

TAUBERIAN THEOREMS FOR WEIGHTED MEANS OF DOUBLE SEQUENCES IN INTUITIONISTIC FUZZY NORMED SPACES

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Received: September 2021 / Accepted: March 2022

Abstract: We define weighted mean summability method of double sequences in intuitionistic fuzzy normed spaces (*IFNS*), and obtain necessary and sufficient Tauberian conditions under which convergence of double sequences in *IFNS* follows from their weighted mean summability. This study reveals also Tauberian results for some known summation methods in the special cases.

Keywords: Tauberian Theorem, Double Sequence, Intuitionistic Fuzzy, Summability Methods, Summation Methods.

MSC: 05A15, 05C38, 15A15, 15A18.

1. INTRODUCTION

Zadeh [14] introduced the concept of fuzzy sets by assigning membership grades to each element in the universe of discourse in order to handle noncategorical and unclassifiable data. Fuzzy sets were studied by many mathematicians and applied to many fields of science. Besides, Atanassov [1] introduced intuitionistic fuzzy sets (IFS) also assigning nonmembership grades to the elements. IFS were also studied by many mathematicians, i.e., ([4], [9], [5]). The concept of intuitionistic fuzzy norm was studied by Saadati and Park [11], and by Lael [7]. convergence of sequences in *IFNS* was studied and some convergence and summation methods were introduced to recover the convergence where ordinary convergence of sequences in *IFNS* fails [6, 8, 12, 13]. In this study, we extend the results of [2] to intuitionistic fuzzy normed spaces. That is, we define weighted mean summability methods for double sequences in *IFNS* and give some Tauberian conditions under which convergence of double sequences in *IFNS* follows from weighted mean summability. This study also reveals Tauberian results for some known summation methods such as Cesàro summability method $(C, 1, 1)$ and Nörlund summability method (\bar{N}, p) in the special cases. We now give some preliminaries for *IFNS*.

Definition 1. [7] *The triplicate (V, μ, ν) is said to be an IF-normed space if V is a real vector space, and μ, ν are F -sets on $V \times \mathbb{R}$ satisfying the following conditions for every $x, y \in V$ and $t, s \in \mathbb{R}$:*

1. $\mu(x, t) = 0$ for all non-positive real number t ,
2. $\mu(x, t) = 1$ for all $t \in \mathbb{R}^+$ if and only if $x = \theta$
3. $\mu(cx, t) = \mu\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,
4. $\mu(x + y, t + s) \geq \min\{\mu(x, t), \mu(y, s)\}$,
5. $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$,
6. $\nu(x, t) = 1$ for all non-positive real number t ,
7. $\nu(x, t) = 0$ for all $t \in \mathbb{R}^+$ if and only if $x = \theta$
8. $\nu(cx, t) = \nu\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,
9. $\max\{\nu(x, t), \nu(y, s)\} \geq \nu(x + y, t + s)$,
10. $\lim_{t \rightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x, t) = 1$.

In this case, we will call (μ, ν) an IF-norm on V .

Throughout the paper (V, μ, ν) will denote an IF-normed space.

Example 2. Let $(V, \|\cdot\|)$ be a normed space and μ_0, ν_0 be F -sets on $V \times \mathbb{R}$ defined by

$$\mu_0(x, t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t + \|x\|}, & t > 0, \end{cases} \quad \nu_0(x, t) = \begin{cases} 1, & t \leq 0, \\ \frac{\|x\|}{t + \|x\|}, & t > 0. \end{cases}$$

Then (μ_0, ν_0) is IF-norm on V .

Definition 3. [11] A double sequence (x_{mn}) in (V, μ, ν) is said to be convergent to $x \in V$ and denoted by $x_{mn} \rightarrow x$, if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that

$$\mu(x_{mn} - x, t) > 1 - \varepsilon \quad \text{and} \quad \nu(x_{mn} - x, t) < \varepsilon$$

for all $m, n \geq n_0$.

Here we note that convergence of double sequences in *IFNS* is meant in the sense of Definition 3 throughout the paper. Similarly, limit and convergence of double sequences of real numbers are taken in the Pringsheim's sense [10].

Definition 4. [11] A double sequence (x_{mn}) in (V, μ, ν) is said to be Cauchy if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that

$$\mu(x_{jk} - x_{mn}, t) > 1 - \varepsilon \quad \text{and} \quad \nu(x_{jk} - x_{mn}, t) < \varepsilon$$

for all $i, j, m, n \geq n_0$.

