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MOTIVATIONS AND BASIC OF FUZZY, INTUITIONISTIC FUZZY AND NEUTROSOPHIC SETS AND NORMS

Vakeel A. KHAN
Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
vakhanmaths@gmail.com

Umme TUBA
Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
ummetuba01@gmail.com

SK Ashadul RAHAMAN
Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
skashadul072@gmail.com

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Abstract: The proposed article mainly comprises of ideas and motivations that led to the generalization of the usual or the crisp sets. Moving forward we intend to look into how the definitions were modified to suit the meaning of the existing definitions and establish link between them, that is to say that we, after giving formal definitions of the aforementioned sets and norms on these new sets would draw parallel between them and the usual sets and the norm. We will then study some riveting algebraic properties of the same.

Keywords: Fuzzy Set, Fuzzy Norm, Fuzzy Metric, Statistical Metric Space, Intuitionistic Fuzzy Set, Intuitionisitic Fuzzy Norm, Intuitionisitic Fuzzy Metric, Neutrosophic Set, Neutrosophic Norm..

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1. FUZZY SETS AND NORMS

The real world problems may not seem practically real when it comes to assigning mathematical values to events. The application part of the mathematical analysis usually in compliance with classical boolean logic works on the concept of "yes or 1" and "no or 0" . One such example can be considered as the set of all buildings that lie in Mumbai for which the mathematical value 1 can be assigned if the buildings lie within and 0 can be assigned if they lie outside. Formally the stated explanation can be defined in terms of a function commonly known as characteristic function by

$$f_{A}:\mathcal{X}\to\{0,1\} \text{ such that } f_{\mathcal{A}}(\mathbf{x})=\begin{cases}1, & \text{if } \mathbf{x}\in\mathcal{A},\\ 0, & \text{otherwise,}\end{cases}$$

where $\mathcal{A}$ and $\mathcal{X}$ denote the set of all buildings in Mumbai and India respectively. If we relax the condition from buildings lying exactly in Mumbai to the buildings that lie in the vicinity of Mumbai, the problem arises to know the extent to which a building lies near the city; then the concept of assigning numbers just 0 or 1 to the buildings become a little tedious because of the reason that some buildings might lie within the city, some may lie on the boundary where as some will be at the outskirts of the city. Therefore while defining the set of these buildings we cannot neglect the ones that lie on the boundary or outside the boundary. Therefore we need a generalized form of this characterisitc function whose range is not limited to just 0 or 1.

To overcome problems of these sort, L.A. Zadeh in 1965 [1] came up with the concept of fuzzy set. According to his theory, in this case the characteristic function should take values between 0 and 1 along with $\{0,1\}$, which he termed as membership function denoted by $\mu_{\mathcal{A}}$, as a generalisation of characteristic function f_A. Formally he defined the fuzzy set as the pair $\mathcal{A}=(\mathcal{X},\mu_{\mathcal{A}})=\{(\mathbf{x},\mu_{\mathcal{A}}(\mathbf{x})):\mathbf{x}\in\mathcal{X}\}$, where $\mathcal{X}$ is the space of objects and $\mu_{\mathcal{A}}:\mathcal{X}\to[0,1]$ is the membership function such that for each element $\mathbf{x}\in\mathcal{X}$, $\mu_{\mathcal{A}}(\mathbf{x})$ denotes the "grade of membership" of $\mathbf{x}$ in $\mathcal{A}$, which can be interpreted as the extent to which an element may lie in the set. In the above defined fuzzy set, $\mathcal{X}$ is also termed as universe of discourse. A fuzzy set can be characterised in terms of its membership function.

In the fuzzy set $\mathcal{A}=(\mathcal{X},\mu_{\mathcal{A}})$, if $\mu_{\mathcal{A}}(\mathbf{x})=0$, $\mu_{\mathcal{A}}(\mathbf{x})=1$ or $0<\mu_{\mathcal{A}}(\mathbf{x})<1$, we say that $\mathbf{x}$ is not a member, full member or a partial member of the fuzzy set respectively. If for some $\mathbf{x}\in\mathcal{X}$ $\mu_{\mathcal{A}}(\mathbf{x})=1$, then $\mathcal{A}$ is known as a normalized fuzzy set. A classical or a crisp set can be seen as a specific case of the fuzzy set. Following is an example of a real world problem to give an insight into how a fuzzy set looks like.

Example 1. *Let $\mathcal{X}$ be the set of varying temperatures between -50 and 70 in degree celcius and $\mathcal{A}$ be the set of high temperatures. Since the set $\mathcal{A}$ does not specify precisely how "high" is considered high, therefore the values of the membership* *function* $\mu_{\mathcal{A}} : \mathcal{X} \to [0,1]$ *which can be defined as follows, takes different values.*

$$\mu_{\mathcal{A}}(\mathbf{x}) = \begin{cases} 0 & \textit{if} -50 < \mathbf{x} \leq 12, \\ 0.1 & \textit{if} \;\; 12 < \mathbf{x} \leq 20, \\ 0.3 & \textit{if} \;\; 20 < \mathbf{x} \leq 24, \\ 0.5 & \textit{if} \;\; 24 < \mathbf{x} \leq 30, \\ 0.7 & \textit{if} \;\; 30 < \mathbf{x} \leq 40, \\ 0.8 & \textit{if} \;\; 40 < \mathbf{x} \leq 55, \\ 1 & \textit{if} \;\; 55 < \mathbf{x} \leq 70. \end{cases}$$

Correspondingly the fuzzy set $\mathcal{A}$ *can be defined as the ordered pair* $\mathcal{A} = (\mathcal{X}, \mu_{\mathcal{A}}) = \{(-50 < \mathbf{x} \leq 12, 0), (12 < \mathbf{x} \leq 20, 0.1), (20 < \mathbf{x} \leq 24, 0.3), (24 < \mathbf{x} \leq 30, 0.5), (30 < \mathbf{x} \leq 40, 0.7), (40 < \mathbf{x} \leq 55, 0.8), (55 < \mathbf{x} \leq 70, 1)\}$.

The natural instinct of defining operations can be thought of in fuzzy settings just as there are well defined set operations in the classical set theory. The standard notions of union, intersection, containment, complement, emptiness and equality used for the crisp sets are associated with fuzzy sets in terms of membership function that are defined as follows :

- Two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$ are identical $\iff \mu_{\mathcal{A}}(\mathbf{x}) = \mu_{\mathcal{B}}(\mathbf{x}), \;\; \forall\, \mathbf{x} \in \mathcal{X}$.

- A fuzzy set $\mathcal{A}$ is said to be empty $\iff \mu_{\mathcal{A}}(\mathcal{X}) = 0$, $\forall\, \mathbf{x} \in \mathcal{X}$.

- A fuzzy set $\mathcal{B}$ contains the fuzzy set $\mathcal{A}$ i.e., $\mathcal{A} \subseteq \mathcal{B} \iff \mu_{\mathcal{A}}(\mathbf{x}) \leq \mu_{\mathcal{B}}(\mathbf{x})$, $\forall\, \mathbf{x} \in \mathcal{X}$.

- The complement of a fuzzy set $\mathcal{A}$ is symbolised by $\bar{\mathcal{A}}$ and interpreted as $\mu_{\bar{\mathcal{A}}} = 1 - \mu_{\mathcal{A}}$.

- Union of two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ is a fuzzy set $\mathcal{C}$ denoted by $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ interpreted by $\mu_{\mathcal{C}}(\mathbf{x}) = \max\{\mu_{\mathcal{A}}(\mathbf{x}), \mu_{\mathcal{B}}(\mathbf{x})\}$, $\forall\, \mathbf{x} \in \mathcal{X}$.

- Intersection of two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ is a fuzzy set $\mathcal{D}$ denoted by $\mathcal{D} = \mathcal{A} \cap \mathcal{B}$ interpreted by membership function as $\mu_{\mathcal{D}}(\mathbf{x}) = \min\{\mu_{\mathcal{A}}(\mathbf{x}), \mu_{\mathcal{B}}(\mathbf{x})\}$, $\forall\, \mathbf{x} \in \mathcal{X}$.

For more details on fuzzy sets and operations defined on them, we refer to [1].

We present the following geometrical interpretations of intersection, union and complement of fuzzy sets. Consider two fuzzy sets $\mathcal{A}, \mathcal{B}$ and universe of discourse $\mathbb{R}$ as shown in figure 1.

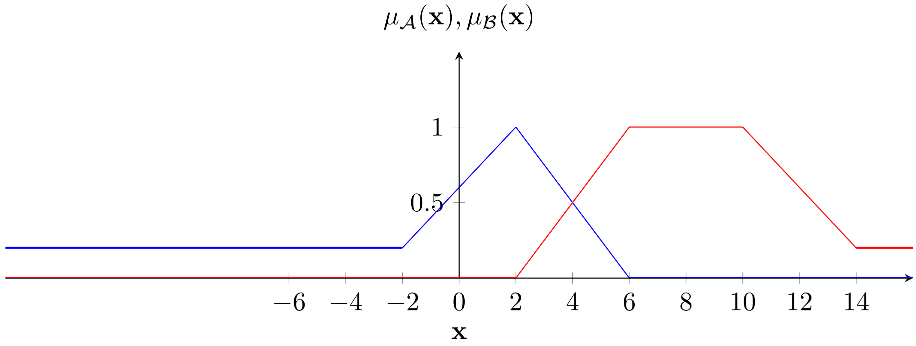

Figure 1: Membership functions of $\mathcal{A}$ and $\mathcal{B}$, $\mu_{\mathcal{A}}$=blue line, $\mu_{\mathcal{B}}$= red line

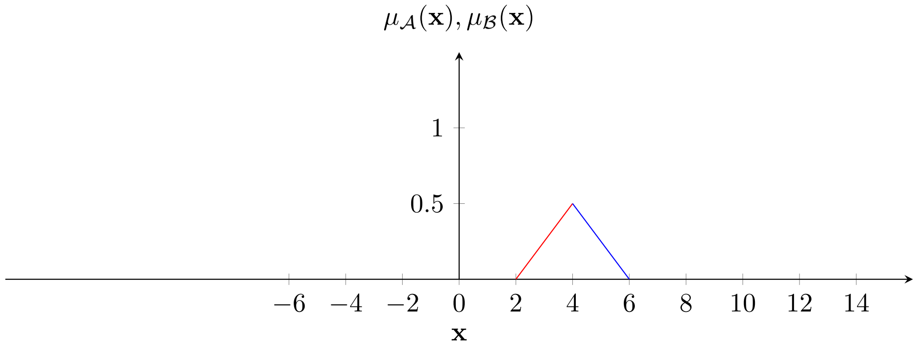

Figure 2: $\mathcal{A} \cap \mathcal{B}$

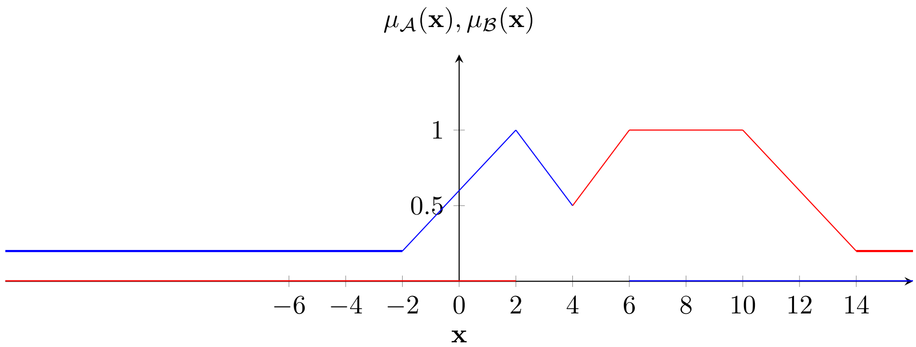

Figure 3: $\mathcal{A} \cup \mathcal{B}$

1.1. ***Fuzzy Metric***

Motivation

In 1906, M. Frechet [2] introduced the notion of distance between two points of an abstract space, which he defined formally for any nonempty crisp set $\mathcal{X}$ as

