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SADDLE POINT CRITERIA FOR SEMIDEFINITE SEMI-INFINITE CONVEX MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract: In this paper, we consider a nonlinear semidefinite semi-infinite convex multiobjective optimization problem where the feasible region is determined by finite number of equality and infinite number of inequality constraints. We establish saddle point necessary and sufficient optimality conditions under some suitable constraint qualification. We establish Karush-Kuhn-Tucker optimality conditions using the saddle point optimality conditions for the differentiable case and construct some examples to illustrate our results.

Keywords: Saddle Point Criteria, Semidefinite Programming, Semi-infinite Programming, Multiobjective Programming.

MSC: 90C22, 90C29, 90C34.

1. INTRODUCTION

Consider the nonlinear semidefinite semi-infinite multiobjective optimization problems (NSD-SIMOP)

$$\begin{aligned}&\min f(X) := (f_1(X), ..., f_p(X)),\\ &\text{subject to } g_t(x) \leq 0,\ t \in T,\ h_i(x) = 0\ (i = 1, ..., r),\ x \in \mathbb{S}^n_+,\end{aligned} \tag{1}$$

where the functions $f_j : \mathbb{S}^n \to \mathbb{R}, j \in P := \{1, 2, .., p\},\ g_t : \mathbb{S}^n \to \mathbb{R},\ t \in T,\ T$ present a compact and infinite set, $h_i : \mathbb{S}^n \to \mathbb{R}, i \in R := \{1, 2, .., r\}$ are real valued functions where $\mathbb{S}^n$ and $\mathbb{S}^n_+$ are sets of $n \times n$ symmetric matrices and positive semidefinite matrices, respectively.

Semidefinite programming is a widely applicable field of optimization [50, 15, 19, 26, 27, 41, 43, 16, 44, 34]. Some recent applications of semidefinite programming are in combinatorial optimization [46], dc microgrids [18], generalized Gauss inequalities [49], polynomial invariants [39], stochastic block models [1], integer convex quadratic minimization [35], analysis of Hoek-Brown material [47], power transmission network expansion planning [17], controlling the renewable microgrid [37], economic dispatch problems [2] etc.

A semi-infinite optimization problem is the minimization of real valued objective functions subject to an arbitrary possibly infinite number of constraint functions. Fundamental theoretical aspects and a wide range of applications of semi-infinite programming have been studied intensively by many researchers during last two decades, a few of them are constraint qualifications [20], optimality conditions [24], duality results [31], saddle point analysis [12], augmented Lagrangian functions [23], exact penalty functions [29], duality gap [32], algorithms and applications in social science [21], engineering [36], robotics [51], air pollutions [52], lapidary cutting [53], and power supply [45].

Multiobjective optimization problems arise in many real world problems [11], when optimal decisions need to be taken in the presence of two or more conflicting objectives. Generally, there does not exist a single solution that simultaneously optimizes each objective. Instead, there exists a set of Pareto optimal solutions. A solution is called nondominated or Pareto optimal or efficient solutions if none of the objective functions can be improved in value without degrading one or more of the other objective values [40]. Without additional subjective preference information, all Pareto optimal solutions are considered equally good. Multiobjective optimization problems are usually solved by scalarization. In scalarization technique, multiobjective optimization problem is converted into a single (scalar) objective optimization problem. In this way the new problem has a real-valued objective function, possibly depending on some parameters. After scalarization, we use widely developed theory and methods for single objective optimization.

Saddle point optimality conditions basically explained by Mangasarian [30] for scalar objective optimization problems, where optimality has been discussed without differentiability under convexity assumption in Euclidean space. Saddle-point optimality criteria method have attracted the attentions of many authors (see, [25, 33, 42, 5, 38, 48]). In 2012, saddle point optimality conditions of scalar convex constraints optimization problems were discussed in real Banach space [6] using hyperplane separation theorem technique, in which a refined solution based on convexity theory was given without differentiability hypothesis. Since in the derivation of necessary and sufficient optimality conditions for a given feasible point to be optimal convexity plays a paramount role. Therefore, convexity assumptions and avoiding differentiability of used functions lead towards wide range of applications. To the best of our knowledge, there are few papers dealing with a multiobjective semidefinite semi-infinite programming.