We note that if sequence (x_{mn}) converges to $x \in V$, then (x_{mn}) is Cauchy in view of the facts that

$$\begin{aligned} \mu(x_{jk} - x_{mn}, t) &\geq \min\{\mu(x_{jk} - x, t/2), \mu(x_{mn} - x, t/2)\} \\ \nu(x_{jk} - x_{mn}, t) &\leq \max\{\nu(x_{jk} - x, t/2), \nu(x_{mn} - x, t/2)\}. \end{aligned}$$

Definition 5. [3] A double sequence (x_{mn}) in (V, μ, ν) is called *q-bounded* if $\liminf_{t \rightarrow \infty} \inf_{m,n} \mu(x_{mn}, t) = 1$ and $\limsup_{t \rightarrow \infty} \sup_{m,n} \nu(x_{mn}, t) = 0$.

Theorem 6. [12] Let (x_n) be a sequence in (V, μ, ν) . If $\{n(x_n - x_{n-1})\}$ is *q-bounded*, then (x_n) is slowly oscillating.

Let $p = (p_j)$ and $q = (q_k)$ be two sequences of nonnegative numbers ($p_0, q_0 > 0$) with

$$P_m = \sum_{j=0}^m p_j \rightarrow \infty \quad (m \rightarrow \infty), \quad Q_n = \sum_{k=0}^n q_k \rightarrow \infty \quad (n \rightarrow \infty).$$

Let $(\alpha, \beta) \in \{(1, 1), (1, 0), (0, 1)\}$. The weighted means $(t_{mn}^{\alpha\beta})$ of a double sequence (x_{mn}) in *IF*-normed space (V, μ, ν) are defined by

$$t_{mn}^{11} = \frac{1}{P_m Q_n} \sum_{j=0}^m \sum_{k=0}^n p_j q_k x_{jk}, \quad t_{mn}^{10} = \frac{1}{P_m} \sum_{j=0}^m p_j x_{jn}, \quad t_{mn}^{01} = \frac{1}{Q_n} \sum_{k=0}^n q_k x_{mk}.$$

The double sequence (x_{mn}) in (V, μ, ν) is said to be $(\bar{N}, p, q; \alpha, \beta)$ summable to $x \in V$ if $\lim_{m,n \rightarrow \infty} t_{mn}^{\alpha\beta} = x$, and denoted by $x_{mn} \rightarrow x (\bar{N}, p, q; \alpha, \beta)$.

Since t_{mn}^{10} is independent of q , we write $(\bar{N}, p, *, 1, 0)$ in place of $(\bar{N}, p, q; 1, 0)$ (see [2]). We note that if we take $p_j = 1, q_k = 1$ for all $j, k \in \mathbb{N}$ in $(\bar{N}, p, q; 1, 1)$

summability, then we obtain $(C, 1, 1)$ summability. Also if we take $(x_{jk}) = (x_j)$ in $(\bar{N}, p, *, 1, 0)$ summability, then we obtain (\bar{N}, p) summability of single sequences. Hence, this paper reveals also Tauberian results for $(C, 1, 1)$ and (\bar{N}, p) summability methods in special cases. Here we note that when the case is $(x_{jk}) = (x_j)$ in $(\bar{N}, p, *, 1, 0)$ summability, the condition of q -boundedness in Theorem 21 are redundant.

2. RESULTS FOR $(\bar{N}, p, q; 1, 1)$ SUMMABILITY IN $IFNS$

In this section we give some Tauberian conditions under which $(\bar{N}, p, q; 1, 1)$ summability implies convergence in $IFNS$. Before giving our Tauberian results, we first show that convergence and q -boundedness imply $(\bar{N}, p, q; 1, 1)$ summability for double sequences in $IFNS$.

Theorem 7. *If double sequence (x_{mn}) in (V, μ, ν) is q -bounded and convergent to $x \in V$, then (x_{mn}) is $(\bar{N}, p, q; 1, 1)$ summable to x .*

Proof. Let double sequence (x_{mn}) in (V, μ, ν) be q -bounded and convergent to $x \in V$. Fix $t > 0$. Let n_0 be from Definition 3 of convergence with $t \leftrightarrow \frac{t}{3}$. Then, we have