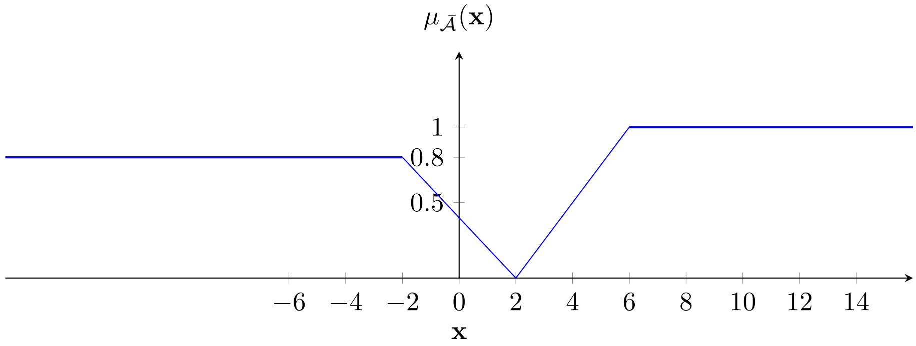

Figure 4: $\bar{\mathcal{A}}$

the function $\delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfying

(i) $\delta(\mathbf{a}, \mathbf{b}) > 0$ for $\mathbf{a} \neq \mathbf{b}$ (positivity),

(ii) $\delta(\mathbf{a}, \mathbf{b}) = 0 \iff \mathbf{a} = \mathbf{b}$ (identification),

(iii) $\delta(\mathbf{a}, \mathbf{b}) = \delta(\mathbf{b}, \mathbf{a})$ (commutativity),

(iv) $\delta(\mathbf{a}, \mathbf{b}) + \delta(\mathbf{b}, \mathbf{c}) \geq \delta(\mathbf{a}, \mathbf{c})$ (triangle inequality),

known as a metric , which denotes the distance between two points $\mathbf{a}$ and $\mathbf{b}$ of $\mathcal{X}$ and is a single real number and the pair $(\mathcal{X}, \delta)$ is known as metric space. If from the above conditions of the metric, we remove the condition of triangle inequality then the distance funtion δ is said to be a semi–metric and the pair $(\mathcal{X}, \delta)$ is a semi–metric space.

The concept of associating a single number to distance between two points may not always be realistically true because of the fact that several discrepancies arise while measuring it multiple times. To avoid such problem we usually take the average of all the measurements which might also be quite tedious. Therefore one can see the metric function as "statistical" rather than a determinate, that is to say for $\mathbf{x} > 0$ one can associate a distribution function $F_{\mathbf{ab}}$ for $\mathbf{a}$ and $\mathbf{b}$ which is interpreted as the probability that the distance between the points is less than $\mathbf{x}$. This concept introduced by K. Menger in 1942 [3] can be fairly seen as a generalization of the usual metric, known as statistical metric.
Following are the notions of certain terms that are required to define the statistical metric.

Definition 2. *Random variable: A random variable usually denoted by X, is a variable in statistics whose possible values depend on the consequence of a certain random occurence.*

Definition 3. *Distribution function: Let X be a random variable, the distribution function of F is a function $F_X : \mathbb{R} \to [0,1]$ such that $F_X(\boldsymbol{x}) = P(X \leq \boldsymbol{x})$, $\forall\ \boldsymbol{x} \in \mathbb{R}$, where $P(X \leq \boldsymbol{x})$ can be interpreted as the probability that $X \leq \boldsymbol{x}$.*

The distribution function F satisfies the following properties:

(i) It is non–decreasing,

(ii) It is left continuous and

(iii) $F_X(\mathbf{x}) \to 0$ when $\mathbf{x} \to -\infty$ and $F_X(\mathbf{x}) \to 1$ when $\mathbf{x} \to \infty$.

One such distribution function $\mathcal{H} : \mathbb{R} \to \{0,1\}$ defined as

$$\mathcal{H}(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} > 0, \\ 0, & \text{if } \mathbf{x} \leq 0, \end{cases} \tag{1}$$

will be used to relate the ordinary metric with the statistical metric.

According to K. Menger, a non–empty set $\mathcal{X}$ is called a statistical semi metric space if for any $\mathbf{a}$ and $\mathbf{b} \in \mathcal{X}$, a real function $F : \mathcal{X} \times \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

(i) $F(\mathbf{a{,}b{;}x}) = 0$ for $\mathbf{x} \leq 0$ and $F(\mathbf{a{,}b{;}x}) \to 1$ when $\mathbf{x} \to \infty$,

(ii) $F(\mathbf{a{,}b{;}}) : \mathbb{R} \to \mathbb{R}$ is a non–decreasing function of $\mathbf{x}$ and left continuous,

(iii) $F(\mathbf{a{,}b{;}x}) = F(\mathbf{b{,}a{;}x})$ and

(iv) $F(\mathbf{a{,}a{;}x}) = 1$ for $\mathbf{x} > 0$,

where the associated real number $F(\mathbf{a{,}b{;}x})$ can be treated as a probability distribution function of the distance between $\mathbf{a}$ and $\mathbf{b}$ i.e., it denotes the probability that the distance between $\mathbf{a}$ and $\mathbf{b}$ is less than $\mathbf{x}$. Another common notion to denote this probability distribution function, which we will use throughout the chapter is $F_{\mathbf{ab}}(\mathbf{x})$.

Following are the explanations of the above mentioned properties:

- For $\mathbf{x} = 0$ or $\mathbf{x} < 0$, the distance between any two points $\mathbf{a{,}b}$ of $\mathcal{X}$ cannot be less than $\mathbf{x}$, therefore the probability that the distance between $\mathbf{a}$ and $\mathbf{b}$ less than $\mathbf{x}$ must be 0 i.e. $F(\mathbf{a{,}b{;}x}) = 0$ for $\mathbf{x} \leq 0$.
  If the real number $\mathbf{x}$ tends to ∞, then for any two points $\mathbf{a}$ and $\mathbf{b}$ of $\mathcal{X}$, the distance between them is always less than $\mathbf{x}$, therefore the probability that the distance between $\mathbf{a}$ and $\mathbf{b}$ less than $\mathbf{x}$ must be 1 i.e., $\lim\limits_{\mathbf{x}\to\infty} F(\mathbf{a{,}b{;}x}) = 1$,
- Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}$ satisfies $\mathbf{x}_1 \leq \mathbf{x}_2$. Fix $\mathbf{a}, \mathbf{b} \in \mathcal{X}$, the probability that the distance between $\mathbf{a}$ and $\mathbf{b}$ less than $\mathbf{x}_1$ is always less than probability that the distance between $\mathbf{a}$ and $\mathbf{b}$ less than $\mathbf{x}_2$ that is, $F(\mathbf{a}, \mathbf{b}; \mathbf{x}_1) \leq F(\mathbf{a}, \mathbf{b}; \mathbf{x}_2)$

whenever $\mathbf{x}_1 \leq \mathbf{x}_2$. Let $\mathbf{x}_0$ and $\mathbf{x} \in \mathbb{R}$ such that $\mathbf{x} \to \mathbf{x}_0$ from the left and fix $\mathbf{a},\mathbf{b} \in \mathcal{X}$. Clearly $\lim_{\mathbf{x}\to\mathbf{x}_0} F(\mathbf{a},\mathbf{b};\mathbf{x}) \leq F(\mathbf{a},\mathbf{b};\mathbf{x}_0)$ i.e. $F(\mathbf{a},\mathbf{b};.)$ is left continuous in $\mathbf{x}$.

- Since $\delta(\mathbf{a},\mathbf{b}) = \delta(\mathbf{b},\mathbf{a})$, therefore $P(\delta(\mathbf{a},\mathbf{b}) < \mathbf{x}) = P(\delta(\mathbf{b},\mathbf{a}) < \mathbf{x})$.
- Since $\delta(\mathbf{a},\mathbf{a}) = 0$ for any $\mathbf{a} \in \mathcal{X}$, the probability that the $\delta(\mathbf{a},\mathbf{a}) < \mathbf{x}$ for any $\mathbf{x} > 0$ is always 1.

It is clearly evident that the conditions of the statistical semi–metric (i), (iii) and (iv) are the generalized form of positivity, commutativity and identification respectively of the usual metric, hence every semi–metric space is a statistical semi–metric space. Since the statistical semi–metric lacks the notion of triangle inequality, therefore one cannot obtain it from the usual metric. Thus there is a need to generalise the triangle inequality to generalize the statistical semi–metric and the concept of new metric thus obtained will be known as statistical metric.

Definition 4. *Let $\mathcal{X}$ be a set such that $\mathcal{X} \neq \phi$ and $\mathcal{F}$ is a function from $\mathcal{X} \times \mathcal{X}$ into the set of distribution functions i.e., for every pair $\boldsymbol{a},\boldsymbol{b} \in \mathcal{X}$ it associates the distribution function $\mathcal{F}(\boldsymbol{a},\boldsymbol{b}) = F_{\boldsymbol{ab}}$. Then the pair $(\mathcal{X}, \mathcal{F})$ is called statisitcal metric space if for every any $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ of $\mathcal{X}$ and real argument $\boldsymbol{x}$ and $\boldsymbol{y}$, the function $F_{\boldsymbol{ab}}$ satisfies the following conditions:*

(i) $F_{\boldsymbol{ab}}(\boldsymbol{x}) = 1$ *for all* $\boldsymbol{x} > 0$ *iff* $\boldsymbol{a} = \boldsymbol{b}$,

(ii) $F_{\boldsymbol{ab}}(\boldsymbol{x}) = 0$ *for* $\boldsymbol{x} \leq 0$,

(iii) $F_{\boldsymbol{ab}}(\boldsymbol{x}) = F_{\boldsymbol{ba}}(\boldsymbol{x}), \quad \forall \ \boldsymbol{x} \in \mathbb{R}$,

(iv) $F_{\boldsymbol{ab}}(\boldsymbol{x}) = 1$ *and* $F_{\boldsymbol{bc}}(\boldsymbol{y}) = 1$, *then* $F_{\boldsymbol{ac}}(\boldsymbol{x}+\boldsymbol{y}) = 1$.

Explanation of the above mentioned properties in (i), (ii) and (iii) have already been provided in statistical semi–metric. The property mentioned in the last point is a generalized form of triangle inequality in statistical sense which can interpreted as; if it is obvious that the probability that the distance between $\mathbf{a},\mathbf{b}$ less than $\mathbf{x}$ is 1 and the probability that the distance between $\mathbf{b},\mathbf{c}$ less than $\mathbf{y}$ is 1, then it is also obvious that the probability that the distance between $\mathbf{a},\mathbf{c}$ less than $\mathbf{x}+\mathbf{y}$ is 1.

The very natural instinct that comes into the mind while studying metric spaces is the concept of a neighbourhood which is defined in terms of the metric. A similar approach can be constructed to define a neighbourhood in the statistical metric space. Scheweizer and Sklar defined the neighbourhood as the set of all points in a non-empty set such that the probability of the distance from a fixed point to the other points less than ϵ is greater than $1 - \mathbf{t}$, where $\epsilon \in (0,1)$ and $\mathbf{t} > 0$. The formal definition was given by them as,

Definition 5. *[4] Let $(\mathcal{X}, \mathcal{F})$ be a statistical metric space and $\boldsymbol{a} \in \mathcal{X}$. For $\epsilon \in (0,1), \boldsymbol{x} > 0$, the $\epsilon, \boldsymbol{x}$ neighbourhod of $\boldsymbol{a}$ is the collection of all $\boldsymbol{b} \in \mathcal{X}$ such that $F_{\boldsymbol{ab}}(\boldsymbol{x}) > 1 - \epsilon$ and is denoted as $N_{\boldsymbol{a}}(\boldsymbol{x}, \epsilon) = \{\boldsymbol{b} : F_{\boldsymbol{ab}}(\boldsymbol{x}) > 1 - \epsilon\}$.*

Here $N_{\mathbf{a}}(\mathbf{x}, \epsilon)$ is the neighbourhood with center at $\mathbf{a}$, radius $\mathbf{x} > 0$ with probability parameter ϵ.

We are well aware with the notion of convergence of a sequence in a metric space being initiated with the help of neighbourhood. Moving on similar lines one can draw parallel and generalize the idea of convergence in a statistical metric space, see [4].

