\circ = (\omega_1^\circ, \omega_2^\circ) \in \limsup_{(x,r)\to(x^\circ,0)} \mathcal{Q}(x,r)$, then there exist sequences $x^k \to x^\circ$, $\omega^k \to \omega^\circ$ in $\mathbb{R}^2$ and $r^k \downarrow 0$, where

$$\begin{aligned}\omega^k =& \lambda_1^k \nabla f_1(x^k) + \lambda_2^k \nabla f_2(x^k) + \mu^k \nabla g(x^k) + \tau^k \nabla h(x^k),\\ =& \lambda_1^k (0,2) + \lambda_2^k (0,-1) + \mu^k (x_2^k e^{x_1^k}, e^{x_1^k}) + \tau^k (0,1),\end{aligned}$$

and multipliers satisfying the conditions

$$\begin{aligned}&|\lambda_1^k (f_1(x^k) - f_1(x^\circ))_+| + |\lambda_2^k (f_2(x^k) - f_2(x^\circ))_+| + |\mu^k g(x^k)_+|\\ &\qquad + |\tau^k h_l(x^k)| \leqq r^k, \lambda_i \geqq 0, \mu \geqq 0, \tau \in \mathbb{R}. \quad (20)\end{aligned}$$

Then, from (20), we have

$$|2\lambda_1^k x_2^k| + |\lambda_2^k x_2^k| + |\mu^k x_2^k e^{x_1^k}| + |\tau^k x_2^k| \leqq r^k, \lambda_i \geqq 0,\ \mu \geqq 0,\ \tau \in \mathbb{R}.$$

Since $\omega_1^k = |\mu^k x_2^k e^{x_1^k}| \leqq r^k$, then $\omega_1^k \to 0$, which implies that $\omega^\circ \in \mathcal{Q}(x^\circ, 0) = \{0\} \times \mathbb{R}$, but x° does not satisfy cone continuity regularity condition, choose $x_1^k = \frac{1}{k}, x_2^k = \frac{1}{k}, \lambda_1^k = \frac{\lambda_2^k}{2}, \mu^k = \frac{1}{x_2^k e^{x_1^k}}$ and $\tau^k = -k$, then we have

$$\omega^k = \lambda_1^k (0,2) + \lambda_2^k (0,-1) + \mu^k (x_2^k e^{x_1^k}, e^{x_1^k}) + \tau^k (0,1) = (1,0) \notin \mathcal{D}(x^\circ),\ \forall\ k.$$

□

In the following lemma we establish relationship between strong Karush-Kuhn-Tucker (SKKT) optimality conditions for (MOP) and feasible points of (MOP).

Lemma 14. *Let x° be a feasible point. Then, x° satisfies the SKKT conditions of (MOP) if and only if $-\lambda_i \nabla f_i(x^\circ) \in \mathcal{Q}(x^\circ, 0)$ for $\lambda_i > 0,\ \forall\ i = 1, ..., p$.*

Proof. Suppose that there exists vector $\lambda \in \mathbb{R}^p$, with $\lambda > 0, \mu \in \mathbb{R}_+^m$ and $\tau \in \mathbb{R}^r$ such that

$$\sum_{i=1}^p \lambda_i \nabla f_i(x^\circ) + \sum_{j=1}^m \mu_j \nabla g_i(x^\circ) + \sum_{l=1}^r \tau_l \nabla h_l(x^\circ) = 0.$$

It follows that, for each $i \in I = \{1, ..., p\}$,

$$-\lambda_i \nabla f_i(x^\circ) = \sum_{t \in I \setminus \{i\}} \lambda_t \nabla f_t(x^\circ) + \sum_{j=1}^m \mu_j \nabla g_i(x^\circ) + \sum_{l=1}^r \tau_l \nabla h_l(x^\circ) \in \mathcal{Q}(x^\circ, 0).$$