Recently, Dorsch *et al.* [13] established a new genericity result for nonlinear semidefinite programming (NLSDP) where almost all linear perturbations of a given NLSDP are shown to be nondegenerate. Further, nondegeneracy for NLSDP studied under transversality constraint qualification, strict complementarity and second-order sufficient condition. Semidefinite programming is a powerful framework from convex optimization that has striking potential for data science applications [54]. Sequential optimality conditions have played a vital role in unifying and extending global convergence results for several classes of algorithms for general nonlinear optimization, Andreani *et al.* [4] extend these concepts for nonlinear semidefinite programming. Andreani *et al.* [3] discuss naive extensions of constant rank-type constraint qualifications to semidefinite programming, which are based on the Approximate Karush-Kuhn-Tucker necessary optimality condition and on the application of the reduction approach.

Motivated by above in this paper, we extend the concept of saddle point optimality conditions for nonlinear semidefinite semi-infinite convex multiobjective optimization problems. We established Karush-Kuhn-Tucker optimality conditions under Slater's constraint qualification through saddle point optimality criteria approach.

The organization of this paper is as follows: in Section 2, we recall some preliminary and basic results. In Section 3, we present our results on saddle point and Karush-Kuhn-Tucker necessary and sufficient optimality conditions for the semidefinite semi-infinite multiobjective convex optimization problems. We also state the relationship between the Pareto solutions and the saddle points for Lagrange function using Slater's constraint qualification. The Last section is dedicated to conclusions.

2. PRELIMINARIES

In this section, we recall some notions and preliminary results that we used in this paper. For basic notions of matrix analysis [8, 10]. Here, $\mathbb{S}^n$ and $\mathbb{S}^n_+$ denote a set of all $n \times n$ symmetric and positive semidefinite matrices, respectively. The inner product between any two $A,\ B \in \mathbb{S}^n$ is defined by $\langle A, B\rangle = tr(A^T B)$ and the associated Frobenius norm is given by $\|A\|_F = \sqrt{tr(A^T A)}$. For any set X, $|X|$ denotes cardinality of the set X. Order relation between two vectors $y,\ z \in \mathbb{R}^p$, follow the following conventions

$$y \leqq z \Longleftrightarrow y_i \leq z_i,\ i = 1, ..., p,$$
$$y \leq z \Longleftrightarrow y \leqq z \quad \text{and} \quad y \neq z,$$
$$y < z \Longleftrightarrow y_i < z_i,\ i = 1, ..., p.$$

We denote the feasible region by F and assume that it is nonempty:

$$F = \{X \in \mathbb{S}_+^n :\ g_t(X) \leq 0\ (t \in T),\quad h_i(X) = 0\ (i = 1, ..., r)\}.$$

Motivated by Bazaraa *et al.* [7], we define the convex function on $\mathcal{S}_+^n$, as follows:

Definition 2.1. *A function $f : S \subset \mathbb{S}_+^n \to \mathbb{R}$ is said to be convex on S if the inequality*

$$f\left(\lambda X + (1 - \lambda)Y\right) \ \leq \lambda f(X) + (1 - \lambda)f(Y)$$

holds for all $X,\ Y \in S$ and for every $\lambda \in [0, 1]$.

Theorem 2.1. *[6] If A_1 and A_2 are two non-empty disjoint convex sets of $\mathbb{R}^n$, then there exists a non-zero element $c := (c_1, \cdots, c_n) \in \mathbb{R}^n \setminus \{0\}$ such that*

$$\sum_{i=1}^{n} c_i u_i \ \leqq \ \sum_{i=1}^{n} c_i v_i,\ \forall\ u = (u_i)_{i=1,2,...,n} \in A_1,\ \forall\ v = (v_i)_{i=1,2,...,n} \in A_2.$$

The following definition is an extension of Pareto optimal point from $\mathbb{R}^n$ to $\mathbb{S}_+^n$, which is extensively studied by Ehrgott [14];

Definition 2.2.

- *A feasible point $X^0 \in F$ is said to be a weak Pareto optimal solution of the NSD-SIMOP (1), iff there is no feasible point $X \in F$, such that*

$$f(X) < f(X^0).$$

- *A feasible point $X^0 \in F$ is said to be a Pareto optimal solution of the NSD-SIMOP (1), iff there is no feasible point $X \in F$, such that*

$$f(X) \leq f(X^0).$$

- *A feasible point $X^0 \in F$ is said to be a locally Pareto optimal solution of the NSD-SIMOP (1), iff there exists a neighbourhood U of X^0 and there is no other feasible point $X \in F \cap U$, such that*

$$f(X) \leq f(X^0).$$

Let $X^0 \in F$ be an arbitrary feasible point for $(SDP - SIMOP)$. Consider the following problem:

$$\text{(Hybrid method) } \min \sum_{i=1}^{p} \eta_i f_i(X),\ \forall\ \eta_i > 0, \tag{2}$$

subject to $f(X) \leqq f(X^0),\ X \in F$.