Definition 6. *A sequence $\{\boldsymbol{a_n}\}$ of points in a statistical metric space $(\mathcal{X}, \mathcal{F})$ converges to some $\boldsymbol{a} \in \mathcal{X}$ if and only if, for each $\epsilon > 0$ and $\boldsymbol{x} > 0$, $\exists$ an integer $K_{\boldsymbol{x},\epsilon}$, such that $\forall\ \boldsymbol{n} > K_{\boldsymbol{x},\epsilon}$, $\boldsymbol{a_n} \in N_{\boldsymbol{a}}(\boldsymbol{x}, \epsilon)$ i.e., $F_{\boldsymbol{aa_n}}(\boldsymbol{x}) > 1 - \epsilon$.*

Remark 7. *On a non–empty set $\mathcal{X}$, every usual metric space is a statistical metric space defined by relation $F_{\boldsymbol{ab}}(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x} - \delta(\boldsymbol{a}, \boldsymbol{b}))$, for every $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{X}$.*

Proof. *Suppose δ is a usual metric on $\mathcal{X}$, $\forall\ \boldsymbol{a,b,c} \in \mathcal{X}$,*

- *If $\delta(\boldsymbol{a,b}) = 0 \iff \boldsymbol{a} = \boldsymbol{b}$. Then $F_{\boldsymbol{ab}}(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x}) = \begin{cases} 0, & \text{if } \boldsymbol{x} \leq 0, \\ 1, & \text{if } \boldsymbol{x} > 0, \end{cases}$.*
  Therefore for $\boldsymbol{x} > 0$, $F_{\boldsymbol{ab}} = 1 \iff \boldsymbol{a} = \boldsymbol{b}$.
- *For $\boldsymbol{x} \leq 0$, from the equation (1) it follows that $F_{\boldsymbol{ab}}(\boldsymbol{x}) = 0$.*
- *Let $\delta(\boldsymbol{a,b}) \geq 0$, now $F_{\boldsymbol{ab}}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} > \delta(\boldsymbol{a,b}), \\ 0, & \text{if } \boldsymbol{x} \leq \delta(\boldsymbol{a,b}). \end{cases}$*

  Since $\delta(\boldsymbol{a,b}) \geq 0$, taking $\boldsymbol{x} > \delta(\boldsymbol{a,b}) \geq 0$, we have $F_{\boldsymbol{ab}}(\boldsymbol{x}) = 1$.
- *Let $\delta(\boldsymbol{a,b}) = \delta(\boldsymbol{b,a})$, then*

$$
\begin{aligned}
F_{\boldsymbol{ab}}(\boldsymbol{x}) &= \begin{cases} 1, & \text{if } \boldsymbol{x} > \delta(\boldsymbol{a,b}), \\ 0, & \text{if } \boldsymbol{x} \leq \delta(\boldsymbol{a,b}), \end{cases} \\
&= \begin{cases} 1, & \text{if } \boldsymbol{x} > \delta(\boldsymbol{b,a}), \\ 0, & \text{if } \boldsymbol{x} \leq \delta(\boldsymbol{b,a}), \quad \forall\ \boldsymbol{x} \end{cases} \\
&= F_{\boldsymbol{ba}}(\boldsymbol{x})
\end{aligned}
$$

- *$F_{\boldsymbol{ab}}(\boldsymbol{x}) = 1$ implies $P(\delta(\boldsymbol{a,b}) < \boldsymbol{x}) = 1$ and $F_{\boldsymbol{bc}}(\boldsymbol{y}) = 1$ implies $P(\delta(\boldsymbol{b,c}) < \boldsymbol{y}) = 1$, since by triangle inequality of usual metric δ we have $\delta(\boldsymbol{a,c}) \leq \delta(\boldsymbol{a,b}) + \delta(\boldsymbol{b,c})$, this implies $P(\delta(\boldsymbol{a,c}) < \boldsymbol{x+y}) = 1$. Hence $F_{\boldsymbol{ac}}(\boldsymbol{x+y}) = 1$.*

Note 1. **In statistical metric space, $F_{\mathbf{ab}}(\mathrm{x}) = 1$ does not imply $\mathrm{a} = \mathrm{b}$ for a real argument x in general.**

□

The above mentioned triangle inequality defined by Menger does not include values of the distribution function which are less than 1, thus rendering it to be a weak condition i.e., to say that the triangle inequality becomes a null condition when $F_{\mathbf{ab}}(\mathbf{x}) < 1$. Thus to overcome the restriction imposed on $F_{\mathbf{ab}}$, we need a stronger triangle inequality that does not restrict its value to just 1. This generalized version of the inequality was defined by Menger [3] wherein he stated the existence of a function T_F defined as

$$T_F(\mathbf{t}_1, \mathbf{t}_2) = \inf\{F_{\mathbf{ab}}(\mathbf{x}+\mathbf{y}) : F_{\mathbf{ac}}(\mathbf{x}) \geq \mathbf{t}_1, F_{\mathbf{bc}}(\mathbf{y}) \geq \mathbf{t}_2\} \tag{2}$$

where $T_F : [0,1] \times [0,1] \to [0,1]$ satisfy the following conditions:

- $T_F(\mathbf{t}_1, \mathbf{t}_2) \leq T_F(\mathbf{s}_1, \mathbf{s}_2)$ for $\mathbf{t}_1 \leq \mathbf{s}_1, \mathbf{t}_2 \leq \mathbf{s}_2$,
- $T_F(\mathbf{t}_1, \mathbf{t}_2) = T_F(\mathbf{t}_2, \mathbf{t}_1)$,
- $T_F(1, 1) = 1$ and
- $T_F(F_{\mathbf{ac}}(\mathbf{t}_1), F_{\mathbf{bc}}(\mathbf{t}_2)) \leq F_{\mathbf{ab}}(\mathbf{t}_1 + \mathbf{t}_2)$.

The above mentioned postulates helped Menger come up with the notion of Menger space which incorporates the concept of a function named t–norm whose properties are derived from those of boolean logic and fuzzy logic which are formally defined as follows.

1.2. *Boolean logic & Fuzzy logic*

A form of algebra involving operations based on the truth values of the variables yes and no, denoted by 1 and 0, respectively, is known as boolean logic. The primary operations concerning boolean logic are mainly conjunction, disjunction and negation.

Let A be a proposition and $\mathcal{P}$ be the collection of statements that may or may not satisfy A, then the truth value $\mathcal{T} : \mathcal{P} \to \{0, 1\}$ is a function defined by

$$\mathcal{T}(p) = \begin{cases} 1, & \text{if } p \text{ satisfies } A \ , \\ 0, & \text{otherwise} \ . \end{cases} \tag{3}$$

The basic operations of Boolean logic are as follows:

- Conjunction (AND), denoted by the symbol $\wedge$, is given by $\mathcal{T}(p_1 \wedge p_2) = \min\{\mathcal{T}(p_1), \mathcal{T}(p_2)\}$.
- Disjunction (OR), denoted by the symbol $\vee$, is given by $\mathcal{T}(p_1 \vee p_2) = \max\{\mathcal{T}(p_1), \mathcal{T}(p_2)\}$.
- Negation (NOT), denoted by the symbol $\neg$, is given by $\neg\mathcal{T}(p_1) = 1 - p_1$.

where p_1 and p_2 are two statements whose truth values are $\mathcal{T}(p_1)$ and $\mathcal{T}(p_2)$ respectively. Following is the truth table of statements p_1, p_2, conjunction, disjunction and their negations.

$\mathcal{T}(p_1)$	$\mathcal{T}(p_2)$	$\mathcal{T}(p_1 \wedge p_2)$	$\mathcal{T}(p_1 \vee p_2)$	$\mathcal{T}(\neg p_1)$	$\mathcal{T}(\neg p_2)$
1	1	1	1	0	0
0	1	0	1	1	0
1	0	0	1	0	1
0	0	0	0	1	1

Properties of Boolean logic

The following are the basic properties of boolean operators $\wedge, \vee$ and $\neg$

- Commutativity of $\wedge$ and $\vee$:
  $p_1 \wedge p_2 = p_2 \wedge p_1$ and $p_1 \vee p_2 = p_2 \vee p_1$.
- Associativity of $\wedge$ and $\vee$:
  $p_1 \wedge (p_2 \wedge p_3) = (p_1 \wedge p_2) \wedge p_3$ and $p_1 \vee (p_2 \vee p_3) = (p_1 \vee p_2) \vee p_3$.
- Distributivity of $\wedge$ over $\vee$:
  $p_1 \wedge (p_2 \vee p_3) = (p_1 \wedge p_2) \vee (p_1 \wedge p_3)$.
- Distributivity of $\vee$ over $\wedge$:
  $p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_1 \vee p_3)$.
- Identity of $\wedge$ and $\vee$:
  $p_1 \wedge 1 = p_1$ and $p_1 \vee 0 = p_1$.
- Annhilator of $\wedge$ and $\vee$:
  $p_1 \wedge 0 = 0$ and $p_1 \vee 1 = 1$.
- Idempotence of $\wedge$ and $\vee$:
  $p_1 \wedge p_1 = p_1$ and $p_1 \vee p_1 = p_1$.
- Absorption of $\wedge$ and $\vee$:
  $p_1 \wedge (p_1 \vee p_2) = p_1$ and $p_1 \vee (p_1 \wedge p_2) = p_1$.
- Complementation of $\wedge$ and $\vee$:
  $p_1 \wedge \neg p_1 = 0$ and $p_1 \wedge \neg p_1 = 1$.
- De Morgan's Law:
  $\neg p_1 \wedge \neg p_2 = \neg(p_1 \vee p_2)$ and $\neg p_1 \vee \neg p_2 = \neg(p_1 \wedge p_2)$.

Fuzzy logic

As observed in crisp logic, any statement is either true or false with their truth values 1 and 0 respectively whereas fuzzy logic demands the statements to be maybe partially true, therefore fuzzy logic is a form of multi-valued logic in which

the truth value of variables may be any real number in $[0,1]$.
Then the truth values $\mu_{\mathcal{A}}$, a generalization of characteristic function $\mathcal{T}$, is identified by the fuzzy set in terms of its membership function, $\mu_{\mathcal{A}} : \mathcal{P} \to [0,1]$.

The basic operations of Fuzzy logic are as follows:

- Conjunction (AND), denoted by the symbol $\wedge$, is given by $\mu_{\mathcal{A}}(p_1) \wedge \mu_{\mathcal{A}}(p_2) = \min\{\mu_{\mathcal{A}}(p_1), \mu_{\mathcal{A}}(p_2)\}$.
- Disjunction (OR), denoted by the symbol $\vee$, is given by $\mu_{\mathcal{A}}(p_1) \vee \mu_{\mathcal{A}}(p_2) = \max\{\mu_{\mathcal{A}}(p_1), \mu_{\mathcal{A}}(p_2)\}$.
- Negation (NOT), denoted by the symbol $\neg$, is given by $\neg\mu_{\mathcal{A}} = 1 - \mu_{\mathcal{A}}$.

These operations can be defined in terms of function as follows:

Definition 8. *An operation $\mathcal{C} : [0,1] \times [0,1] \to [0,1]$ is said to be a fuzzy conjunction if $\mathcal{C}(1,1) = 1, \mathcal{C}(0,0) = \mathcal{C}(0,1) = \mathcal{C}(1,0) = 0$, $\mathcal{C}(\mathbf{x},\mathbf{y}) \leq \mathcal{C}(\mathbf{x},\mathbf{z})$ if $\mathbf{y} \leq \mathbf{z}$ and $\mathcal{C}(\mathbf{u},\mathbf{v}) \leq \mathcal{C}(\mathbf{w},\mathbf{v})$ if $\mathbf{u} \leq \mathbf{w}$.*

The zero element for the fuzzy conjunction is 0.

Definition 9. *An operation $\mathcal{D} : [0,1] \times [0,1] \to [0,1]$ is said to be a fuzzy disjunction if $\mathcal{D}(0,0) = 0, \mathcal{D}(1,1) = \mathcal{D}(0,1) = \mathcal{D}(1,0) = 1$, $\mathcal{D}(\mathbf{x},\mathbf{y}) \leq \mathcal{D}(\mathbf{x},\mathbf{z})$ if $\mathbf{y} \leq \mathbf{z}$ and $\mathcal{D}(\mathbf{u},\mathbf{v}) \leq \mathcal{D}(\mathbf{w},\mathbf{v})$ if $\mathbf{u} \leq \mathbf{w}$.*

The zero element for the fuzzy disjunction is 1.

To generalize the triangle inequality from the statistical metric space to Menger space, we define the following concepts.