For converse part, let $-\lambda_i \nabla f_i(x^\circ) \in \mathcal{Q}(x^\circ, 0)$ for $\lambda_i > 0,\ i \in I$. Then, for each $i \in I$, there exist $\lambda^i \in \mathbb{R}_+^p, \mu^i \in \mathbb{R}_+^m$ and $\tau^i \in \mathbb{R}^r$ such that

$$-\lambda_i \nabla f_i(x^\circ) = \sum_{t=1}^p \lambda_t^i \nabla f_t(x^\circ) + \sum_{j=1}^m \mu_j^i \nabla g_j(x^\circ) + \sum_{l=1}^r \tau_l^i \nabla h_l(x^\circ) \in \mathcal{Q}(x^\circ, 0).$$

which implies that,

$$\lambda_i \nabla f_i(x^\circ) + \sum_{t=1}^{p} \lambda_t^i \nabla f_t(x^\circ) + \sum_{j=1}^{m} \mu_j^i \nabla g_j(x^\circ) + \sum_{l=1}^{r} \tau_l^i \nabla h_l(x^\circ) = 0, \forall\ i \in I. \quad (21)$$

If we add all equations included in (23) from $i = 1$ to p and put

$$\bar{\lambda}_t = \lambda_i + \sum_{t=1}^{p} \lambda_t^i,\ t = 1, ..., p, \bar{\mu}_j = \sum_{j=1}^{m} \mu_j^i, \bar{\tau}_j = \sum_{l=1}^{r} \tau_l^i,$$

then we get

$$\sum_{t=1}^{p} \bar{\lambda}_t \nabla f_t(x^\circ) + \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(x^\circ) + \sum_{l=1}^{r} \bar{\tau}_l \nabla h_l(x^\circ) = 0.$$

This completes the proof. □

Theorem 15. *If $x^\circ \in \Omega$ is a limit point of an SCAKKT sequence $(x^k) \subset \mathbb{R}^n$ and SCAKKT-regularity holds, then x° is an SKKT point.*

Proof. Let x° be a SCAKKT point. To show that the SKKT conditions hold at x°, it is sufficient to prove $-\lambda_i^\circ \nabla f_i(x^\circ) \in \mathcal{Q}(x^\circ, 0)$ for $\lambda_i^\circ > 0,\ \forall\ i \in I = \{1, ..., p\}$, as in Lemma 14. Since x° satisfies SCAKKT conditions, then there exist sequences $x^k \subset \mathbb{R}^n$ and $\{(\lambda^k, \mu^k, \tau^k)\} \subset \mathbb{R}_+^p \times \mathbb{R}_+^m \times \mathbb{R}^r$, such that $\lambda_i^k > 0, i \in I, \mu_j^k = 0$ for $j \notin J(x^\circ) = \{j : g_j(x^\circ) = 0\}$ and

$$\omega^k = \sum_{i=1}^{p} \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k) \to 0, \quad (22)$$

under the conditions

$$\sum_{i=1}^{p} |\lambda_i^k (f_i(x^k) - f_i(x^\circ))_+| + \sum_{j=1}^{m} |\mu_j^k g_i(x^k)_+| + \sum_{l=1}^{r} |\tau_l^k h_l(x^k)| \leqq r^k,\ r^k \downarrow 0.$$

It follows that, for each $i \in I$,

$$\omega^k - \lambda_i^k \nabla f_i(x^k) = \sum_{t \in I \setminus \{i\}} \lambda_t^k \nabla f_t(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k). \quad (23)$$

Thus, for each i, we have

$$\omega^k - \lambda_i^k \nabla f_i(x^k) \in \mathcal{Q}(x^k, r^k). \quad (24)$$

From (23) to (25), we get

$$-\lambda_i^\circ \nabla f_i(x^\circ) \in \limsup_{k \to \infty} \mathcal{Q}(x^k, r^k) \subset \limsup_{(x^k, r^k) \to (x^\circ, 0)} \mathcal{Q}(x^k, r^k) \subset \mathcal{Q}(x^\circ, 0),$$

as SCAKKT-regularity holds at x°. □

Corollary 16. *If $x^\circ \in \Omega$ is a local efficient solution to (MOP) and verifies the SCAKKT-regularity conditions, then x° is a SKKT point.*

The following lemma is extension of [11, Lemma 4.3] to (MOP), which is required to establish relationship between SCAKKT-regularity conditions and Abadie's constraint qualification.