Theorem 2.2. *[14, Theorem 4.7] A feasible solution $X^0 \in F$ is an optimal solution of (2) if and only if X^0 is a Pareto optimal solution of $(SDP - SIMOP)$.*

Consider the following optimization problem

$$\text{(SIP)} \min f(X), \qquad (3)$$
$$\text{subject to } g_t(X) \le 0,\ t \in T,\ h_i(X) = 0, i = 1, 2, ..., r.$$

where the function $f : \mathbb{S}^n \to \mathbb{R}$, remaining notions are defined as the previous ones.

Definition 2.3. *(Slater's condition) [9, 22, 28] In the (SIP) Slater's condition holds if for every set of $[\frac{n(n+1)}{2}]+1$ points $t_0, ..., t_{n(n+1)/2} \in T$, there exists a point $X \in F$ such that $g_{t_j}(X) < 0,\ j = 0, ..., n(n+1)/2$.*

Theorem 2.3. *[9, 22, 28] If Slater's condition holds in (SIP), X^0 is a feasible solution of (SIP) and $T(X^0) = \{t_j \in T : g_{t_j}(X^0) = 0\}$. Then, X^0 is an optimal solution of (SIP) if and only if there exists a finite set $T^0 \subset T$ containing at most $n(n+1)/2$ elements such that X^0 is an optimal solution of finite (SIP),*

$$\textit{(finite-SIP)} \min f(X), \qquad (4)$$
$$\textit{subject to } g_{t_j}(X) \le 0,\ \forall\ t_j \in T^0 \cap T(X^0),\ h_i(X) = 0, i = 1, 2, ..., r.$$

Inspired by Bazaraa *et al.* [7], we extend following results from $\mathbb{R}^n$ to $\mathbb{S}^n_+$, as follows:

Theorem 2.4. *If f is a differentiable convex function on $\mathbb{S}^n_+$, then the minimum (global) of f over $\mathbb{S}^n_+$ is attained at point $X^0 \in \mathbb{S}^n_+$ if and only if $0 = \nabla f(X^0)$.*

Theorem 2.5. *If the functions f_1 and f_2 are continuously differentiable, then*

$$\nabla(f_1 + f_2)(X) = \nabla f_1(X) + \nabla f_2(X)\ \forall\ X \in \mathbb{S}^n_+.$$

3. SADDLE POINT AND KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

In this section, we establish saddle point and Karush-Kuhn-Tucker type optimality conditions for considered $(SDP - SIMOP)$ in $\mathbb{S}^n_+$.

Lemma 3.1. *If the functions f and g are convex over X, then the set*

$$A = \{(f(x) + \alpha, g(x) + \beta) : x \in X \text{ and } \alpha,\ \beta \in \mathbb{R}_+\}$$

is also convex.

Proof. Let us choose two points arbitrarily in A say

$$P = (f(x_1) + \alpha_1, g(x_1) + \beta_1),$$
$$Q = (f(x_2) + \alpha_2, g(x_2) + \beta_2).$$

Now, let $t \in [0,1]$ be arbitrary fixed and consider

$$\begin{aligned} tP + (1-t)Q =& (tf(x_1) + (1-t)f(x_2) + t\alpha_1 + (1-t)\alpha_2, \\ & tg(x_1) + (1-t)g(x_2) + t\beta_1 + (1-t)\beta_2). \end{aligned}$$

Then, it is sufficient to prove that $tP + (1-t)Q \in A$. Since f and g are convex functions and X is a convex set, therefore $tx_1 + (1-t)x_2 \in X$ and

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\leqq tf(x_1) + (1-t)f(x_2), \\ g(tx_1 + (1-t)x_2) &\leqq tg(x_1) + (1-t)g(x_2). \end{aligned}$$