Definition 10. *[4] Let $* : [0,1] \times [0,1] \longrightarrow [0,1]$ be a binary operation. Then $*$ is known as a t-norm if for each $0 \leq \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \leq 1$,*
(i) $\mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a}$ and $\mathbf{a} * (\boldsymbol{b} * \boldsymbol{c}) = (\mathbf{a} * \mathbf{b}) * \mathbf{c}$,
(ii) $\mathbf{a} * 1 = \mathbf{a}$,
(iii) If $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$ then $\mathbf{a} * \mathbf{b} \leq \mathbf{c} * \mathbf{d}$.

A t-norm is known as a continuous t-norm if it satisfies contintuity as a function. The idempotents of a t-norm are those $\mathbf{a} \in [0,1]$ satisfying $\mathbf{a} * \mathbf{a} = \mathbf{a}$.

Definition 11. *[4] Let $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ be a binary operation. Then $\diamond$ is known as a t-conorm if for each $0 \leq \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \leq 1$,*
(i) $\mathbf{a} \diamond \mathbf{b} = \mathbf{b} \diamond \mathbf{a}$ and $\mathbf{a} \diamond (\boldsymbol{b} \diamond \boldsymbol{c}) = (\mathbf{a} \diamond \mathbf{b}) \diamond \mathbf{c}$,
(ii) $\mathbf{a} \diamond 0 = \mathbf{a}$,
(iii) If $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$ then $\mathbf{a} \diamond \mathbf{b} \leq \mathbf{c} \diamond \mathbf{d}$.

A t-conorm is called a continuous t-conorm if it satisfies contintuity as a function. The idempotents of a t-conorm are those $\mathbf{a} \in [0,1]$ satisfying $\mathbf{a} \diamond \mathbf{a} = \mathbf{a}$.

Triangular norms and conorms are operations which extend the ideas of the two valued logical conjunction and logical disjunction to fuzzy logic. The monotonicity property of the t-norm and t-conorm ensures that the degree of truth of conjunction and disjunction does not decrease when the truth values of the conjuncts and the disjuncts increase respectively. Their repective identities 1 and 0 corresponds to the explication as that of truth and false. The continuity of t-norm and t-conorm suggests that minor changes in truth values of conjuncts and disjuncts changes the truth values of conjunction and disjunction microscopically. They can be seen as dual notions of each other.
t-norm and t-conorm are employed to construct the intersection and union of fuzzy sets or as a basis for fuzzy set operations.

Example 12. *For* $0 \leq \mathbf{a}, \mathbf{b} \leq 1$, $\mathbf{a} * \mathbf{b} = \min\{\mathbf{a}, \mathbf{b}\}, \mathbf{a} * \mathbf{b} = \mathbf{a}.\mathbf{b}$ *and their duals* $\mathbf{a} \diamond \mathbf{b} = \max\{\mathbf{a}, \mathbf{b}\}, \mathbf{a} \diamond \mathbf{b} = \mathbf{a} + \mathbf{b} - \mathbf{ab}$ *are examples of continous t-norm and t-conorm respectively.*

Example 13.

$$\mathbf{a} * \mathbf{b} = \begin{cases} \mathbf{a} & \textit{if} \;\; \mathbf{b} = 1 \;, \\ \mathbf{b} & \textit{if} \;\; \mathbf{a} = 1 \;, \\ 0 & \textit{if} \;\; \mathbf{a} \neq 1, \mathbf{b} \neq 1 \end{cases}$$

and its dual

$$\mathbf{a} \diamond \mathbf{b} = \begin{cases} \mathbf{a} & \textit{if} \;\; \mathbf{b} = 0 \;, \\ \mathbf{b} & \textit{if} \;\; \mathbf{a} = 0 \;, \\ 1 & \textit{if} \;\; \mathbf{a} \neq 0, \mathbf{b} \neq 0 \end{cases}$$

is an example of a right continuous t-norm and t-conorm.

Example 14.

$$\mathbf{a} * \mathbf{b} = \begin{cases} \min\{\mathbf{a}, \mathbf{b}\} & \textit{if} \; 1 < \mathbf{a} + \mathbf{b} \;, \\ 0 & \textit{if} \; 1 \geq \mathbf{a} + \mathbf{b} \end{cases}$$

and its dual

$$\mathbf{a} \diamond \mathbf{b} = \begin{cases} \max\{\mathbf{a}, \mathbf{b}\} & \textit{if} \; 1 < \mathbf{a} + \mathbf{b} \;, \\ 1 & \textit{if} \; 1 \geq \mathbf{a} + \mathbf{b} \end{cases}$$

is an example of a left continuous t-norm and t-conorm.

Now using t-norm, the generalized concept of statistical metric space is described as below:

Definition 15. *Let $\mathcal{X} \neq \phi$ and $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to F$ where F is the set of distribution functions i.e. for every pair $\mathbf{a}, \mathbf{b} \in \mathcal{X}$ it associates the distribution function $\mathcal{F}(\mathbf{a}, \mathbf{b}) = F_{\mathbf{ab}}$. Then the pair $(\mathcal{X}, \mathcal{F})$ is called Menger metric space if for every any $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of $\mathcal{X}$ and real arguments $\mathbf{x}$ and $\mathbf{y}$, the function $F_{\mathbf{ab}}$ complies with the following conditions:*

(i) $F_{\mathbf{ab}}(\mathbf{x}) = 1$ *for all* $\mathbf{x} > 0$ *iff* $\mathbf{a} = \mathbf{b}$,

(ii) $F_{\mathbf{ab}}(\mathbf{x}) = 0$ *for* $\mathbf{x} \leq 0$,

(iii) $F_{\mathbf{ab}}(\mathbf{x}) = F_{\boldsymbol{ba}}(\mathbf{x}) \ \ \forall \ \ \mathbf{x} \in \mathbb{R}$,

(iv) $F_{\mathbf{ab}}(\mathbf{t}_1 + \mathbf{t}_2) \geq F_{\mathbf{ac}}(\mathbf{t}_1) * F_{\mathbf{bc}}(\mathbf{t}_2)$.

We have seen earlier in this chapter how the concept of associating a single number to the distance between two points in usual metric space was generalized to statistical metric space in terms of distribution function. An another approach was defined to measure this distance in the settings of fuzzy notion which incorporated the idea of assigning values from $[0, 1]$ to multiple statements proclaiming something concerning the distance. The idea lies behind the concept of finding the degree of truth for which the underlying distance is smaller than a given real number.

By lemma 2 page number 340 [5], any metric δ on a set $\mathcal{X} \neq \phi$, is uniquely determined by relation defined by $R_\delta \subset \mathcal{X} \times \mathcal{X} \times \mathbb{R}$ so that $\forall\ \mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $\mathbf{t} \in \mathbb{R}$, the relation $R_\delta(\mathbf{x}, \mathbf{y}, \mathbf{t})$ is valid iff $\delta(\mathbf{x}, \mathbf{y}) < \mathbf{t}$.

In [5], the authors came forward with the notion of fuzzy metric space to which George and Veeramani made a slight modification [6] and defined it as following:

Definition 16. *Let $\mathcal{X}$ be set such that $\mathcal{X} \neq \phi$, $*$ is a continuous t–norm and M is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times [0, \infty)$, then the 3–tuple $(\mathcal{X}, M, *)$ is called a fuzzy metric space if for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$ and $\mathbf{t}, \mathbf{s} \in [0, \infty)$, if M fulfils the following requirements:*

(i) $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) > 0$ *and* $M(\mathbf{x}, \mathbf{y}, 0) = 0$,

(ii) $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) = 1, \forall\ \mathbf{t} > 0 \iff \mathbf{x} = \mathbf{y}$,

(iii) $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) = M(\mathbf{y}, \mathbf{x}, \mathbf{t})$,

(iv) $M(\mathbf{x}, \mathbf{z}, \mathbf{t} + \mathbf{s}) \geq M(\mathbf{x}, \mathbf{y}, \mathbf{t}) * M(\mathbf{y}, \mathbf{z}, \mathbf{s})$,

(v) $M(\mathbf{x}, \mathbf{y}, .)$ *is continuous* .

Here M is known as a fuzzy metric in the fuzzy metric space $(\mathcal{X}, M, *)$. The fuzzy metric space is known as Menger fuzzy metric space if $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) \to 1$ when $\mathbf{t} \to \infty$ where $M(\mathbf{x}, \mathbf{y}, \mathbf{t})$ becomes $F_{\mathbf{xy}}(\mathbf{t})\ \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $\mathbf{t} > 0$. In the sequel we will use the notion of Menger fuzzy metric space.

It is quite evident that the properties mentioned in (i), (ii) and (iii) are generalised forms of non–negativity, identity and symmetry respectively of usual

metric. These generalisations do not suggest the properties to be fuzzy but instead focuses on the fuzziness of the distance parameter ($\mathbf{t}$). The fuzzy metric transforms into a fuzzy pseudo–metric when the condition in (ii) is replaced by $M(\mathbf{x},\mathbf{x},\mathbf{t}) = 1 \ \forall \ x \ \in \mathcal{X}$ and $\mathbf{t} > 0$. Condition (iv) is a generalization of the triangle inequality which can be interpreted as, if the degree of our belief that the distance between $\mathbf{x}$ and $\mathbf{y}$ less than $\mathbf{t}$ is certain and simultaneously the degree of our belief that the distance between $\mathbf{y}$ and $\mathbf{z}$ less than $\mathbf{s}$ is certain, then we are certain that the degree of our belief that the distance between $\mathbf{x}$ and $\mathbf{z}$ will be less than $\mathbf{t}+\mathbf{s}$. We know that a function is continuous if and only if it is both right and left continuous. Left continuity can be thought of as degree of belief of the distance between $\mathbf{x}$ and $\mathbf{y}$ strictly less than $\mathbf{t}$ and right continuity can be thought of as degree of belief of the distance between $\mathbf{x}$ and $\mathbf{y}$ strictly less than or equal to $\mathbf{t}$. It is obvious that metric space implies fuzzy metric space.
The condition where the distance parameter $\mathbf{t}$ tends to infinity, ensures the finiteness of the distance. If we understand the real function $M(\mathbf{x},\mathbf{y},\mathbf{t})$ as a degree of certainity that the distance $\delta(\mathbf{x},\mathbf{y})$ is less than $\mathbf{t}$, it is obvious that for any $\mathbf{s} \geq \mathbf{t}$, the inequality $M(\mathbf{x},\mathbf{y},\mathbf{s}) \geq M(\mathbf{x},\mathbf{y},\mathbf{t})$ holds. Therefore it can be stated that $M(\mathbf{x},\mathbf{y},.)$ is a non–decreasing function of $\mathbf{t}$.