Lemma 17. *For all $x^\circ \in \Omega$ and $v \in T^\circ_\Omega(x^\circ)$, there exist sequences $\{x^k\} \subset \mathbb{R}^n, \{\omega^k\} \subset \mathbb{R}^n, \{\lambda^k\} \subset \mathbb{R}^p_+, \{\mu^k\} \subset \mathbb{R}^m_+, \{\tau^k\} \subset \mathbb{R}^r, w \in \mathbb{R}^p, \rho_k > 0,$ and $r^k \subset \mathbb{R}$ with conditions $x^k \to x^\circ, \lambda^k \to \lambda^\circ, r^k \downarrow 0, w > 0,$ $\rho_k \to \infty$ and $\lambda^k > 0\ \forall\ k,$ such that*

(1) $v_i^k = \sum\limits_{t \in I\setminus\{i\}} \lambda_t^k \nabla f_t(x^k) + \sum\limits_{j=1}^m \mu_j^k \nabla g_j(x^k) + \sum\limits_{l=1}^r \tau_l^k \nabla h_l(x^k) \to v,\ \forall\ i = 1, ..., p,$

(2) $\sum\limits_{i=1}^p |\lambda_i^k (f_i(x^k) - f_i(x^\circ))_+| + \sum\limits_{j=1}^m |\mu_j^k g_i(x^k)_+| + \sum\limits_{l=1}^r |\tau_l^k h_l(x^k)| \leqq r^k,$

(3) $\lambda_i^k = w_i + \rho_k (f_i(x^k) - f_i(x^\circ))_+,\ \mu_j^k = \rho_k g_j(x^k)_+$ *and* $\tau_l^k = \rho_k h_l(x^k).$

Proof. Let $v \in T^\circ_\Omega(x^\circ)$, then from [14, Theorem 6.11], there exist smooth functions $\lambda_i^\circ f_i(x)$, for each $i \in I = \{1, ..., p\}$, such that

$$-\lambda_i^\circ \nabla f_i(x^\circ) = v \tag{25}$$

and $\lambda_i^\circ f_i(x^\circ)$ attains its global minimum uniquely at $x^\circ \in \Omega$. Consider, for each $k \in \mathbb{N}$, the following optimization problem as in Theorem4:

$$\text{Min } F_k(x), \quad \text{subject to } x \in \bar{\mathbb{B}}(x^\circ, \delta), \tag{26}$$

where

$$F_k(x) = \sum_{i=1}^p w_i f_i(x) + \frac{\rho_k}{2}\Big\{\sum_{i=1}^p (f_i(x) - f_i(x^\circ))_+^2 + \sum_{j=1}^m g_j(x)_+^2 + \sum_{l=1}^r h_l(x)^2\Big\}.$$

Since $\bar{\mathbb{B}}(x^\circ, \delta)$ is compact and $F_k(x)$ is continuous, then from Weierstrass theorem there exists a solution, for (26). Let x^k be the solution, then

$$\begin{aligned}\sum_{i=1}^p w_i \nabla f_i(x^k) + \sum_{i=1}^p \rho_k (f_i(x^k) - f_i(x^\circ))_+ \nabla f_i(x^k) + \sum_{j=1}^m \rho_k g_j(x^k)_+ \nabla g_j(x^k) \\ + \sum_{l=1}^r \rho_k h_l(x^k) \nabla h_l(x^k) = 0.\end{aligned} \tag{27}$$

Let $\lambda_i^k = w_i + \rho_k (f_i(x^k) - f_i(x^\circ))_+,\ \mu_j^k = \rho_k g_j(x^k)_+$ and $\tau_l^k = \rho_k h_l(x^k)$. Then, (27) implies that

$$\nabla F_k(x^k) = \sum_{i=1}^p \lambda_i^k \nabla f_i(x^k) + \sum_{j=1}^m \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^r \tau_l^k \nabla h_l(x^k) = 0. \tag{28}$$

Now, from [13, Theorem 2.1] we get

$$\sum_{i=1}^{p} w_i f_i(x^k) \leqq \sum_{i=1}^{p} w_i f_i(x^k)+\frac{\rho_k}{2}\Big\{\sum_{i=1}^{p}(f_i(x^k)-f_i(x^0))_+^2+\sum_{j=1}^{m} g_j(x^k)_+^2+\sum_{l=1}^{r} h_l(x^k)^2\Big\}$$
$$\leqq F_k(x^\circ) = \sum_{i=1}^{p} w_i f_i(x^\circ). \quad (29)$$

$F_k(x^k)$ is bounded in $\bar{\mathbb{B}}(x^\circ,\delta)$, then from (29), we have

$$\sum_{i=1}^{p} |\lambda_i^k (f_i(x^k) - f_i(x^\circ))_+| + \sum_{j=1}^{m} |\mu_j^k g_i(x^k)_+| + \sum_{l=1}^{r} |\tau_l^k h_l(x^k)| \leqq r^k, \text{ for some } r^k \downarrow 0.$$