So we may write

$$\begin{aligned} tf(x_1) + (1-t)f(x_2) &= f(tx_1 + (1-t)x_2) + h_1(t), \\ tg(x_1) + (1-t)g(x_2) &= g(tx_1 + (1-t)x_2) + h_2(t). \end{aligned}$$

where $h_1(t),\ h_2(t)$ are non-negative real numbers depending on t. Now,

$$\begin{aligned} tP + (1-t)Q =& (f(tx_1 + (1-t)x_2) + (t\alpha_1 + (1-t)\alpha_2) + h_1(t), \\ & g(tx_1 + (1-t)x_2) + (t\beta_1 + (1-t)\beta_2) + h_2(t)), \\ & = (f(tx_1 + (1-t)x_2) + m(t),\ g(tx_1 + (1-t)x_2) + n(t)). \end{aligned}$$

where $m(t) = (t\alpha_1 + (1-t)\alpha_2) + h_1(t) > 0$ and $n(t) = (t\beta_1 + (1-t)\beta_2) + h_2(t) > 0$. So, we have

$$\begin{aligned} tP + (1-t)Q = (f(tx_1 + (1-t)x_2) + m(t),\ g(tx_1 + (1-t)x_2) + n(t)) \in A \\ \text{as } tx_1 + (1-t)x_2 \in X. \end{aligned}$$

Hence, the set A is convex. □

Theorem 3.2. *Let f_i $(i = 1, ..., p)$, g_{t_j}, $t_j \in T$, be convex and let h_i $(i = 1, ..., r)$ be affine functions. If X^0 is an efficient solution where Slater condition holds in (NSD-SIMOP). Then, there exist real numbers $\eta_1^f, ..., \eta_p^f$, $\eta_{t_j}^g$ $(t_j \in T^0 \cap T(X^0))$, $\eta_1^h, ..., \eta_r^h$, not all zero, having the properties:*

$$\sum_{i=1}^{p} \eta_i^f f_i(X^0) \ \leq \ \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X), \qquad (5)$$

$$\forall\ X \in (\mathbb{S}_+^n)_0,\ \eta_i^f \geq 0,\ \eta_{t_j}^g \geq 0,\ \eta_{t_j}^g g_{t_j}(X^0) = 0\ (t_j \in T^0 \cap T(X^0)),$$

where $(\mathbb{S}_+^n)_0 = \cap_{i=1}^{p} Dom f_i \ \cap \bigcap_{t_j \in T^0 \cap T(X^0)} Dom(g_{t_j}).$

Proof. Let X^0 be an efficient solution of $(NSD-SIMOP)$. Then, from Theorem 2.2, X^0 is an optimal solution of problem (2). Since Slater condition holds, then from Theorem 2.3, problem (2) reduces to the following problem

$$\min \sum_{i=1}^{p} \eta_i f_i(X),\ \forall\ \eta_i > 0,\ \text{subject to } f(X) \leqq f(X^0), \tag{6}$$
$$\text{and } X \in F := \{X \in \mathbb{S}^n_+ : g_{t_j}(X) \leq 0,\ t_j \in T^0 \cap T(X^0),\ h_i(X) = 0\}.$$

Now, we proceed with the method of [6, Theorem 3.1] which is as follows: consider the following set

$$B = \Bigg\{ \Bigg(\sum_{i=1}^{p} \eta_i f_i(X) - \sum_{i=1}^{p} \eta_i f_i(X^0) + \alpha_0^f,\ f_1(X) - f_1(X^0) + \alpha_1^f, ..., f_p(X) - f_p(X^0) + \alpha_p^f,$$
$$g_{t_1}(X) + \alpha_{t_1}^g, ..., g_{t_j}(X) + \alpha_{t_j}^g\ (t_j \in T^0 \cap T(X^0)), h_1(X), ..., h_r(X) \Bigg);\ X \in (\mathbb{S}^n_+)_0,$$
$$\alpha_i^f > 0,\ \alpha_{t_j}^g > 0\ \forall\ i \Bigg\}. \tag{7}$$

Since X^0 is an optimal solution of (2) and $\alpha_0^f > 0$, then first component in the set B is always $\sum_{i=1}^{p} \eta_i f_i(X) - \sum_{i=1}^{p} \eta_i f_i(X^0) + \alpha_0^f > 0$. Due to this the set B does not contain the origin and from Lemma 3.1, B is a convex set, also. Non-emptiness of the set B is obvious from the definition by noting that X^0 is in $(\mathbb{S}^n_+)_0$. Since each singleton set is a convex set so the set containing the origin only is also a convex set. Then, from Theorem 2.1, there exists a homogeneous hyperplane, that is, there exist real numbers, not all zero, $\eta_0^f,\ \eta_i^f\ (i = 1, ..., p),\ \eta_{t_j}^g\ (t_j \in T^0 \cap T(x^0)),\ \eta_i^h\ (i = 1, ..., r)$ such that

$$\eta_0^f \Big(\sum_{i=1}^{p} \eta_i f_i(X) - \sum_{i=1}^{p} \eta_i f_i(X^0) + \alpha_0^f\Big) + \sum_{i=1}^{p} \eta_i^f (f_i(X) - f_i(X^0) + \alpha_i^f)$$
$$+ \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g (g_{t_j}(X) + \alpha_{t_j}^g) + \sum_{i=1}^{r} \eta_i^h h_i(X) \geq 0, \tag{8}$$