Example 17. *Let $(\mathcal{X},\delta)$ be a metric space and $\mathbf{a}*\mathbf{b} = \min\{\mathbf{a},\mathbf{b}\} \ \forall \ \mathbf{a},\mathbf{b} \in [0,1]$. Define $M(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\mathbf{t}}{\mathbf{t}+\delta(\mathbf{x},\mathbf{y})}$. Then $(\mathcal{X},M,*)$ is a fuzzy metric space which is also known as standard fuzzy metric space induced by δ.*

Proof. *For all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathcal{X}$ and $\mathbf{s},\mathbf{t} > 0$ we have,*

- *Since $\delta(\mathbf{x},\mathbf{y}) \geq 0$ therefore $M(\mathbf{x},\mathbf{y},\mathbf{t}) > 0$ and for $\mathbf{t} = 0$ we have $M(\mathbf{x},\mathbf{y},\mathbf{t}) = 0$.*
- *Let $\mathbf{x} = \mathbf{y}$ and $\mathbf{t} > 0 \implies \delta(\mathbf{x},\mathbf{y}) = 0$, therefore $M(\mathbf{x},\mathbf{y},\mathbf{t}) = 1$. Conversely suppose $M(\mathbf{x},\mathbf{y},\mathbf{t}) = 1$ for $\mathbf{t} > 0$ thus $\frac{t}{\mathbf{t}+\delta(\mathbf{x},\mathbf{y},)} = 1 \implies \delta(\mathbf{x},\mathbf{y}) = 0$.*
- *Since $\delta(\mathbf{x},\mathbf{y}) = \delta(\mathbf{y},\mathbf{x})$, therefore $M(\mathbf{x},\mathbf{y},\mathbf{t}) = M(\mathbf{y},\mathbf{x},\mathbf{t})$.*
- *Let $\min\{M(\mathbf{x},\mathbf{y},\mathbf{t}), M(\mathbf{y},\mathbf{z},\mathbf{s})\} = M(\mathbf{x},\mathbf{y},\mathbf{t})$ i.e.,*

$$M(\mathbf{x},\mathbf{y},\mathbf{t}) \leq M(\mathbf{y},\mathbf{z},\mathbf{s}) \implies \frac{\mathbf{t}}{\mathbf{t}+\delta(\mathbf{x},\mathbf{y})} \leq \frac{\mathbf{s}}{\mathbf{s}+\delta(\mathbf{y},\mathbf{z})} \implies \frac{\delta(\mathbf{y},\mathbf{z})}{\mathbf{s}} \leq \frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{t}}. \tag{4}$$

Now if $\mathbf{x} = \mathbf{z}$ we have $\mathbf{t} \leq \mathbf{s}$. Thus for $\mathbf{x} \neq \mathbf{z}$, we have

$$\begin{aligned} 2\frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{t}} &\geq \frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{t}} + \frac{\delta(\mathbf{y},\mathbf{z})}{\mathbf{s}} \\ &\geq \frac{2}{\mathbf{t}+\mathbf{s}}(\delta(\mathbf{x},\mathbf{y}) + \delta(\mathbf{y},\mathbf{z})) \\ &\geq 2\frac{\delta(\mathbf{x},\mathbf{z})}{\mathbf{t}+\mathbf{s}}. \end{aligned}$$

$\implies \mathbf{t}\delta(\mathbf{x},\mathbf{z}) \leq (\mathbf{t}+\mathbf{s})\delta(\mathbf{x},\mathbf{y}) \implies 1+\frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{t}} \geq 1+\frac{\delta(\mathbf{x},\mathbf{z})}{\mathbf{t}+\mathbf{s}}$. *Therefore* $\frac{\mathbf{t}}{\mathbf{t}+\delta(\mathbf{x},\mathbf{y})} \leq \frac{\mathbf{t}+\mathbf{s}}{(\mathbf{t}+\mathbf{s})+\delta(\mathbf{x},\mathbf{z})}$ *which implies* $M(\mathbf{x},\mathbf{y},\mathbf{t}) \leq M(\mathbf{x},\mathbf{z},\mathbf{t}+\mathbf{s})$. *The statement can be verified on similar grounds if the above mentioned minimum is* $M(\mathbf{y},\mathbf{z},\mathbf{s})$.

- *It is evident that* $M(\mathbf{x},\mathbf{y},.)$ *is continuous in* $\mathbf{t}$.

□

Example 18. *Let* $(\mathcal{X},\delta)$ *be a metric space and* $\mathbf{a}*\mathbf{b}=\mathbf{ab}\ \forall\ \mathbf{a},\mathbf{b}\in[0,1]$. *Define* $M(\mathbf{x},\mathbf{y},\mathbf{t}) = \left[\exp\left(\frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{t}}\right)\right]^{-1}\ \forall\ \mathbf{x},\mathbf{y}\in\mathcal{X}$ *and* $\mathbf{t}>0$. *Then* $(\mathcal{X},M,*)$ *is a fuzzy metric space.*

Just as we mentioned the concepts of open ball and convergence in statistical metric space prior in the chapter; we move forward and define same concepts on fuzzy metric space which was defined by George and Veeramani.

Definition 19. *[6] An open ball* $B(\mathbf{x},\mathbf{r},\epsilon)$ *with center* $\mathbf{x}\in\mathcal{X},\mathbf{r}>0$ *with respect to parameter of fuzziness* $0<\epsilon<1$ *in a fuzzy metric space* $(\mathcal{X},M,*)$ *is defined as* $B(\mathbf{x},\mathbf{r},\epsilon)=\{\mathbf{z}\in\mathcal{X}:M(\mathbf{x},\mathbf{z},\mathbf{r})>1-\epsilon\}$.

Definition 20. *[6] Consider the fuzzy metric space* $(\mathcal{X},M,*)$. *A sequence* $\{\mathbf{x_n}\}\in\mathcal{X}$ *converges to some* $\mathbf{x}\in\mathcal{X}$ *if for all* $0<\epsilon<1$ *and* $\mathbf{t}>0$, $M(\mathbf{x_n},\mathbf{x},\mathbf{t})>1-\epsilon$.

We suggest the readers to further look into the topological properties of fuzzy metric space defined by George and Veeramani in [6].

2. FUZZY NORM

The primary motivation to develop the norm was to measure the length of a vector in a vector space $\mathcal{X}$. It can be thought of as a function on the vector space formally defined as $f:\mathcal{X}\to\mathbb{R}$ satisfying some standard properties. To be precise in order to visualise the concept, one can imagine it to be as the distance of a vector from the origin.

Definition 21. *Let* $\mathcal{X}$ *be a linear space over* K. *A functional* $||.||:\mathcal{X}\to K$ *is called a norm if* $\forall\ \mathbf{x},\mathbf{y}\in\mathcal{X}$ *and* $\lambda\in K$, *it fulfils the following requirements:*

- $||\mathbf{x}||\geq 0$,
- $||\mathbf{x}+\mathbf{y}||\leq||\mathbf{x}||+||\mathbf{y}||$,
- $||\lambda\mathbf{x}||=|\lambda|||\mathbf{x}||$,
- $||\mathbf{x}||=0$ *iff* $\mathbf{x}=0$.

Consequently the pair $(\mathcal{X},||.||)$ *is called a normed space.*

A natural generalization of the classical norm can be defined on parallel lines in fuzzy setting, keeping in mind the original properties of the norm are intact. The key difference is that in the classical sense, the norm is a non–negative real valued function which defines the length of a vector whereas fuzzy norm denotes the degree of truth of the length of a vector or the degree of truth that the distance of a vector from origin which lies in $[0,1]$.

Primarily introduced by Katsaras [7] ; fuzzy normed spaces(FNS) was then studied by various authors with Felbin [9] being the one to introduce the concept of fuzzy norm on a vector space by associating an element of the underlying vector space with a fuzzy real number. In 2003, Bag and Samanta [8] established the base of an even general fuzzy norm. To generalise this concept of fuzzy norm , Saadati and Vaezpour brought forward the definition of fuzzy normed space.

Definition 22. *[10] Let $\mathcal{X}$ be a linear space over $K(\mathbb{R}$ or $\mathbb{C})$ and $*$ is a continuous t-norm, then a fuzzy subset, $N : \mathcal{X} \times [0,\infty) \to [0,1]$ is said to be a fuzzy norm on the space $\mathcal{X}$ if for all $\mathbf{x},\mathbf{y} \in \mathcal{X}$ and $\mathbf{s},\mathbf{t} \in [0,\infty)$, the following conditions hold;*

(i) $N(\mathbf{x},\mathbf{t}) > 0$ *and* $N(\mathbf{x},,0) = 0$,

(ii) $N(\mathbf{x},\mathbf{t}) = 1, \ \forall\ \mathbf{t} > 0 \iff \mathbf{x} = 0$,

(iii) $N(\lambda\mathbf{x},\mathbf{t}) = N\big(\mathbf{x}, \frac{\mathbf{t}}{|\lambda|}\big)\ \forall\ \lambda > 0$,

(iv) $N(\mathbf{x},\mathbf{t}) * N(\mathbf{y},\mathbf{s}) \leq N(\mathbf{x}+\mathbf{y},\mathbf{t}+\mathbf{s})$,

(v) $N(\mathbf{x},.) : [0,\infty) \to [0,1]$ *is left continuous and*

(vi) $N(\mathbf{x},\mathbf{t}) \to 1$ *when* $\mathbf{t} \to \infty$.

The three tuple $(\mathcal{X}, N, *)$ is said to be fuzzy normed space.
Condition (iii) can be interpreted as; for $\alpha > 0$, $N(\alpha\mathbf{x},\mathbf{t})$ denotes the degree of truth that $||\alpha\mathbf{x}|| < \mathbf{t}$. Now, $||\alpha\mathbf{x}|| < \mathbf{t} \implies ||\mathbf{x}|| < \frac{\mathbf{t}}{|\alpha|}$. Therefore for $\alpha > 0$, $N(\alpha\mathbf{x},\mathbf{t})$ can also be stated as the degree of truth that length of vector $\mathbf{x}$ is less than $\frac{\mathbf{t}}{|\alpha|}$.

Remark 23. *Clearly every normed space is a fuzzy normed space.*

Every fuzzy norm N on $\mathcal{X}$ induces a fuzzy metric M given by the relation $M(\mathbf{x},\mathbf{y},\mathbf{t}) = N(\mathbf{x}-\mathbf{y},\mathbf{t})$ for all $\mathbf{x},\mathbf{y},\in \mathcal{X}$.

Example 24. *Suppose $(\mathcal{X}, ||.||)$ be a normed space. Define $\mathbf{a} * \mathbf{b} = \mathbf{a}.\mathbf{b}$, $\ \forall\ 0 \leq \mathbf{a},\mathbf{b} \leq 1$ and*

$$N(\mathbf{x},\mathbf{t}) = \frac{\mathbf{k}\mathbf{t}^{\mathbf{n}}}{\mathbf{k}\mathbf{t}^{\mathbf{n}} + \mathbf{m}||\mathbf{x}||} \text{ for any } \mathbf{k},\mathbf{m},\mathbf{n} \in \mathbb{N}.$$

Clearly $(\mathcal{X}, N, *)$ is a fuzzy normed space.

Example 25. *Consider the normed space* $(\mathcal{X}, ||.||)$. *Define* $\mathbf{a} * \mathbf{b} = \mathbf{a}.\mathbf{b}$, $\forall\ 0 \leq \mathbf{a}, \mathbf{b} \leq 1$ *and*

$$N(\mathbf{x}, \mathbf{t}) = \frac{1}{\left(\exp \frac{||\mathbf{x}||}{\mathbf{t}}\right)} \forall\ \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{t} > 0.$$

Clearly $(\mathcal{X}, N, *)$ *defines a fuzzy normed space.*

Proof. *We only verify the triangle inequality since the other conditions follow easily.*
Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *and* $\mathbf{t} > 0$, *then*

$$\begin{aligned} N(\mathbf{x}, \mathbf{t}) * N(\mathbf{y}, \mathbf{s}) &= \left(\exp \frac{||\mathbf{x}||}{\mathbf{t}}\right)^{-1} . \left(\exp \frac{||\mathbf{y}||}{\mathbf{s}}\right)^{-1} \\ &= \left(\exp \frac{||\mathbf{x}||}{\mathbf{t}} + \frac{||\mathbf{y}||}{\mathbf{s}}\right)^{-1}. \end{aligned}$$

Since

$$\frac{||\mathbf{x}||}{\mathbf{t}} + \frac{||\mathbf{y}||}{\mathbf{s}} \geq \frac{||\mathbf{x}||}{\mathbf{t}+\mathbf{s}} + \frac{||\mathbf{y}||}{\mathbf{t}+\mathbf{s}} \geq \frac{||\mathbf{x}+\mathbf{y}||}{\mathbf{t}+\mathbf{s}}.$$

Therefore $\left(\exp \frac{||\mathbf{x}||}{\mathbf{t}}\right)^{-1} . \left(\exp \frac{||\mathbf{y}||}{\mathbf{s}}\right)^{-1} \leq \left(\exp \frac{||\mathbf{x}+\mathbf{y}||}{\mathbf{t}+\mathbf{s}}\right)^{-1}$, *which implies* $N(\mathbf{x}, \mathbf{t}) * N(\mathbf{y}, \mathbf{s}) \leq N(\mathbf{x}+\mathbf{y}, \mathbf{t}+\mathbf{s})$. □

Definition 26. *[8] An open ball in the fuzzy normed space* $(\mathcal{X}, N, *)$ *of radius* $\mathbf{r} > 0$, *centered at* $\mathbf{x} \in \mathcal{X}$ *with respect to parameter of fuzziness* $0 < \epsilon < 1$ *is depicted by* $B(\mathbf{x}, \mathbf{r}, \epsilon) = \{z \in \mathcal{X} : N(\mathbf{x} - \mathbf{z}, \mathbf{r}) > 1 - \epsilon\}$.