$$\begin{aligned} \mu(t_{mn}^{11} - x, t) &= \mu\left(\frac{1}{P_m Q_n} \sum_{j=0}^m \sum_{k=0}^n p_j q_k x_{jk} - x, t\right) = \mu\left(\sum_{j=0}^m \sum_{k=0}^n p_j q_k (x_{jk} - x), P_m Q_n t\right) \\ &\geq \\ &\min\left\{\mu\left(\sum_{j=0}^m \sum_{k=0}^{n_0} p_j q_k (x_{jk} - x), P_m Q_n t/3\right), \mu\left(\sum_{j=0}^{n_0} \sum_{k=n_0+1}^n p_j q_k (x_{jk} - x), P_m Q_n t/3\right), \right. \\ &\left. \mu\left(\sum_{j=n_0+1}^m \sum_{k=n_0+1}^n p_j q_k (x_{jk} - x), P_m Q_n t/3\right)\right\} \\ &\geq \min\left\{\min_{\substack{j \in \mathbb{N} \\ 0 \leq k \leq n_0}} \mu\left(x_{jk} - x, \frac{Q_n t}{n_0 A 3}\right), \min_{\substack{k \in \mathbb{N} \\ 0 \leq j \leq n_0}} \mu\left(x_{jk} - x, \frac{P_m t}{n_0 B 3}\right), \min_{n_0 < k, j} \mu\left(x_{jk} - x, \frac{t}{3}\right)\right\} \\ &\geq \min\left\{\inf_{j, k \in \mathbb{N}} \mu\left(x_{jk} - x, \frac{Q_n t}{n_0 A 3}\right), \inf_{j, k \in \mathbb{N}} \mu\left(x_{jk} - x, \frac{P_m t}{n_0 B 3}\right), \min_{n_0 < j, k} \mu\left(x_{jk} - x, \frac{t}{3}\right)\right\} \\ &\text{and} \end{aligned}$$

$$\begin{aligned} \nu(t_{mn}^{11} - x, t) &= \nu\left(\sum_{j=0}^m \sum_{k=0}^n p_j q_k (x_{jk} - x), P_m Q_n t\right) \\ &\leq \max\left\{\sup_{j, k \in \mathbb{N}} \nu\left(x_{jk} - x, \frac{Q_n t}{n_0 A 3}\right), \sup_{j, k \in \mathbb{N}} \nu\left(x_{jk} - x, \frac{P_m t}{n_0 B 3}\right), \max_{n_0 < k, j} \nu\left(x_{jk} - x, \frac{t}{3}\right)\right\}. \end{aligned}$$

where $A = \max_{0 \leq k \leq n_0} q_k$, $B = \max_{0 \leq j \leq n_0} p_j$. Also, by q -boundedness of (x_{mn}) there exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} \inf_{j, k \in \mathbb{N}} \mu\left(x_{jk} - x, \frac{Q_n t}{n_0 A 3}\right) &> 1 - \varepsilon, & \sup_{j, k \in \mathbb{N}} \nu\left(x_{jk} - x, \frac{Q_n t}{n_0 A 3}\right) &< \varepsilon \\ \inf_{j, k \in \mathbb{N}} \mu\left(x_{jk} - x, \frac{P_m t}{n_0 B 3}\right) &> 1 - \varepsilon, & \sup_{j, k \in \mathbb{N}} \nu\left(x_{jk} - x, \frac{P_m t}{n_0 B 3}\right) &< \varepsilon \end{aligned}$$

for $n, m > n_1$. Hence, combining all above we get

$$\mu(t_{mn}^{11} - x, t) > 1 - \varepsilon, \quad \nu(t_{mn}^{11} - x, t) < \varepsilon$$

for $m, n > \max\{n_0, n_1\}$, which implies $t_{mn}^{11} \rightarrow x$ by Definition 3. \square

The converse statement of Theorem 7 is not valid which can be seen by the next example. That is, $(\bar{N}, p, q; 1, 1)$ summability does not imply convergence in *IFNS*.

Example 8. Let us consider *IFNS* $(\mathbb{R}, \mu_0, \nu_0)$ where μ_0, ν_0 are from Example 2. Sequence (x_{mn}) defined by $x_{mn} = (-1)^{m+n}$ is in $(\mathbb{R}, \mu_0, \nu_0)$. (x_{mn}) is $(\bar{N}, p, q; 1, 1)$ summable to 0 with $p_j = 1, q_k = 1(j, k \in \mathbb{N})$, but it is not convergent in $(\mathbb{R}, \mu_0, \nu_0)$. See also [12, Example 3.3].

Our aim is to give the conditions under which $(\bar{N}, p, q; 1, 1)$ summability implies convergence in *IFNS*. Let SVA_+ be the set of all nonnegative sequences $p = (p_j)$ with $p_0 > 0$ satisfying

$$\liminf_{m \rightarrow \infty} \left| \frac{P_{\lambda_m}}{P_m} - 1 \right| > 0 \quad \text{for all } \lambda > 0 \text{ with } \lambda \neq 1$$

where $\lambda_m = [\lambda m]$ and $[\cdot]$ denote integral part [2]. In [2, Lemma 2.2], equivalent assertions for the set SVA_+ are obtained.