Since,

$$v_i^k = \sum_{t\in I\setminus\{i\}} \lambda_t^k \nabla f_t(x^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x^k) + \sum_{l=1}^{r} \tau_l^k \nabla h_l(x^k) \quad (30)$$

therefore from (25), (28) and (30), we get

$$v_i^k = -\lambda_i^k \nabla f_i(x^k) \to v.$$

Hence the theorem. □

Theorem 18. *SCAKKT-regularity implies Abadie's constraint qualification.*

Proof. We have to show that $\mathcal{P}(x^\circ) \subset T_\Omega(x^\circ)$, for this, first we show $N_\Omega(x^\circ) \subset \mathcal{P}^\circ(x^\circ)$, which is equivalent to $N_\Omega(x^\circ) \subset \mathcal{Q}(x^\circ,0)$. Let $v \in N_\Omega(x^\circ)$, then from property of normal cone[14], there are sequences $\{x^k\}$ and $\{v^k\}$ such that

$$x^k \to x^\circ, v^k \to v, \text{ and } v^k \in T_\Omega^\circ(x^k).$$

Now, from Lemma (13), for each $v^k \in T_\Omega^\circ(x^k)$ there exist sequences $x^{k,\ell}$ and $v_i^{k,\ell}$ satisfying the Lemma 17. Therefore, for all $k,\ \ell \in \mathbb{N}$, we have

$$\lim_{\ell\to\infty} v_i^{k,\ell} = \lim_{\ell\to\infty}\Bigg\{\sum_{t\in I\setminus\{i\}} \lambda_t^{k,\ell}\nabla f_t(x^{k,\ell}) + \sum_{j=1}^{m} \mu_j^{k,\ell}\nabla g_j(x^{k,\ell})$$
$$+\sum_{l=1}^{r} \tau_l^{k,\ell}\nabla h_l(x^{k,\ell})\Bigg\} = v^k \quad (31)$$

where $\lambda_i^{k,\ell} = w_i + \rho_\ell(f_i(x^{k,\ell}) - f_i(x^\circ))_+, i = 1,...,p,\ \mu_j^{k,\ell} = \rho_\ell g_j(x^{k,\ell})_+, \forall j = 1,...,m$, and $\tau_l^{k,\ell} = \rho_\ell h_l(x^{k,\ell}),\ \forall l = 1,...,r$. Thus, for all $k \in \mathbb{N}$, there exist $\ell(k)$ such that

1. $||x^k - x^{k,\ell(k)}|| < \frac{1}{2^k}$;
2. $v_i^{k,\ell(k)} = \sum\limits_{t \in I \setminus \{i\}} \lambda_t^{k,\ell(k)} \nabla f_t(x^{k,\ell(k)}) + \sum\limits_{j=1}^{m} \mu_j^{k,\ell(k)} \nabla g_j(x^{k,\ell(k)}) + \sum\limits_{l=1}^{r} \tau_l^{k,\ell(k)} \nabla h_l(x^{k,\ell(k)})$
3. $||v^k - v_i^{k,\ell(k)}|| < \frac{1}{2^k}$;
4. $\lambda_i^{k,\ell(k)} - w_i = \rho_{\ell(k)}(f_i(x^{k,\ell(k)}) - f_i(x^\circ))_+, i = 1, ..., p,\ \mu_j^{k,\ell(k)} = \rho_{\ell(k)} g_j(x^{k,\ell(k)})_+, \forall j = 1, ..., m$, and $\tau_l^{k,\ell(k)} = \rho_{\ell(k)} h_l(x^{k,\ell(k)}),\ \forall l = 1, ..., r$.