Definition 27. *[8] A sequence* $\{\mathbf{x_n}\}$ *in the fuzzy normed space* $(\mathcal{X}, N, *)$ *is convergent to a point* $\mathbf{x} \in \mathcal{X}$, *if* $N(\mathbf{x_n} - \mathbf{x}, \mathbf{t}) > 1 - \epsilon$, $\forall\ 0 < \epsilon < 1$ *and* $\mathbf{t} > 0$,
.

3. MOTIVATION OF INTUITIONISTIC FUZZY SET

The example defined earlier in this chapter discusses about the "belongingness" of elements in a set but when it comes to real life, the problems aren't as seamless. One can expand the idea of "vicinity" used in case of fuzzy set in terms of its membership function. For instance, one can consider the buildings that lie within the area of radius 200 km with center at Mumbai to be near the city and those that lie outside the area of radius 250 km of Mumbai to be far from the city. Here a little confusion arsises as to what should be done regarding the buildings that lie between the 200 km and 250 km belt. i.e., there exists a confusion about the buildings' nearness or farness. This develops the base of construction of Intuitionistic fuzzy theory which was introduced by Atanassov [11]. In view of our example, his theory suggests to define a function of degree of nearness for buildings within 200 km as membership function, a function of degree of farness

for buildings that lie outside the radius of 250 km as a non-membership function and hesitation function for those which lie between the areas of radii 200 km and 250 km.
The generalization of fuzzy to intuitionistic fuzzy brings in numerous real life applications to existence because of its correspondence with belonginess, non-belongingness and hesitation. These two sets may not be a complement of each other in general. The notion of intuitionistic fuzzy set is therefore an even more meaningful set which comprises of membership function, non-membership function and hesitation function. For readers intrested in learning more about these sets, we direct to [12, 13, 14].

Definition 28. *Let $\mathcal{X}$ be a non-empty set. An intuitionistic fuzzy set $\mathcal{A}$ in $\mathcal{X}$ is a set of the form $\mathcal{A} = \{< \mathbf{x}, \mu_{\mathcal{A}}(\mathbf{x}), \nu_{\mathcal{A}}(\mathbf{x}) >: \mathbf{x} \in \mathcal{X}\}$, where $\mu_{\mathcal{A}}, \nu_{\mathcal{A}}, \pi_{\mathcal{A}} : \mathcal{X} \to [0, 1]$, denote the membership, non-membership and hesitation function respectively such that $\nu_{\mathcal{A}}(\mathbf{x}) + \mu_{\mathcal{A}}(\mathbf{x}) \in [0, 1]$ for every $\mathbf{x} \in \mathcal{X}$ and $\pi_{\mathcal{A}}(\mathbf{x}) = 1 - \mu_{\mathcal{A}}(\mathbf{x}) - \nu_{\mathcal{A}}(\mathbf{x})$.*

Here $\mu_{\mathcal{A}}(\mathbf{x})$ can be interpreted as the degree that $\mathbf{x}$ belongs to $\mathcal{A}$, $\nu_{\mathcal{A}}(\mathbf{x})$ as the degree that $\mathbf{x}$ does not belong to $\mathcal{A}$ and $\pi_{\mathcal{A}}(\mathbf{x})$ as the hesitation function of $\mathbf{x}$ in $\mathcal{A}$, which suggests the confusion as to whether the element $\mathbf{x}$ belongs to $\mathcal{A}$ or not. If $\pi_{\mathcal{A}}(\mathbf{x}) = 0 \ \forall \ \mathbf{x} \in \mathcal{A}$, then the intuitionistic fuzzy set reduces to fuzzy set i.e. every fuzzy set is an intuitionisitic fuzzy set but converse may not be true.

Example 29. *Consider the set of non-negative real numbers $\mathbb{R}^+$. Define the set $\mathcal{A} = \{< \mathbf{x}, \mu_{\mathcal{A}}(\mathbf{x}), \nu_{\mathcal{A}}(\mathbf{x}) > | \mathbf{x} \in \mathbb{R}^+\}$ where $\mu_{\mathcal{A}}(\mathbf{x}) = \frac{1}{1+\mathbf{x}}, \nu_{\mathcal{A}}(\mathbf{x}) = \frac{\mathbf{x}}{2(1+\mathbf{x})}$ and the hesitation function as $\pi_{\mathcal{A}}(\mathbf{x}) = \frac{\mathbf{x}}{2(1+\mathbf{x})}$. One can see that conditions required by the membership, non-membership and hesitation functions for $\mathcal{A}$ to be an intuitionistic fuzzy set are met i.e. $\nu_{\mathcal{A}}(\mathbf{x}) + \mu_{\mathcal{A}}(\mathbf{x}) \in [0, 1]$ for every $\mathbf{x} \in \mathbb{R}^+$ and $\pi_{\mathcal{A}}(\mathbf{x}) = 1 - \mu_{\mathcal{A}}(\mathbf{x}) - \nu_{\mathcal{A}}(\mathbf{x})$. Thus $\mathcal{A}$ is an intuitionistic fuzzy set.*

In fuzzy metric space, the metric defines the degree of nearness between two points. A natural generalization of this metric can be seen as the one which incorporates the degree of non-nearness betwixt two points less than a real number(with the aid of continuous t-conorm) along with the degree of nearness. Post the introduction of fuzzy metric space by George and Veeramani in terms of continuous t-norm [6], J.H Park [15] put forward a generalization of the fuzzy metric in intuitionistic fuzzy sets aided by continuous t-norm and t-conorm, which he termed as intuitionistic fuzzy metric.

Definition 30. *[15] Let $\mathcal{X}$ be a non-empty abstract set. $*$ and $\diamond$ be continuous t-norm and t-conorm respectively and if the fuzzy sets $M, N : \mathcal{X}^2 \times [0, \infty) \to [0, 1]$ meet the following prerequisites $\forall \ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$ and $\forall \ \mathbf{s}, \mathbf{t} > 0$:*
(i) $N(\mathbf{x}, \mathbf{y}, \mathbf{t}) + M(\mathbf{x}, \mathbf{y}, \mathbf{t}) \leq 1$,
(ii) $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) > 0$,
(iii) $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) = 1, \ \forall \ t > 0 \iff \mathbf{x} = \mathbf{y}$,
(iv) $M(\mathbf{x}, \mathbf{y}, \mathbf{t}) = M(\mathbf{y}, \mathbf{x}, \mathbf{t})$,
(v) $M(\mathbf{y}, \mathbf{z}, \mathbf{s}) * M(\mathbf{x}, \mathbf{y}, \mathbf{t}) \leq M(\mathbf{x}, \mathbf{z}, \mathbf{t} + \mathbf{s})$,
(vi) $M(\mathbf{x}, .)$ *is continuous and non-decreasing in* $\mathbf{t} \in [0, \infty)$,

(vii) $N(\mathbf{x},\mathbf{y},\mathbf{t}) > 0$,
(viii) $N(\mathbf{x},\mathbf{y},\mathbf{t}) < 1$,
(ix) $N(\mathbf{x},\mathbf{y},\mathbf{t}) = 0,\ \forall\ \mathbf{t} > 0 \iff \mathbf{x} = \mathbf{y}$,
(x) $N(\mathbf{x},\mathbf{y},\mathbf{t}) = N(\mathbf{y},\mathbf{x},\mathbf{t})$
(xi) $N(\mathbf{y},\mathbf{z},\mathbf{s}) \diamond N(\mathbf{x},\mathbf{y},\mathbf{t}) \geq N(\mathbf{x},\mathbf{z},\mathbf{t}+\mathbf{s})$,
(xii) $N(\mathbf{x},.)$ *is continuous and non-increasing in* $\mathbf{t} \in [0,\infty)$,
(xiii) $\lim_{\mathbf{t}\to\infty} M(\mathbf{x},\mathbf{y},\mathbf{t}) = 1$ *and* $\lim_{\mathbf{t}\to\infty} N(\mathbf{x},\mathbf{y},\mathbf{t}) = 0$.
Then the tuple (M,N) *is known as intuitionistic fuzzy metric and* $(\mathcal{X},M,N,*,\diamond)$ *is known as an intuitionistic fuzzy metric space. In this space* $M(\mathbf{x},\mathbf{y},\mathbf{t})$ *indicates the degree of nearness and* $N(\mathbf{x},\mathbf{y},\mathbf{t})$*, degree of non-nearness between two points* $\mathbf{x},\mathbf{y}$ *of* $\mathcal{X}$ *relative to* $\mathbf{t}$*. From the definition it is clear that every fuzzy metric space* $(\mathcal{X},M,*)$ *induces intuitionistic fuzzy metric space* $(\mathcal{X},M,1-M,*,\diamond)$ *where* $*$ *and* $\diamond$ *satisfy* $\mathbf{x}\diamond\mathbf{y} = 1-((1-\mathbf{x})*(1-\mathbf{y}))\ \forall\ \mathbf{x},\mathbf{y} \in \mathcal{X}$.

Example 31. *Consider the ordinary metric space* $(\mathcal{X},\delta)$ *and denote* $\mathbf{a}*\boldsymbol{b} = \mathbf{a}.\mathbf{b}$, $\mathbf{a}\diamond\mathbf{b} = \min\{\mathbf{a}+\mathbf{b},1\}\ \forall\ \mathbf{a},\mathbf{b} \in [0,1]$*. Define fuzzy sets* $M,N : \mathcal{X}^2 \times [0,\infty) \to [0,1]$ *as,* $M(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\mathbf{kt}}{\mathbf{kt}+\mathbf{m}\delta(\mathbf{x},\mathbf{y})}$ *and* $N(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{kt}+\delta(\mathbf{x},\mathbf{y})}\ \forall\ \mathbf{k},\mathbf{m} \in \mathbb{R}^+$*. It can be shown easily that* $(\mathcal{X},M,N,*,\diamond)$ *is an intuitionistic fuzzy metric space.*

In the example provided above, the intuitionistic fuzzy metrics M,N *are induced by the ordinary metric* δ*. The statement however is not true in general for all t–norm and t–conorm. For more details we refer to [15].*

Definition 32. *[15] Consider the intuitionistic fuzzy metric space* $(\mathcal{X},M,N,*,\diamond)$*. Then for* $\mathbf{x} \in \mathcal{X}$*, the open ball centered at* $\mathbf{x}$*, radius* $\mathbf{r} > 0$ *with respect to parameter of fuzziness* $\epsilon \in (0,1)$*, is the set* $B(\mathbf{x},\mathbf{r},\epsilon) = \{\mathbf{z} \in \mathcal{X} : M(\mathbf{x},\mathbf{z},\mathbf{r}) > 1-\epsilon, N(\mathbf{x},\mathbf{z},\mathbf{r}) < \epsilon\}$.

Definition 33. *[15] Let* $(\mathcal{X},M,N,*,\diamond)$ *be an intuitionistic fuzzy metric space. Then a sequence* $\{\mathbf{x_n}\}$ *of points in* $\mathcal{X}$ *converges to a point* $\mathbf{x}_0 \in \mathcal{X}$*, if for all* $\epsilon \in (0,1)$ *and* $\mathbf{t} > 0$*, there exists* $\mathbf{n}_0 \in \mathbb{N}$ *such that* $\forall\ \mathbf{n} \geq \mathbf{n}_0$*,* $M(\mathbf{x_n},\mathbf{x}_0,\mathbf{t}) > 1-\epsilon$ *and* $N(\mathbf{x_n},\mathbf{x}_0,\mathbf{t}) < \epsilon$.

A fairly natural extension of the fuzzy norm to intuitionistic fuzzy norm was introduced by Saadati and Park in [16], which he defined as follows.