Theorem 9. Let $p, q \in SVA_+$ and double sequence (x_{mn}) be in (V, μ, ν) . Assume that $x_{mn} \rightarrow x$ $(\bar{N}, p, q; 1, 1)$. Then, $x_{mn} \rightarrow x$ if and only if for all $t > 0$

$$\sup_{\lambda > 1} \liminf_{m, n \rightarrow \infty} \mu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t \right) = 1 \quad (1)$$

and

$$\inf_{\lambda > 1} \limsup_{m, n \rightarrow \infty} \nu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t \right) = 0. \quad (2)$$

Proof. Necessity. Let $x_{mn} \rightarrow x$. Fix $t > 0$. For any $\lambda > 1$, and for sufficiently large m, n such that $\lambda_m > m, \lambda_n > n$ we have (see [2, Equation (3.3)])

$$\begin{aligned} & \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) = \quad (3) \\ & t_{\lambda_m, \lambda_n}^{11} - x_{mn} + \frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}) + \frac{1}{\frac{Q_{\lambda_n}}{Q_n} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}) \\ & + \frac{1}{\left(\frac{P_{\lambda_m}}{P_m} - 1\right) \left(\frac{Q_{\lambda_n}}{Q_n} - 1\right)} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11} - t_{\lambda_m, n}^{11} + t_{mn}^{11}) \end{aligned}$$

Since $p, q \in SVA_+$ and $t_{mn}^{11} \rightarrow x$ we have

$$\mu(t_{\lambda_m, \lambda_n}^{11} - x_{mn}, t) \rightarrow 1, \quad \nu(t_{\lambda_m, \lambda_n}^{11} - x_{mn}, t) \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty,$$

$$\begin{aligned} \mu \left(\frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}), t \right) &\geq \mu \left(t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}, \left\{ \liminf_{m \rightarrow \infty} \left| \frac{P_{\lambda_m}}{P_m} - 1 \right| \right\} t \right) \rightarrow 1 \text{ as } \min\{m, n\} \rightarrow \infty \\ \nu \left(\frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}), t \right) &\leq \nu \left(t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}, \left\{ \liminf_{m \rightarrow \infty} \left| \frac{P_{\lambda_m}}{P_m} - 1 \right| \right\} t \right) \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mu \left(\frac{1}{\frac{Q_{\lambda_m}}{Q_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}), t \right) &\geq \mu \left(t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}, \left\{ \liminf_{m \rightarrow \infty} \left| \frac{Q_{\lambda_m}}{Q_m} - 1 \right| \right\} t \right) \rightarrow 1 \text{ as } \min\{m, n\} \rightarrow \infty \\ \nu \left(\frac{1}{\frac{Q_{\lambda_m}}{Q_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}), t \right) &\leq \nu \left(t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}, \left\{ \liminf_{m \rightarrow \infty} \left| \frac{Q_{\lambda_m}}{Q_m} - 1 \right| \right\} t \right) \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty \end{aligned} \tag{5}$$

we obtain

$$\begin{aligned} \lim_{m, n \rightarrow \infty} \mu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t \right) &= 1 \\ \lim_{m, n \rightarrow \infty} \nu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t \right) &= 0. \end{aligned}$$

which implies (1) and (2). *Sufficiency.* Let (1) and (2) be satisfied. Fix $t > 0$. Then for given $\varepsilon > 0$ we have:

- There exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \mu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t/5 \right) &> 1 - \varepsilon \\ \nu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t/5 \right) &< \varepsilon \end{aligned}$$

for all $m, n > n_0$.

- There exists $n_1 \in \mathbb{N}$ such that $\mu \left(t_{\lambda_m, \lambda_n}^{11} - x, t/5 \right) > 1 - \varepsilon$ and $\nu \left(t_{\lambda_m, \lambda_n}^{11} - x, t/5 \right) < \varepsilon$ for $m, n > n_1$.
- There exists $n_2 \in \mathbb{N}$ such that

$$\mu \left(\frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}), t/5 \right) > 1 - \varepsilon \text{ and } \nu \left(\frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}), t/5 \right) < \varepsilon$$

for $m, n > n_2$ by (4).

- There exists $n_3 \in \mathbb{N}$ such that

$$\mu \left(\frac{1}{\frac{Q_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}), t/5 \right) > 1 - \varepsilon \text{ and } \nu \left(\frac{1}{\frac{Q_{\lambda_m}}{Q_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}), t/5 \right) < \varepsilon$$

for $m, n > n_3$ by (5).