$\forall\ X \in (\mathbb{S}^n_+)_0,\ \alpha_i^f > 0\ (i = 0, 1, ..., p),\ \alpha_{t_j}^g > 0\ (t_j \in T^0 \cap T(X^0))$. Now, taking $X = X^0$, one $\alpha_{t_j}^g \uparrow \infty$ remaining all $\alpha_{t_j}^g \downarrow 0$, and $\alpha_i^f \downarrow 0$. Then, we get corresponding to that $t_j,\ \eta_{t_j}^g \geq 0$. Continuing this process, we get

$$\eta_0^f \geq 0,\ \eta_i^f \geq 0 \text{ and } \eta_{t_j}^g \geq 0.$$

Thus, relation (8) becomes

$$\eta_0^f \Big(\sum_{i=1}^{p} \eta_i f_i(X) - \sum_{i=1}^{p} \eta_i f_i(X^0)\Big) + \sum_{i=1}^{p} \eta_i^f (f_i(X) - f_i(X^0))$$
$$+ \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g (g_{t_j}(X)) + \sum_{i=1}^{r} \eta_i^h h_i(X) \geq 0,$$

$$\implies \sum_{i=1}^{p} f_i(X)(\eta_0^f \eta_i + \eta_i^f) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X) \geq \sum_{i=1}^{p} f_i(X^0)(\eta_0^f \eta_i + \eta_i^f),$$

$$\implies \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X) \geq \sum_{i=1}^{p} \eta_i^f f_i(X^0), \quad (9)$$

where we again write $\eta_0^f \eta_i + \eta_i^f$ *by* η_i^f. Since X^0 is feasible, therefore

$$\eta_{t_j}^g g_{t_j}(X^0) \leq 0, \ \forall \ t_j \in T^0 \cap T(X^0), \quad (10)$$

substituting $X = X^0$ in inequality (9), we get

$$\sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X^0) \geq 0. \quad (11)$$

Now, from (10) and (11), we have $\eta_{t_j}^g g_{t_j}(X^0) = 0, \ \forall \ t_j \in T^0 \cap T(X^0)$, which completes the proof. □

Example 3.3. *Consider the problem*

$$min\ f(X) = (f_1(X),\ f_2(X),\ f_3(X)),\ subject\ to\ g_t(X) \leq 0,$$

where $f_1(X) = x_1^2,\ f_2(X) = x_2^2,\ f_3(X) = x_3^2,$ *and* $g_t(X) = (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 - 2 + t,\ t \in T = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\},$ $\mathbb{N}$ *is set of all natural numbers.*

Clearly the feasible region is given by $F := \left\{ X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}_+^2 : \ g_t(X) = (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 - 2 + t \leq 0, t \in T = \{\frac{1}{n} : n \in \mathbb{N}\} \right\}$ *and the common domain is*

$$(\mathbb{S}_+^2)_0 = \bigcap_{i=1}^{3} Dom\ f_i(X) \bigcap_{t_j \in T^0 \cap T(X^0)} Dom\ g_{t_j}(X)$$

Since, $X^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is a Pareto optimal solution, so* $T^0 \cap T(X^0) = \{1\}$*. Then, for* $\eta_1^f = 0,\ \eta_2^f > 0,\ \eta_3^f = 0,$ *and* $\eta_1^g = 0$*, the following inequality satisfies*

$$\begin{aligned} &\eta_1^f f_1(X^0) + \eta_2^f f_2(X^0) + \eta_3^f f_3(X^0) \\ &= 0 \leq \eta_1^f X_1^2 + \eta_2^f X_2^2 + \eta_3^f X_3^2 + \eta_1^g[(X_1 - 1)^2 + (X_2 - 1)^2 + (X_3 - 1)^2 - 2] \\ &= \eta_1^f f_1(X) + \eta_2^f f_2(X) + \eta_3^f f_3(X) + \eta_1^g g_1(X), \forall\ X \in (\mathbb{S}_+^n)_0. \end{aligned}$$

Hence, the result is verified.