Definition 34. *[16] Let* $\mathcal{X}$ *be a linear space,* $*$ *and* $\diamond$ *are continuous t–norm and t–conorm and* $\mu,\nu : \mathcal{X}\times[0,\infty) \to [0,1]$ *are fuzzy sets. Then* $(\mathcal{X},\mu,\nu,*,\diamond)$ *is called an intuitionistic fuzzy normed space (IFNS) if for every* $\mathbf{x},\mathbf{y} \in \mathcal{X}$ *and* $\mathbf{s},\mathbf{t} > 0$ *the following requirements are met:*
(i) $\nu(\mathbf{x},\mathbf{t}) + \mu(\mathbf{x},\mathbf{t}) \leq 1$,
(ii) $\mu(\mathbf{x},\mathbf{t}) > 0$,
(iii) $\mu(\mathbf{x},\mathbf{t}) = 1 \iff \mathbf{x} = 0$,
(iv) $\mu(\lambda\mathbf{x},t) = \mu\big(\mathbf{x},\frac{\mathbf{t}}{|\lambda|}\big)$, $\lambda \neq 0$,
(v) $\mu(\mathbf{x}+\mathbf{y},\mathbf{t}+\mathbf{s}) \geq \mu(\mathbf{x},\mathbf{t}) * \mu(\mathbf{y},\mathbf{s})$,
(vi) $\mu(\mathbf{x},.) : (0,\infty) \to [0,1]$ *is continuous* ,

(vii) $\mu(\mathbf{x},\mathbf{t}) \to 1$ *when* $\mathbf{t} \to \infty$, *and* $\mu(\mathbf{x},\mathbf{t}) \to 0$, *when* $\mathbf{t} \to 0$,
(viii) $\nu(\mathbf{x},\mathbf{t}) < 1$,
(ix) $\nu(\mathbf{x},\mathbf{t}) = 0 \iff \boldsymbol{x} = 0$,
(x) $\nu(\lambda\mathbf{x},\mathbf{t}) = \nu\big(\mathbf{x}, \frac{\mathbf{t}}{|\lambda|}\big)$, $\lambda \neq 0$,
(xi) $\nu(\mathbf{x}+\mathbf{y},\mathbf{t}+\mathbf{s}) \leq \nu(\mathbf{x},\mathbf{t}) \diamond \nu(\mathbf{y},\mathbf{s})$,
(xii) $\nu(\mathbf{x},.) : (0,\infty) \to [0,1]$ *is continuous,*
(xiii) $\nu(\mathbf{x},\mathbf{t}) \to 0$ *when* $\mathbf{t} \to \infty$ *and* $\nu(\mathbf{x},\mathbf{t}) \to 1$ *when* $\mathbf{t} \to 0$.

Here (μ,ν) *is called intuitionistic fuzzy norm. From the definition it is clear that every fuzzy normed space* $(\mathcal{X}, N, *)$ *is an IFNS* $(\mathcal{X}, N, 1-N, *, \diamond)$ *where* $*$ *and* $\diamond$ *satisfy* $\mathbf{x} \diamond \mathbf{y} = 1 - ((1-\mathbf{x}) * (1-\mathbf{y}))$ $\forall$ $\mathbf{x},\mathbf{y} \in \mathcal{X}$. *Every IFNS* $(\mathcal{X},\mu,\nu,*,\diamond)$ *induces an intuitionistic fuzzy metric space where* $M(\mathbf{x},\mathbf{y},\mathbf{t}) = \mu(\mathbf{x}-\mathbf{y},\mathbf{t})$ *and* $N(\mathbf{x},\mathbf{y},\mathbf{t}) = \nu(\mathbf{x}-\mathbf{y},\mathbf{t})$ $\forall$ $\mathbf{x},\mathbf{y} \in \mathcal{X}$ *and* $\mathbf{t} > 0$.

Example 35. *Consider the ordinary normed space* $(\mathcal{X}, ||.||_p)$ *and denote* $\mathbf{a} * \boldsymbol{b} = \min\{\mathbf{a},\mathbf{b}\}$, $\mathbf{a} \diamond \mathbf{b} = \max\{\mathbf{a},\mathbf{b}\}$ $\forall$ $\mathbf{a},\mathbf{b} \in [0,1]$. *Now define fuzzy sets* $\mu,\nu : \mathcal{X} \times [0,\infty) \to [0,1]$ *such that,* $\mu(\mathbf{x},\mathbf{t}) = \frac{\mathbf{kt}}{\boldsymbol{kt}+\boldsymbol{m}||\mathbf{x}||_p}$ *and* $\nu(\mathbf{x},\mathbf{t})) = \frac{||\mathbf{x}||_p}{\mathbf{kt}+||\mathbf{x}||_p}$ *for all* $\mathbf{k},\mathbf{m} \in \mathbb{R}^+$. *It can be shown easily that the tuple* $(\mathcal{X},\mu,\nu,*,\diamond)$ *is an IFNS.*

In the example provided above, the intuitionistic fuzzy norms μ,ν *are induced by the ordinary norm* $||.||_p$ *and therefore the space* $(\mathcal{X},\mu,\nu,*,\diamond)$ *is called induced intuitionistic fuzzy normed.*

Definition 36. *[16] Let* $(\mathcal{X},\mu,\nu,*,\diamond)$ *be an IFNS . Then for* $\mathbf{x} \in \mathcal{X}$*, the open ball centered at* $\mathbf{x}$*, radius* $\mathbf{r} > 0$ *appropos parameter of fuzziness* $\epsilon \in (0,1)$*, is the set* $B(\mathbf{x},\mathbf{r},\epsilon) = \{\mathbf{z} \in \mathcal{X} : \mu(\mathbf{x}-\mathbf{z},\boldsymbol{r}) > 1-\epsilon, \nu(\mathbf{x}-\mathbf{z},\mathbf{r}) < \epsilon\}$.

Definition 37. *[16] Let* $(\mathcal{X},\mu,\nu,*,\diamond)$ *be an IFNS. Then a sequence* $\{\mathbf{x_n}\}$ *of points in* $\mathcal{X}$ *converges to a point* $\mathbf{x}_0 \in \mathcal{X}$*, if* $\forall$ $\epsilon \in (0,1), \mathbf{t} > 0$, $\exists$ $\mathbf{n}_0 \in \mathbb{N}$ *such that* $\forall$ $\mathbf{n} \geq \mathbf{n}_0$, $\mu(\mathbf{x_n}-\mathbf{x}_0,\mathbf{t}) > 1-\epsilon$ *and* $\nu(\mathbf{x_n}-\mathbf{x}_0,\mathbf{t}) < \epsilon$.

4. MOTIVATION OF NEUTROSOPHIC SET

Even though the concept of intuitionistic fuzzy theory is a generalized version of fuzzy theory and thus overcomes a real life situation by incorporating a non–membership function, it is still limited within the boundaries of deficient information and fails to process the indeterminate and inconsistent information. In other words, the membership and non–membership functions in the intuitionistic fuzzy sets are dependent on each other i.e. if one of them increases, the other one decreases which may not be the case in practical world problems.
These intuitionistic fuzzy sets however may fail to function in real life, for instance, if we consider the number of people who are willing to get vaccinated, not get vaccinated and unsure are 70, 20 and 10 out of 100 respectively then the concept goes beyond the reach of intuitionistic fuzzy set since all the scenarios in the given example occur simultaneously yet independently. Keeping this in mind, a neutral approach to solving these problems was introduced by Smarandache

[19], which he termed as neutrosophic. The name neutrosophic actually comes from two words, " neutro " meaning neutral and " sophic " meaning knowledge or information. The actual definition of the neutrosophic sets was given based on the independency of membership, non–membership and hesitation function. The above mentioned example can be written in terms of neutrosophic notation as $\mathbf{x}(0.7, 0.2, 0.1)$.

Definition 38. *[19] Let $\mathcal{X}$ be a non–empty set. A neutrosophic set $\mathcal{A}$ in $\mathcal{X}$ is a set $\mathcal{A} = \{(\mathbf{x}, \mu_{\mathcal{A}}(\mathbf{x}), \nu_{\mathcal{A}}(\mathbf{x}), \pi_{\mathcal{A}}(\mathbf{x})) : \mathbf{x} \in \mathcal{X}\}$, where $\mu_{\mathcal{A}}, \nu_{\mathcal{A}}, \pi_{\mathcal{A}} : \mathcal{X} \to [0,1]$, denote the membership, non-membership and hesitation function respectively such that $0 \leq \nu_{\mathcal{A}}(\mathbf{x}) + \mu_{\mathcal{A}}(\mathbf{x}) + \pi_{\mathcal{A}}(\mathbf{x}) \leq 3$ for every $\mathbf{x} \in \mathcal{X}$.*
This condition ensures that the neutrsophic components $\nu_{\mathcal{A}}, \mu_{\mathcal{A}}$ and $\pi_{\mathcal{A}}$ are independent of each other.

Example 39. *Let $\mathcal{X} = \mathbb{R}^+$. Define $\mathcal{A} = \{(\mathbf{x}, \mu_{\mathcal{A}}(\mathbf{x}), \nu_{\mathcal{A}}(\mathbf{x}), \pi_{\mathcal{A}}(\mathbf{x}))|\mathbf{x} \in \mathbb{R}^+\}$ as $\mu_{\mathcal{A}}(\mathbf{x}) = \frac{1}{1+\mathbf{x}}$,*
$\nu_{\mathcal{A}}(\mathbf{x}) = \frac{1}{(1+\mathbf{x})^2}$ and the hesitation function as $\pi_{\mathcal{A}}(\mathbf{x}) = \frac{1}{(1+\mathbf{x})^3}$. One can see that conditions required by the membership, non-membership and hesitation functions for $\mathcal{A}$ to be a neutrosophic set are met i.e., $0 \leq \mu_{\mathcal{A}}(\mathbf{x}) + \nu_{\mathcal{A}}(\mathbf{x}) + \pi_{\mathcal{A}}(\mathbf{x}) \leq 3$ for every $\mathbf{x} \in \mathbb{R}^+$. Thus $\mathcal{A}$ is a neutrosophic set.

Definition 40. *[18] Let $\mathcal{X}$ is an arbitrary space and $\mathcal{M} = \{(\mathbf{x}, \mu(\mathbf{x}), \nu(\mathbf{x}), \pi(\mathbf{x})) : \mathbf{x} \in \mathcal{X}\}$ be a neutrosophic set such that $\mathcal{M} : \mathcal{X}^2 \times R^+ \to [0,1]$. Let $*$ and $\diamond$ be the continuous t–norm and t–conorm respectively. The four-tuple $(\mathcal{X}, \mathcal{M}, *, \diamond)$ is known as Neutrosophic metric space (NMS) if these requirements are met $\forall\ \mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathcal{X}$ and $\mathbf{s}, \mathbf{t} \in \mathbb{R}^+$:*

(i) $0 \leq \mu(\mathbf{x}, \mathbf{y}, \mathbf{t}) \leq 1,\ 0 \leq \nu(\mathbf{x}, \mathbf{y}, \mathbf{t}) \leq 1,\ 0 \leq \pi(\mathbf{x}, \mathbf{y}, \mathbf{t}) \leq 1,$

(ii) $\mu(\mathbf{x}, \mathbf{y}, \mathbf{t}) + \nu(\mathbf{x}, \mathbf{y}, \mathbf{t}) + \pi(\mathbf{x}, \mathbf{y}, \mathbf{t}) \leq 3,$

(iii) $\mu(\mathbf{x}, \mathbf{y}, \mathbf{t}) = 1$ *for* $\mathbf{t} > 0 \iff \mathbf{x} = \mathbf{y},$

(iv) $\mu(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \mu(\mathbf{y}, \mathbf{x}, \mathbf{t}),$

(v) $\mu(\mathbf{x}, \mathbf{w}, \mathbf{t} + \mathbf{s}) \geq \mu(\mathbf{x}, \mathbf{y}, \mathbf{t}) * \mu(\mathbf{y}, \mathbf{w}, \mathbf{s}),$

(vi) $\mu(\mathbf{x}, \mathbf{y}, .) : (0, \infty) \to [0,1]$ *is continuous* ,

(vii) $\mu(\mathbf{x}, \mathbf{y}, \mathbf{t}) \to 1$ *when* $\boldsymbol{t} \to \infty$

(viii) $\nu(\mathbf{x}, \mathbf{y}, \mathbf{t}) = 0\ \forall\ \boldsymbol{t} > 0 \iff \boldsymbol{x} = \boldsymbol{y},$

(ix) $\nu(\mathbf{x},\mathbf{y},\mathbf{t}) = \nu(\mathbf{y},\mathbf{x},\mathbf{t})$,

(x) $\mu(\mathbf{x},\mathbf{w},\mathbf{t}+\mathbf{s}) \leq \mu(\mathbf{x},\mathbf{y},\mathbf{t}) \diamond \mu(\mathbf{y},\mathbf{w},\mathbf{s})$,

(xi) $\nu(\mathbf{x},\mathbf{y},.) : (0,\infty) \to [0,1]$ *is continuous* ,

(xii) $\nu(\mathbf{x},\mathbf{y},\mathbf{t}) \to 0$ *when* $\mathbf{t} \to \infty$,

(xiii) $\pi(\mathbf{x},\mathbf{y},\mathbf{t}) = 0 \ \forall \ \mathbf{t} > 0$ *iff* $\mathbf{x} = \mathbf{y}$,

(xiv) $\pi(\mathbf{x},\mathbf{y},\mathbf{t}) = \pi(\mathbf{y},\mathbf{x},\mathbf{t})$,

(xv) $\pi(\mathbf{x},\mathbf{w},\mathbf{t}+\mathbf{s}) \leq \pi(\mathbf{x},\mathbf{y},\mathbf{t}) \diamond \pi(\mathbf{y},\mathbf{w},\mathbf{s})$,

(xvi) $\pi(\mathbf{x},\mathbf{y},.) : (0,\infty) \to [0,1]$ *is continuous* ,

(xvii) $\pi(\mathbf{x},\mathbf{y},\mathbf{t}) \to 0$ *when* $\mathbf{t} \to \infty$,

(xviii) If $0 \geq \mathbf{t}$, *then* $\mu(\mathbf{x},\mathbf{y},\mathbf{t}) = 0$, $\nu(\mathbf{x},\mathbf{y},\mathbf{t}) = 1$, $\pi(\mathbf{x},\mathbf{y},\mathbf{t}) = 1$.