- There exists $n_4 \in \mathbb{N}$ such that

$$\begin{aligned} \mu \left(\frac{1}{\left(\frac{P_{\lambda_m}}{P_m} - 1\right)} \frac{1}{\left(\frac{Q_{\lambda_n}}{Q_n} - 1\right)} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11} - t_{\lambda_m, n}^{11} + t_{mn}^{11}), t/5 \right) &> 1 - \varepsilon \\ \nu \left(\frac{1}{\left(\frac{P_{\lambda_m}}{P_m} - 1\right)} \frac{1}{\left(\frac{Q_{\lambda_n}}{Q_n} - 1\right)} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11} - t_{\lambda_m, n}^{11} + t_{mn}^{11}), t/5 \right) &< \varepsilon \end{aligned}$$

for $m, n > n_4$.

Then, by equation (3) we get

$$\begin{aligned} &\mu(x_{mn} - x, t) \\ &\geq \min \left\{ \mu(t_{\lambda_m, \lambda_n}^{11} - x, t/5), \mu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t/5 \right), \right. \\ &\quad \mu \left(\frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11}), t/5 \right), \mu \left(\frac{1}{\frac{Q_{\lambda_m}}{Q_m} - 1} (t_{\lambda_m, \lambda_n}^{11} - t_{\lambda_m, n}^{11}), t/5 \right), \\ &\quad \left. \mu \left(\frac{1}{\left(\frac{P_{\lambda_m}}{P_m} - 1\right)} \frac{1}{\left(\frac{Q_{\lambda_n}}{Q_n} - 1\right)} (t_{\lambda_m, \lambda_n}^{11} - t_{m, \lambda_n}^{11} - t_{\lambda_m, n}^{11} + t_{mn}^{11}), t/5 \right) \right\} \\ &> 1 - \varepsilon \end{aligned}$$

and, similarly,

$$\nu(x_{mn} - x, t) < \varepsilon$$

for $m, n > \max\{n_0, n_1, n_2, n_3, n_4\}$, and this implies $x_{mn} \rightarrow x$. The proof is completed. \square

We can give following theorem similar to Theorem 9. The proof is given in a similar way by making the changes $\lambda_m \leftrightarrow m$ and $\lambda_n \leftrightarrow n$.

Theorem 10. *Let $p, q \in SVA_+$ and double sequence (x_{mn}) be in (V, μ, ν) . Assume that $x_{mn} \rightarrow x$ $(\bar{N}, p, q; 1, 1)$. Then, $x_{mn} \rightarrow x$ if and only if for all $t > 0$*

$$\sup_{0 < \lambda < 1} \liminf_{m, n \rightarrow \infty} \mu \left(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^m \sum_{k=\lambda_n+1}^n p_j q_k (x_{mn} - x_{jk}), t \right) = 1$$

and

$$\inf_{0 < \lambda < 1} \limsup_{m, n \rightarrow \infty} \nu \left(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^m \sum_{k=\lambda_n+1}^n p_j q_k (x_{mn} - x_{jk}), t \right) = 0.$$

Now we define slow oscillation for double sequences in *IFNS*.

Definition 11. A double sequence (x_{mn}) in (V, μ, ν) is said to be slowly oscillating in the sense (1,1) if

$$\sup_{\lambda > 1} \liminf_{m, n \rightarrow \infty} \min_{\substack{m < j \leq \lambda_m \\ n < k \leq \lambda_n}} \mu(x_{jk} - x_{mn}, t) = 1$$

and

$$\inf_{\lambda > 1} \limsup_{m, n \rightarrow \infty} \max_{\substack{m < j \leq \lambda_m \\ n < k \leq \lambda_n}} \nu(x_{jk} - x_{mn}, t) = 0,$$

for all $t > 0$.

A double sequence (x_{mn}) is slowly oscillating in the sense (1,1) if and only if for all $t > 0$ and for all $\varepsilon \in (0, 1)$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$, depending on t and ε , such that

$$\mu(x_{jk} - x_{mn}, t) > 1 - \varepsilon \quad \text{and} \quad \nu(x_{jk} - x_{mn}, t) < \varepsilon$$

whenever $n_0 \leq m < j \leq \lambda_m$ and $n_0 \leq n < k \leq \lambda_n$.