Clearly,

$$\lim_{k\to\infty} x^{k,\ell(k)} = x^\circ,\ \lim_{k\to\infty} v_i^{k,\ell(k)} = v.$$

Also for sufficiently large k, we have

$$\sum_{i=1}^{p} |\lambda_i^{k,\ell(k)} (f_i(x^{k,\ell(k)}) - f_i(x^\circ))_+| + \sum_{j=1}^{m} |\mu_j^{k,\ell(k)} g_i(x^{k,\ell(k)})_+| + \sum_{l=1}^{r} |\tau_l^{k,\ell(k)} h_l(x^{k,\ell(k)})| \leqq r^{k,\ell(k)},\ r^{k,\ell(k)} \downarrow 0. \quad (32)$$

Therefore,

$$v_i^{k,\ell(k)} \in \mathcal{Q}(x^{k,\ell(k)}, r^{k,\ell(k)}),$$

that is, we have sequences

$$x^{k,\ell(k)} \to x^\circ,\ v_i^{k,\ell(k)} \to v \text{ with } v_i^{k,\ell(k)} \in \mathcal{Q}(x^{k,\ell(k)}, r^{k,\ell(k)}).$$

From SCAKKT-regularity condition and definition of outer limit we have

$$v \in \limsup_{(x,r)\to(x^\circ,0)} \mathcal{Q}(x,r) \subset \mathcal{Q}(x^\circ, 0).$$

Then,

$$N_\Omega(x^\circ) \subset \mathcal{Q}(x^\circ, 0) = \mathcal{P}^\circ(x^\circ),$$

which implies

$$\mathcal{P}(x^\circ) = \mathcal{Q}^\circ(x^\circ, 0) \subset N_\Omega^\circ(x^\circ).$$

From [14, Theorem 6.28], we have

$$N_\Omega^\circ(x^\circ) \subset T_\Omega(x^\circ).$$

Hence,

$$\mathcal{P}(x^\circ) \subset T_\Omega(x^\circ),$$

as we wanted to show. □

In the following example we show that Abadie's CQ is strictly weaker than SCAKKT-regularity condition.

Example 19. *Consider the problem*

$$min\ (f_1(x_1,x_2), f_2(x_1,x_2)),\ \ s.t.\ g_i(x_1,x_2) \leqq 0,\ \ i = 1,2,3,$$

at point $x^\circ = (0,0)$, *where*

$$f_1(x_1,x_2) = -x_1, f_2(x_1,x_2) = -x_2, g_1(x_1,x_2) = -x_1,$$
$$g_2(x_1,x_2) = -x_2 e^{x_2} \text{ and } g_3(x_1,x_2) = -x_1 x_2.$$

Feasible set $\Omega_1 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_1 \geq 0\}$. *Then,* $\mathcal{P}(x^\circ) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_1 \geq 0\} = T_{\Omega_1}(x^\circ)$, *Abadie's constraint qualification holds at point* x°. *Now, we prove that* $\mathcal{Q}(x,r)$ *is not outer semicontinuous at* x°, *for this choose* $x^k = (-\frac{1}{k}, \frac{1}{k}), \lambda_1^k = \frac{1}{k}, \lambda_2^k = \frac{1}{k}, \mu_1^k = 0, \mu_2^k = 0, \mu_3^k = k$. *Then,*

$$r^k = |\lambda_1^k(f_1(x^k) - f_1(x^\circ))| + |\lambda_2^k(f_2(x^k) - f_2(x^\circ))| + |\mu_1^k g_1(x^k)| + |\mu_2^k g_2(x^k)| + |\mu_3^k g_3(x^k)| = \frac{2}{k^2} + \frac{1}{k} \to 0$$

and

$$v_1^k = \lambda_2^k \nabla f_2(x^k) + \mu_1^k \nabla g_1(x^k) + \mu_2^k \nabla g_2(x^k) + \mu_3^k \nabla g_3(x^k) \to (-1,1).$$

$v_1^k = (-1,1) \in \mathcal{Q}(x^k, r^k)$ *for all* $k \in \mathbb{N}$, *means* $(-1,1) \in \limsup_{(x,r)\to(x^\circ,0)} \mathcal{Q}(x,r)$, *but* $(-1,1) \notin \mathcal{Q}(x^\circ,0) = \{(x_1,x_2) \in \mathbb{R}^n : x_1 \leqq 0, x_2 \leqq 0\}$. *Thus, SCAKKT-regularity is not satisfied.*

5. CONCLUSIONS

In this paper, we have established SCAKKT sequential optimality conditions, which are different from ASKKT and AKKT optimality conditions. We have introduced a constraint qualification that is weaker constraint qualification than CCR condition for multiobjective sequential optimality conditions and strong constraint qualification to Abadie's constraint qualification. Since sequential optimality conditions are useful for algorithmic consequences, therefore algorithmic development is still open for future research in this direction.

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