The functions $\mu(\mathbf{x},\mathbf{y},\mathbf{t}), \nu(\mathbf{x},\mathbf{y},\mathbf{t})$ *and* $\pi(\mathbf{x},\mathbf{y},\mathbf{t})$ *denote the degree of nearness, non-nearness and neutralness between* $\mathbf{x},\mathbf{y}$ *relative to* $\mathbf{t}$ *respectively and* $(\mathcal{X},\mathcal{M},*,\diamond)$ *is called* NMS.

Example 41. *Consider the ordinary metric space* $(\mathcal{X},\delta)$ *and denote* $\mathbf{a} * \mathbf{b} = \min\{\mathbf{a},\mathbf{b}\}$, $\mathbf{a} \diamond \mathbf{b} = \max\{\mathbf{a},\mathbf{b}\}$ *for all* $0 \leq \mathbf{a},\mathbf{b} \leq 1$. *Define,* $\mu(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\mathbf{kt}}{\mathbf{kt}+\delta(\mathbf{x},\mathbf{y})}$, $\nu(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{kt}+\delta(\mathbf{x},\mathbf{y})}$ *and* $\pi(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\delta(\mathbf{x},\mathbf{y})}{\mathbf{kt}}$, $\forall \ \mathbf{x},\mathbf{y} \in \mathcal{X}$ *and* $\mathbf{k},\mathbf{t} > 0$. *It can be so that* $(\mathcal{X},\mathcal{M},*,\diamond)$ *is* NMS *induced by the metric* δ.

Definition 42. *Let* $(\mathcal{X},\mathcal{M},*,\diamond)$ *is a* NMS. *Then for* $\mathbf{x} \in \mathcal{X}$, *the open ball centered at* $\mathbf{x}$, *radius* $\mathbf{r} > 0$ *apropos parameter of fuzziness* $0 < \epsilon < 1$, *is the set* $B(\mathbf{x},\mathbf{r},\epsilon) = \{\mathbf{z} \in \mathcal{X} : \mu(\mathbf{x},\mathbf{z},\mathbf{r}) > 1-\epsilon, \nu(\mathbf{x},\mathbf{z},\mathbf{r}) < \epsilon, \pi(\mathbf{x},\mathbf{z},\mathbf{r}) < \epsilon\}$.

Definition 43. *Let* $(\mathcal{X},\mathcal{M},*,\diamond)$ *is a* NMS. *A sequence* $\{\mathbf{x_n}\}$ *of points in* $\mathcal{X}$ *converges to* $\mathbf{x}_0 \in \mathcal{X}$, *if* $\forall \ \epsilon \in (0,1)$, $\mathbf{t} > 0$, *there exists* $\mathbf{n}_0 \in \mathbb{N}$ *such that* $\forall \ \mathbf{n} \geq \mathbf{n}_0$, $\mu(\mathbf{x_n},\mathbf{x}_0,\mathbf{t}) > 1-\epsilon, \nu(\mathbf{x_n},\mathbf{x}_0,\mathbf{t}) < \epsilon$ *and* $\pi(\mathbf{x_n},\mathbf{x}_0,\mathbf{t}) < \epsilon$.

For more details refer to [18]. Now we introduce the concept of length of a vector in vector space in the neutrosophic setting which was defined in [17].

Definition 44. *[17] Let $\mathcal{X}$ be a linear space; $\mathcal{N} = \{(\mathbf{x}, \mu(\mathbf{x}), \nu(\mathbf{x}), \pi(\mathbf{x})) : \mathbf{x} \in \mathcal{X}\}$ be a neutrosophic set such that $\mathcal{N} : \mathcal{X} \times \mathbb{R}^+ \to [0,1]$. Let $*$ and $\diamond$ be the continuous t–norm and t–conorm respectively. The four-tuple $(\mathcal{X}, \mathcal{N}, *, \diamond)$ is said to be Neutrosophic normed space (NNS) if the following requirements are met for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $\mathbf{s}, \mathbf{t} \in \mathbb{R}^+$*

(i) $0 \leq \mu(\mathbf{x}, \mathbf{t}), \nu(\mathbf{x}, \mathbf{t}), \pi(\mathbf{x}, \mathbf{t}) \leq 1$,

(ii) $\nu(\mathbf{x}, \mathbf{t}) + \mu(\mathbf{x}, \mathbf{t}) + \pi(\mathbf{x}, \mathbf{t}) \leq 3$,

(iii) $\mu(\mathbf{x}, \mathbf{t}) = 1$, *for all* $\mathbf{t} > 0 \iff x = 0$,

(iv) $\mu(\lambda\mathbf{x}, \mathbf{t}) = \mu\left(\mathbf{x}, \frac{\mathbf{t}}{|\lambda|}\right)$, $\forall$ *non–zero* λ

(v) $\mu(\mathbf{x} + \mathbf{y}, \mathbf{t} + \mathbf{s}) \geq \mu(\mathbf{x}, \mathbf{t}) * \mu(\mathbf{y}, \mathbf{s})$,

(vi) $\mu(\mathbf{x}, .) : (0, \infty) \to [0,1]$ *is continuous,*

(vii) $\mu(\mathbf{x}, \mathbf{t}) \to 1$, *when* $\mathbf{t} \to \infty$

(viii) $\nu(\mathbf{x}, \mathbf{t}) = 0$, *for all* $\mathbf{t} > 0 \iff \mathbf{x} = 0$,

(ix) $\nu(\lambda\mathbf{x}, \mathbf{t}) = \nu\left(\mathbf{x}, \frac{\mathbf{t}}{|\lambda|}\right)$, $\forall$ *non–zero* λ

(x) $\nu(\mathbf{x} + \mathbf{y}, \mathbf{t} + \mathbf{s}) \leq \nu(\mathbf{x}, \mathbf{t}) \diamond \nu(\mathbf{y}, \mathbf{s})$,

(xi) $\mu(\mathbf{x}, .) : (0, \infty) \to [0,1]$ *is continuous,*

(xii) $\nu(\mathbf{x}, \mathbf{t}) \to 0$, *when* $\mathbf{t} \to \infty$

(xiii) $\pi(\mathbf{x}, \mathbf{t}) = 0$, *for all* $\mathbf{t} > 0 \iff \mathbf{x} = 0$,

(xiv) $\pi(\lambda\mathbf{x}, \mathbf{t}) = \pi\left(\mathbf{x}, \frac{\mathbf{t}}{|\lambda|}\right)$, $\forall$ *non–zero* λ

(xv) $\pi(\mathbf{x} + \mathbf{y}, \mathbf{t} + \mathbf{s}) \leq \pi(\mathbf{x}, \mathbf{t}) \diamond \pi(\mathbf{y}, \mathbf{s})$,

(xvi) $\mu(\mathbf{x}, .) : (0, \infty) \to [0,1]$ *is continuous,*

(xvii) $\pi(\mathbf{x}, \mathbf{t}) \to 0$, *when* $\mathbf{t} \to \infty$,

(xviii) *If* $0 \geq \mathbf{t}$, *then* $\mu(\mathbf{x}, \mathbf{t}) = 0$, $\nu(\mathbf{x}, \mathbf{t}) = 1$, $\pi(\mathbf{x}, \mathbf{t}) = 1$

$(\mathcal{X}, \mathcal{N}, *, \diamond)$ is known as neutrosophic normed space. Clearly every neutrosophic normed space $(\mathcal{X}, \mathcal{N}, *, \diamond)$ is a neutrsophic metric space $(\mathcal{X}, \mathcal{M}, *, \diamond)$.

Example 45. *Consider the ordinary normed space* $(\mathcal{X}, ||.||)$ *and denote* $\mathbf{a} * \mathbf{b} = \min\{\mathbf{a}, \mathbf{b}\}$, $\mathbf{a} \diamond \mathbf{b} = \max\{\mathbf{a}, \mathbf{b}\}$ *for all* $0 \leq \mathbf{a}, \mathbf{b} \leq 1$. *Define* $\mathcal{N} : \mathcal{X} \times \mathbb{R}^+ \to [0, 1]$ *such that,* $\mu(\mathbf{x}, \mathbf{t}) = \frac{\mathbf{kt}}{\mathbf{kt}+||\mathbf{x}||}$, $\nu(\mathbf{x}, \mathbf{t}) = \frac{||\mathbf{x}||}{\mathbf{kt}+||\mathbf{x}||}$ *and* $\pi(\mathbf{x}, \mathbf{t}) = \frac{||\mathbf{x}||}{\mathbf{kt}}$ $\forall$ $\mathbf{x} \in \mathcal{X}$ *and* $\mathbf{k}, \mathbf{t} > 0$. *It can be shown easily that the four tuple* $(\mathcal{X}, \mathcal{N}, *, \diamond)$ *is a* NNS *induced by the norm* $||.||$.

Definition 46. *[17] Let* $(\mathcal{X}, \mathcal{N}, *, \diamond)$ *is a* NNS. *Then for* $\mathbf{x} \in \mathcal{X}$, *the open ball centered at* $\mathbf{x}$, *radius* $\mathbf{r} > 0$ *apropos parameter of fuzziness* $0 < \epsilon < 1$, *is the set* $B(\mathbf{x}, \mathbf{r}, \epsilon) = \{\mathbf{z} \in \mathcal{X} : \mu(\mathbf{x} - \mathbf{z}, \mathbf{r}) > 1 - \epsilon, \nu(\mathbf{x} - \mathbf{z}, \mathbf{r}) < \epsilon, \pi(\mathbf{x} - \mathbf{z}, \mathbf{r}) < \epsilon\}$.

Definition 47. *[17] Let* $(\mathcal{X}, \mathcal{N}, *, \diamond)$ *is a* NNS. *A sequence* $\{\mathbf{x_n}\}$ *of points in* $\mathcal{X}$ *converges to* $\mathbf{x}_0 \in \mathcal{X}$, *if* $\forall$ $\epsilon \in (0, 1)$, $\mathbf{t} > 0$, *there exists* $\mathbf{n}_0 \in \mathbb{N}$ *such that* $\forall$ $\mathbf{n} \geq \mathbf{n}_0$, $\mu(\mathbf{x_n} - \mathbf{x}_0, \mathbf{t}) > 1 - \epsilon, \nu(\mathbf{x_n} - \mathbf{x}_0, \mathbf{t}) < \epsilon$ *and* $\pi(\mathbf{x_n} - \mathbf{x}_0, \mathbf{t}) < \epsilon$.

CONCLUSION

The article explores the motivations behind the aforementioned topics and gives an insight into how one can perceive the notions of fuzzy, intuitionistic fuzzy and neutrosophic sets, metrics and norms. It intends to explain in detail how one can construct a new definition based on a logical idea and then establish relation between them. The chapter thoroughly discusses the motivations of fuzzy sets, fuzzy metric, fuzzy norm, intuitionistic fuzzy set, intuitionistic fuzzy metric, intuitionistic fuzzy norm, neutrosophic set, neutrosophic metric and neutrosophic norm.

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