Theorem 12. Let $p, q \in SVA_+$ and $x \in V$. If double sequence (x_{mn}) in (V, μ, ν) is slowly oscillating in the sense (1,1) and $x_{mn} \rightarrow x$ $(\bar{N}, p, q; 1, 1)$, then $x_{mn} \rightarrow x$.

Proof. Let $x_{mn} \rightarrow x$ $(\bar{N}, p, q; 1, 1)$ and (x_{mn}) be slowly oscillating in the sense (1,1). Fix $t > 0$. Then for given $\varepsilon \in (0, 1)$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\mu(x_{jk} - x_{mn}, t) > 1 - \varepsilon \quad \text{and} \quad \nu(x_k - x_n, t) < \varepsilon$$

whenever $n_0 \leq m < j \leq \lambda_m$ and $n_0 \leq n < k \leq \lambda_n$ by slow oscillation. Hence, we have

$$\mu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t \right) \geq \min_{\substack{m < j \leq \lambda_m \\ n < k \leq \lambda_n}} \mu(x_{jk} - x_{mn}, t) > 1 - \varepsilon$$

and

$$\nu \left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}), t \right) \leq \max_{\substack{m < j \leq \lambda_m \\ n < k \leq \lambda_n}} \nu(x_{jk} - x_{mn}, t) < \varepsilon.$$

So conditions (1) and (2) are satisfied. Then, by Theorem 9 we get $x_{mn} \rightarrow x$. \square

Definition 13. A double sequence (x_{mn}) in (V, μ, ν) is said to be slowly oscillating in the sense $(1,0)$ if

$$\sup_{\lambda > 1} \liminf_{m, n \rightarrow \infty} \min_{m < j \leq \lambda_m} \mu(x_{jn} - x_{mn}, t) = 1$$

and

$$\inf_{\lambda > 1} \limsup_{m, n \rightarrow \infty} \max_{m < j \leq \lambda_m} \nu(x_{jn} - x_{mn}, t) = 0,$$

for all $t > 0$.

Definition 14. A double sequence (x_{mn}) in (V, μ, ν) is said to be slowly oscillating in the sense $(0,1)$ if

$$\sup_{\lambda > 1} \liminf_{m, n \rightarrow \infty} \min_{n < k \leq \lambda_n} \mu(x_{mk} - x_{mn}, t) = 1$$

and

$$\inf_{\lambda > 1} \limsup_{m, n \rightarrow \infty} \max_{n < k \leq \lambda_n} \nu(x_{mk} - x_{mn}, t) = 0,$$

for all $t > 0$.

Theorem 15. If a double sequence (x_{mn}) in (V, μ, ν) is slowly oscillating in the sense $(1,0)$ and $(0,1)$, then it is slowly oscillating in the sense $(1,1)$.

Proof. By the facts that

$$\begin{aligned} \mu(x_{jk} - x_{mn}, t) &\geq \min\{\mu(x_{jk} - x_{mk}, t/2), \mu(x_{mk} - x_{mn}, t/2)\} \\ \nu(x_{jk} - x_{mn}, t) &\leq \max\{\nu(x_{jk} - x_{mk}, t/2), \nu(x_{mk} - x_{mn}, t/2)\} \end{aligned}$$

and by slow oscillation of (x_{mn}) in the sense $(1,0)$ and $(0,1)$, we conclude that (x_{mn}) is slowly oscillating in the sense $(1,1)$. \square

By Theorem 9, Theorem 12 and Theorem 15, we get following theorem.

Theorem 16. Let $p, q \in SVA_+$ and $x \in V$. If double sequence (x_{mn}) in (V, μ, ν) is slowly oscillating in the senses $(1,0) \mathcal{E} (0,1)$ and $x_{mn} \rightarrow x (\bar{N}, p, q; 1, 1)$, then $x_{mn} \rightarrow x$

In view of the theorem above and Theorem 6, we give the next theorem.

Theorem 17. Let $p, q \in SVA_+$ and $x \in V$. If $x_{mn} \rightarrow x (\bar{N}, p, q; 1, 1)$ and sequences $\{m(x_{mn} - x_{m-1, n})\}$, $\{n(x_{mn} - x_{m, n-1})\}$ are q -bounded, then $x_{mn} \rightarrow x$.

We note that (P_m) is regularly varying of index $\rho > 0$ if [2]

$$\lim_{m \rightarrow \infty} \frac{P_{\lambda m}}{P_m} = \lambda^\rho, \quad (\lambda > 0).$$

It is noted in [2], SVA_+ contains all nonnegative sequences $p = (p_j)$ such that (P_m) is regularly varying of positive index and

$$\limsup_{m \rightarrow \infty} \frac{P_{\lambda m} - P_m}{P_m} = \lambda^\rho - 1. \quad (6)$$

Theorem 18. Let $p = (p_j)$ be nonnegative sequence with $p_0 > 0$ and (P_m) is regularly varying of positive indices. If $\left\{ \frac{P_m}{p_m}(x_{mn} - x_{m-1,n}) \right\}$ is q -bounded, then (x_{mn}) is slowly oscillating in the sense (1,0).