Now, we construct a function with respect to X^0

$$L_{X^0}(X, \eta^f, \eta^g, \eta^h) = \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X), \quad (12)$$

$\eta^f = (\eta_i^f) \in \mathbb{R}^p$, $\eta^g = (\eta_{t_j}^g) \in \mathbb{R}^{|T^0 \cap T(X^0)|}$ and $\eta^h = (\eta_i^h) \in \mathbb{R}^r$, which is called Lagrangian function.

Remark 3.1. *The necessary conditions (5) with $X^0 \in F$ are equivalent to the fact that the point $(X^0, \eta^f, \eta^g, \eta^h)$ is a saddle point for the Lagrange function (12) on $(\mathbb{S}_+^n)_0 \times \mathbb{R}_+^p \times \mathbb{R}_+^{|T^0 \cap T(X^0)|} \times \mathbb{R}^r$, with respect to minimization on $(\mathbb{S}_+^n)_0$ and maximization on $\mathbb{R}_+^p \times \mathbb{R}_+^{|T^0 \cap T(X^0)|} \times \mathbb{R}^r$, that is,*

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\eta}_i^f f_i(X^0) + \sum_{t_j \in T^0 \cap T(X^0)} \bar{\eta}_{t_j}^g g_{t_j}(X^0) + \sum_{i=1}^{r} \bar{\eta}_i^h h_i(X^0) \\
&\leq \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X) \\
&\implies L_{X^0}(X^0, \bar{\eta}^f, \bar{\eta}^g, \bar{\eta}^h) \leq L_{X^0}(X, \eta^f, \eta^g, \eta^h), \forall\, X \in (\mathbb{S}_+^n)_0, \qquad (13)
\end{aligned}$$

and for every $(X, \bar{\eta}^f, \bar{\eta}^g, \bar{\eta}^h) \in (\mathbb{S}_+^n)_0 \times \mathbb{R}_+^p \times \mathbb{R}_+^{|T^0 \cap T(X^0)|} \times \mathbb{R}^r$.

Remark 3.2. *The necessary optimality conditions (5) with $\eta^f \neq 0$, and $X^0 \in F$ are also sufficient for X^0 to be a Pareto optimal solution to (NSD-SIMOP). If $\eta^f = 0$, then the optimality conditions concern only the constraint functions, without providing any information about the function which has to be minimized. So, there is a natural requirement of certain additional conditions called constraint qualifications which ensure that $\eta^f \neq 0$.*

Definition 3.1. *(Slater's constraint qualification) The Slater's constraint qualification is an instance of a constraint qualification which is easily verifiable in several particular applications. We say Slater's constraint qualification holds if the following two conditions are satisfied. The second condition is optional and is due to presence of equality constraints.*

- *There exist a point $X^* \in F$ such that $g_{t_j}(X^*) < 0, \ \forall\, t_j \in T^0 \cap T(X^0)$.*
- *Condition called interiority condition, if*

$$0 \in int \left\{ (h_1(X), h_2(X), \cdots, h_r(X));\ X \in (\mathbb{S}_+^n)_0 \right\}.$$

Now, we get to a useful new result using Slater's constraint qualification as follows:

Theorem 3.4. *Let* f_i $(i = 1, ..., p)$, g_t $(t \in T)$ *be convex and let* h_i $(i = 1, ..., r)$, *be affine functions such that Slater's condition as well as Slater's constraint qualification are satisfied. Then, the point* X^0 *is a Pareto optimal solution for (NSD-SIMOP) if and only if there exist* $p + |T^0 \cap T(X^0)| + r$ *real numbers* $\eta_1^f, \cdots, \eta_p^f, \eta_{t_j}^g$ $(t_j \in T^0 \cap T(X^0))$, $\eta_1^h, \cdots, \eta_r^h$, *such that*

$$\sum_{i=1}^{p} \eta_i^f f_i(X^0) \leq \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X), \tag{14}$$

$$\text{and } \eta^f \geqq 0,\ \eta^f \neq 0,\ \eta^g \geqq 0,\ \eta_i^g g_i(X^0) = 0\ \forall\ i \in T^0 \cap T(X^0), \forall\ X \in (\mathbb{S}_+^n)_0.$$