Proof. Let $\left\{ \frac{P_m}{p_m}(x_{mn} - x_{m-1,n}) \right\}$ be q -bounded. Then, for given $\varepsilon > 0$ there exists $M_\varepsilon > 0$ so that

$$t > M_\varepsilon \Rightarrow \inf_{m,n \in \mathbb{N}} \mu \left(\frac{P_m}{p_m}(x_{mn} - x_{m-1,n}), t \right) > 1 - \varepsilon \quad \text{and} \quad \sup_{m,n \in \mathbb{N}} \nu \left(\frac{P_m}{p_m}(x_{mn} - x_{m-1,n}), t \right) < \varepsilon.$$

For every $t > 0$ choose $\lambda < \left(1 + \frac{t}{M_\varepsilon}\right)^{1/\rho}$. Then for $n_0 < m < j \leq \lambda m$ we have

$$\begin{aligned} \mu(x_{jn} - x_{mn}, t) &= \mu \left(\sum_{r=m+1}^j (x_{rn} - x_{r-1,n}), t \right) \\ &\geq \min_{m+1 \leq r \leq j} \mu \left(x_{rn} - x_{r-1,n}, \frac{p_r}{P_j - P_m} t \right) \\ &= \min_{m+1 \leq r \leq j} \mu \left(\frac{P_r}{p_r}(x_{rn} - x_{r-1,n}), \frac{P_r}{P_j - P_m} t \right) \\ &\geq \min_{m+1 \leq r \leq j} \mu \left(\frac{P_r}{p_r}(x_{rn} - x_{r-1,n}), \frac{P_m}{P_{\lambda m} - P_m} t \right) \\ &= \min_{m+1 \leq r \leq j} \mu \left(\frac{P_r}{p_r}(x_{rn} - x_{r-1,n}), \frac{t}{\frac{P_{\lambda m} - P_m}{P_m}} \right) \\ &\geq \min_{m+1 \leq r \leq j} \mu \left(\frac{P_r}{p_r}(x_{rn} - x_{r-1,n}), \frac{t}{\lambda^\rho - 1} \right) \\ &\geq \inf_{m,n \in \mathbb{N}} \mu \left(\frac{P_m}{p_m}(x_{mn} - x_{m-1,n}), \frac{t}{\lambda^\rho - 1} \right) \\ &> 1 - \varepsilon \end{aligned}$$

and

$$\nu(x_{jn} - x_{mn}, t) < \sup_{m,n \in \mathbb{N}} \nu \left(\frac{P_m}{p_m}(x_{mn} - x_{m-1,n}), \frac{t}{\lambda^\rho - 1} \right) < \varepsilon.$$

in view of (6). Hence, (x_{mn}) is slowly oscillating in the sense (1,0). \square

Similarly we can give next theorem for slow oscillation in the sense (0,1).

Theorem 19. Let $q = (q_k)$ be nonnegative sequence with $q_0 > 0$ and (Q_n) is regularly varying of positive indices. If $\left\{ \frac{Q_n}{q_n}(x_{m,n} - x_{m,n-1}) \right\}$ is q -bounded, then (x_{mn}) is slowly oscillating in the sense (0,1).

In view of Theorem 16, Theorem 18 and Theorem 19 we get following theorem.

Theorem 20. Let $p = (p_j)$ and $q = (q_k)$ be nonnegative sequences with $p_0 > 0, q_0 > 0$, and (P_m) and (Q_n) be regularly varying of positive indices. If $x_{mn} \rightarrow x(\bar{N}, p, q; 1, 1)$ and sequences

$\left\{ \frac{P_m}{p_m}(x_{mn} - x_{m-1,n}) \right\}, \left\{ \frac{Q_n}{q_n}(x_{m,n} - x_{m,n-1}) \right\}$ are q -bounded, then $x_{mn} \rightarrow x$.

3. RESULTS FOR $(\bar{N}, p, *, 1, 0)$ SUMMABILITY IN $IFNS$

Similar to the results of the previous section we can give results for $(\bar{N}, p, *, 1, 0)$ summability of double sequences in $IFNS$. The proofs are also similar, hence they are omitted. Similar to Theorem 7, we give the next theorem.

Theorem 21. *If double sequence (x_{mn}) in (V, μ, ν) is q -bounded and convergent to $x \in V$, then (x_{mn}) is $(\bar{N}, p, *, 1, 0)$ summable to x .*

Similar to Theorem 9, we give the next theorem.

Theorem 22. *Let $p \in SVA_+$ and double sequence (x_{mn}) be in (V, μ, ν) . Assume that $x_{mn} \rightarrow x$ $(\bar{N}, p, *, 1, 0)$. Then, $x_{mn} \rightarrow x$ if and only if for all $t > 0$*

$$\sup_{\lambda > 1} \liminf_{m, n \rightarrow \infty} \mu \left(\frac{1}{P_{\lambda m} - P_m} \sum_{j=m+1}^{\lambda m} p_j(x_{jn} - x_{mn}), t \right) = 1$$

and

$$\inf_{\lambda > 1} \limsup_{m, n \rightarrow \infty} \nu \left(\frac{1}{P_{\lambda m} - P_m} \sum_{j=m+1}^{\lambda m} p_j(x_{jn} - x_{mn}), t \right) = 0.$$

Proof. The proof is done in a similar way as in the proof of Theorem 9 by using the equation (see [2, Equation (3.13)])

$$\frac{1}{P_{\lambda m} - P_m} \sum_{j=m+1}^{\lambda m} p_j(x_{jn} - x_{mn}) = t_{\lambda m, n}^{10} - x_{mn} + \frac{1}{(P_{\lambda m}/P_m) - 1} (t_{\lambda m, n}^{10} - t_{m, n}^{10})$$

($\lambda > 1; \lambda_m > m$)

instead of the equation (3). \square

In view of the equation (see [2, Equation (3.14)])

$$\frac{1}{P_m - P_{\lambda m}} \sum_{j=\lambda m+1}^m p_j(x_{jn} - x_{mn}) = t_{m, n}^{10} - x_{mn} + \frac{1}{(P_m/P_{\lambda m}) - 1} (t_{m, n}^{10} - t_{\lambda m, n}^{10})$$

($0 < \lambda < 1; \lambda_m < m$)

we can give the next theorem as analogue of Theorem 10.

Theorem 23. *Let $p \in SVA_+$ and double sequence (x_{mn}) be in (V, μ, ν) . Assume that $x_{mn} \rightarrow x$ $(\bar{N}, p, *, 1, 0)$. Then, $x_{mn} \rightarrow x$ if and only if for all $t > 0$*

$$\sup_{0 < \lambda < 1} \liminf_{m, n \rightarrow \infty} \mu \left(\frac{1}{P_m - P_{\lambda m}} \sum_{j=\lambda m+1}^m p_j(x_{mn} - x_{jn}), t \right) = 1$$

and

$$\inf_{0 < \lambda < 1} \limsup_{m, n \rightarrow \infty} \nu \left(\frac{1}{P_m - P_{\lambda m}} \sum_{j=\lambda m+1}^m p_j(x_{mn} - x_{jn}), t \right) = 0.$$

Similar to Theorem 12, we give the next theorem.

Theorem 24. Let $p \in SVA_+$ and $x \in V$. If double sequence (x_{mn}) in (V, μ, ν) is slowly oscillating in the sense $(1, 0)$ and $x_{mn} \rightarrow x$ $(\bar{N}, p, *, 1, 0)$, then $x_{mn} \rightarrow x$.

In view of the above theorem and Theorem 6, Theorem 18 we give the next theorems.

Theorem 25. Let $p \in SVA_+$ and (x_{mn}) be in (V, μ, ν) . If $x_{mn} \rightarrow x$ $(\bar{N}, p, *, 1, 0)$ and sequence $\{m(x_{mn} - x_{m-1,n})\}$ is q -bounded, then $x_{mn} \rightarrow x$.

Theorem 26. Let $p = (p_j)$ be a nonnegative sequences with $p_0 > 0$ and (P_m) be regularly varying of positive indice. If $x_{mn} \rightarrow x$ $(\bar{N}, p, *, 1, 0)$ and sequence $\left\{\frac{P_m}{p_m}(x_{mn} - x_{m-1,n})\right\}$ is q -bounded, then $x_{mn} \rightarrow x$.

4. CONCLUSION

In this present research article, we defined weighted mean summability method of double sequences in intuitionistic fuzzy normed spaces (*IFNS*). This study reveals also Tauberian results for some known summation methods in the special cases, and obtained necessary and sufficient Tauberian conditions under which convergence of double sequences in *IFNS*.

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