Proof Let X^0 be a Pareto optimal solution of (NSD-SIMOP). Then, from Theorem 3.2, there exist $\eta_1^f, \cdots, \eta_p^f, \eta_{t_j}^g$ $(t_j \in T^0 \cap T(X^0))$, $\eta_1^h, \cdots, \eta_r^h$, not all zero, such that (5) holds. If we suppose $\eta^f = 0$, then from Slater's constraint qualification, there exists $X^* \in F$, such that $g_{t_j}(X^*) < 0$ $(\forall\ t_j \in T^0 \cap T(X^0))$, then substituting $X = X^*$ in (5), we get $\sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_i(X^*) \geq 0$. Since $\eta_{t_j}^g \geq 0$, so we must have $\eta_{t_j}^g = 0$ $(\forall\ t_j \in T^0 \cap T(X^0))$, therefore (5) takes the form

$$\sum_{i=1}^{r} \eta_i^h h_i(X) \ \geq\ 0\ \forall\ X \in (\mathbb{S}_+^n)_0,$$

and all components of η^h are not zero, which is a contradiction of the interiority conditions of Slater's constraint qualification. Hence, $\eta^f \neq 0$, that is, some components of η^f are greater than zero.

Conversely, suppose X^0 is not a Pareto optimal solution of (NSD-SIMOP), then there exists $\hat{X}(\neq X^0) \in F$, such that

$$f(\hat{X}) \leq f(X^0). \tag{15}$$

Now, from relation (14) for $\hat{X} \in F$, we have

$$\sum_{i=1}^{p} \eta_i^f f_i(X^0) \ \leq\ \sum_{i=1}^{p} \eta_i^f f_i(\hat{X}),$$

which is a contradiction of inequality (15). Hence, X^0 is a Pareto optimal solution for (NSD-SIMOP).

Theorem 3.5. *Under the hypothesis of Theorem 3.4, an element* $X^0 \in \mathbb{S}_+^n$ *is a Pareto optimal solution of (NSD-SIMOP) if and only if there exist* $\eta^f = (\eta_1^f, \cdots, \eta_p^f) \in \mathbb{R}^p$, $\eta^g \in \mathbb{R}^{|T^0 \cap T(X^0)|}$ *and* $\eta^h = (\eta_1^h, \cdots, \eta_r^h) \in \mathbb{R}^r$ *such that* $(X^0, \eta^f, \eta^g, \eta^h)$ *is a* *saddle point for the Lagrange function on* $(\mathbb{S}^n_+)_0 \times \mathbb{R}^p_+ \times \mathbb{R}^{|T^0 \cap T(X^0)|}_+ \times \mathbb{R}^r$*, that is*

$$\sum_{i=1}^{p} \eta_i^f f_i(X^0) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X^0) + \sum_{i=1}^{r} \eta_i^h h_i(X^0)$$
$$\leq \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X)$$

for all $(X, \eta^f, \eta^g, \eta^h) \in (\mathbb{S}^n_+)_0 \times \mathbb{R}^p_+ \times \mathbb{R}^{|T^0 \cap T(X^0)|}_+ \times \mathbb{R}^r$.

Proof The proof is obvious from Theorem 3.4.

Now, we establish traditional optimality conditions where the differentiability of all the functions are required.

Theorem 3.6. *(Karush-Kuhn-Tucker Conditions) Under the hypotheses of Theorem 3.4, if we assume that the functions* f_i, g_t, h_i *are continuously differentiable real valued functions, then the optimality conditions for* $X^0 \in F$ *are equivalent to the conditions*

$$0 = \sum_{i=1}^{p} \eta_i^f \nabla f_i(X^0) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g \nabla g_{t_j}(X^0) + \sum_{i=1}^{r} \eta_i^h \nabla h_i(X^0). \tag{16}$$

Proof. From equation (9) if $X^0 \in F$ is the minimum point of the function, then

$$\sum_{i=1}^{p} \eta_i^f f_i(X^0) \ \leq \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X). \tag{17}$$

Since $g_{t_j}(X^0) \leq 0$ $(t_j \in T^0 \cap T(X^0))$, $h_i(X^0) = 0 (i = 1, 2, ..., r)$. Therefore, inequality (17) takes the form

$$\sum_{i=1}^{p} \eta_i^f f_i(X^0) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j}(X^0) + \sum_{i=1}^{r} \eta_i^h h_i(X^0)$$
$$\leq \sum_{i=1}^{p} \eta_i^f f_i(X) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{r_i}^g g_{t_j}(X) + \sum_{i=1}^{r} \eta_i^h h_i(X).$$

Now, from Theorem 2.4 at the minimum point of the Lagrangian function, we must have

$$0 = \nabla \left(\sum_{i=1}^{p} \eta_i^f f_i + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g g_{t_j} + \sum_{i=1}^{r} \eta_i^h h_i \right) (X^0).$$

From the previous results and additive property of the gradients, we get

$$0 = \sum_{i=1}^{p} \eta_i^f \nabla f_i(X^0) + \sum_{t_j \in T^0 \cap T(X^0)} \eta_{t_j}^g \nabla g_{t_j}(X^0) + \sum_{i=1}^{r} \eta_i^h \nabla h_i(X^0).$$

Hence, we get the required result. □

Example 3.7. *Consider the following problem*

$$min\ f(X) = (f_1(X), f_2(X), f_3(X)),\ subject\ to$$
$$g_t(X) \leq 0,\ h(X) = 0,\ at\ a\ feasible\ point\ X^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where, $f_1(X) = x_1,\ f_2(X) = x_2,\ f_3(X) = x_3,$ *and* $g_t(X) = x_1 + x_2 + x_3 - 1 + t\ (t \in T = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\})$, *where* $\mathbb{N}$ *is the set of all natural numbers, and* $h(X) = -2x_1 - 3x_2 - x_3,\ withX := \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}^2_+$.

Since X^0 is a Pareto optimal solution for the considered problem as well as satisfying Slater's condition and Slater's constraint qualification also $T^0 \cap T(X^0) = \{1\}$. $\eta^f = (\eta^f_1, \eta^f_2, \eta^f_3) \neq 0,\ \eta^f \geqq 0$ and $\eta^g_1 g_1(X^0) = 0$. Now, taking $\eta^f_1 = 2\eta^h,\ \eta^f_2 = 3\eta^h,\ \eta^f_3 = \eta^h,\ \eta^g_1 = 0$ and $\eta^h > 0$, we have

$$\eta^f_1 \nabla f_1(X^0) + \eta^f_2 \nabla f_2(X^0) + \eta^f_3 \nabla f_3(X^0) + \eta^g_1 \nabla g_1(X^0) + \eta^h \nabla h(X^0)$$
$$= \eta^f_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \eta^f_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \eta^f_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \eta^g_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \eta^h \begin{bmatrix} -2 & -3 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, the result is verified.

Now, we consider the case in which constraints are given by inequalities only, that is,

$$F_1 = \{X \in \mathbb{S}^n_+ : g_t(X) \leq 0,\ \forall\ t \in T\}.$$

Remark 3.3. *Slater's constraint qualification in the absence of equality constraints is as follows: There exists a feasible point* X^* *such that* $g_{t_j}(X^*) < 0,\ \forall\ t_j \in T^0 \cap T(X^0)$.

Corollary 3.8. *Let* $f_1, ..., f_p,\ g_t\ (t \in T)$, *be real, convex and differentiable functions on* $\mathbb{S}^n$. *Further, assume that Slater's constraint qualification in the absence of equality constraints holds. Then, a feasible point* X^0 *is a Pareto optimal solution of the problem (1)* with inequality constraints only *if and only if there exist real numbers* $\eta^f_1, ..., \eta^f_p,\ \eta^g_t\ (t_j \in T^0 \cap T(X^0))$, *such that*

$$\sum_{i=1}^{p} \eta^f_i \nabla f_i(\bar{X}) + \sum_{t_j \in T^0 \cap T(X^0)} \eta^g_{t_j} \nabla g_{t_j}(X^0) = 0, \tag{18}$$

$$\eta^f \geqq 0,\ \eta^f \neq 0,\ \eta^g_{t_j} \geq 0,\ \eta^g_{t_j} g_{t_j}(X^0) = 0,\ \forall\ t_j \in T^0 \cap T(X^0).$$

4. CONCLUSIONS

In this article, we have established saddle point optimality conditions for nonlinear semidefinite semi-infinite multiobjective convex optimization problems

(NSD-SIMOP). We used Slater's condition as well as Slater's constraint qualification from [6] and derived saddle point necessary and sufficient Pareto optimality conditions for the considered problem where multipliers of the objective functions do not vanish simultaneously. We established Karush-Kuhn-Tucker optimality conditions from saddle point optimality conditions for the differentiable case and presented some examples to verify our results.

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