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LYAPUNOV EXPONENT USING EULER'S ALGORITHM WITH APPLICATIONS IN OPTIMIZATION PROBLEMS

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Abstract: The difference and differential equations have played an eminent part in nonlinear dynamics systems, but in the last two decades one-dimensional difference maps are considered in the forefront of nonlinear systems and the optimization of transportation problems. In the nineteenth century, the nonlinear systems have paved a significant role in analyzing nonlinear phenomena using discrete and continuous time interval. Therefore, it is used in every branch of science such as physics, chemistry, biology, computer science, mathematics, neural networks, traffic control models, etc. This paper deals with the maximum Lyapunov exponent property of the nonlinear dynamical systems using Euler's numerical algorithm. The presents experimental as well as numerical analysis using time-series diagrams and Lyapunov functional plots. Moreover, due to the strongest property of Lyapunov exponent in nonlinear system it may have some application in the optimization of transportation models.

Keywords: Optimization, Euler's algorithm, Lyapunov exponent, logistic map, chaos.

MSC: 39A23, 39A28, 39A30.

1. INTRODUCTION

The dynamical system is collection of possible states in which one may apply any function and can determine novel states using the older states. That means, a dynamical system is described by a method that portrays what will be the next result of future behavior for the current situation of the system. The logistic map $rx(1-x)$, is assumed as one of the most important one-dimensional difference maps described by V. F. Verhulst (1845 and 1847). Poincare [1], was a great physicist who first examined the irregularity in the dynamical systems which depends on the initial situation of the system. Further, it was noticed that there is major relation between the Lyapunov exponent and the dynamics of a nonlinear system because it depends on the initial conditions of nonlinear system. Due to its sensitive dependency on initial conditions, it is used in various real-life applications. Further, it is measured as the strongest property like Fork-width scaling, bifurcation scaling and period-3 window. Furthermore, the researcher may read in detail about the Lyapunov exponent property from the following articles: Holmgren [2], Devaney ([3], [4]), Alligood et al. [5], Ausloos and Dirickx [6], Robinson [7], Andrecut [8], etc.

Due to its important role in the nonlinear dynamical systems, it is used in various applications of science and engineering. In 2010, H. Kocak and K. J. Palmer [9] described the Lyapunov exponent and its dependency on initial conditions. They found that such type of property exists away from the critical point of the dynamical system and is shown for some restricted group of difference and differential maps in which the positive value stands for the chaotic behavior the dynamical system and the negative value denotes the stability in fixed and periodic states of the system. In 2004, C. Abraham et. al. [10] examined the maximum Lyapunov exponent and determines its sensitive behavior. They also determined some boundary conditions for the sensitivity constant. H. Shao and Y. Shi [11] studied the relationship between stability, Lyapunov exponent and the sensitivity for non-autonomous type discrete difference systems. They described some new terminologies like strongest sensitive independency for a point and a set, stability measurement using Lyapunov exponent, and asymptotical stability. In 2001, B. Demir and S. Kocak [12] provided two examples, in the first example they proved positivity of Lyapunov exponent at a point which does not depends on the initial conditions and in the second example they examined negative Lyapunov exponent at a point which depends on the initial conditions. But the maximum Lyapunov property is used in various applications of science. M. A. Abdul et. al. [13] studied the Lyapunov exponent in Duffing maps which depends on the initial conditions. They determined the region for the positive Lyapunov exponent as well as negative

Lyapunov exponent. They produced the time series Lyapunov plot in programming software, "Matlab". Further, for more study on the Lyapunov exponent and its various applications one may refer to: J. Urais [14], M. Opstal [15], etc.

In 2018, Ashish et. al. [16] studied the dynamics the nonlinear one-dimensional logistic map using superior feedback iterative method and examined some dynamical properties such as fixed and periodic states, period-3 window and maximum Lyapunov exponent. They found that Lyapunov exponent property plays a vital role to increase the stability in chaos using superior feedback iterative technique. Further, they presented an advanced chaos-based transportation problem using Lyapunov exponent. Further, in 2021 Ashish et al. [17] established the discrete chaotification behavior in generalized logistic map. Lyapunov exponent property was measured analytically as well as experimentally followed by Lyapunov time series plots. For more recent study one may also refer to Ashish et. al. ([18], [19], [20], [21], [22]).

This paper is divided in two four sections. Section one gives the introduction and literature review about the Lyapunov exponent and its application in science and engineering. Section 2 presents the numerical analysis and the derivation of the Lyapunov exponent method followed by some time-series diagrams and Examples. In Section 3, we give the experimental analysis of the Lyapunov exponent value followed by some Lyapunov plot and the bifurcation plot. Finally, the paper is summarized in the Section 4.

2. LYAPUNOV EXPONENT IN EULER'S ALGORITHM

In this section we deal with the Lyapunov exponent behavior versus time-series representation of the logistic map using Euler's numerical algorithm. For this, let us take some particular values of the Euler's parameter h and the growth rate parameter r. It is described that for the negative value of the Lyapunov exponent the iterative orbit approaches to stable fixed and periodic states while in case of positive Lyapunov exponent the iterative orbit diverges from each other. Therefore, this section starts with following Euler Numerical algorithm system:

$$E_h(s,r) = s + hf_r(s), \tag{1}$$

where r and h are greater than zero. Then, putting the standard one-dimensional logistic map $f_r(x) = rx(1-x)$ in (1) and solving, we get the following Euler's type logistic relation

$$E_h(s,r) = (1 + hr)s(1 - s), \tag{2}$$

where s belongs to the closed interval $[0, 1]$ and the relation (2) is called as Euler's type novel logistic system. Now, to derive the Lyapunov exponent formula, let us start with the two initiators s and $s + \epsilon$, where $0 < \epsilon < 1$. Then, we obtain the divergence μ between the two iterative orbits, that is,

$$E_h^n(s+\epsilon,r) - E_h^n(s,r) = \mu,$$
$$E_h^n(s+\epsilon,r) - E_h^n(s,r) = \epsilon e^{n\delta},$$
$$\frac{E_h^n(s+\epsilon,r) - E_h^n(s,r)}{\epsilon} = e^{n\delta}$$

Since the value of the difference depends on the initial difference ϵ between two orbits, therefore, taking $\epsilon \to 0$, we get

$$\lim_{\epsilon\to\infty} \frac{E_h^n(s+\epsilon,r) - E_h^n(s,r)}{\epsilon} = e^{n\delta},$$
$$\text{that is,} \quad \lim_{\epsilon\to\infty} (E_h^n)'(s,r) = e^{n\delta}.$$
$$\text{thus,} \quad \Delta = \frac{1}{n} log|(E_h^n)'(s,r)|, \tag{3}$$

where $(E_h^n)'(s,r)$, denotes the first derivative of the n^{th} degree difference map. The derivative of system is determined using Devaney's chain rule method of derivative, that is, for an iterative orbit $s_1, s_2 = E_n(s_1,r), s_3 = E_n(s_2,r)....$ and so on, we obtain

$$(E_h^n)'(s_1,r) = E_h'(s_n,r).E_h'(s_{n-1},r).....E_h'(s_1,r) \tag{4}$$

Then, from (3) and (4) we have

$$\Delta = \frac{1}{n} log|E_h'(s_n,r).E_h'(s_{n-1},r).....E_h'(s_1,r)|,$$
$$\Delta = \frac{1}{n}[log|E_h'(s_n,r)| + log|E_h'(s_{n-1},r)| + ... + log|E_h'(s_1,r)|,$$
$$\Delta = \frac{1}{n}\sum_{i=1}^{n} log|E_h'(s_i,r)|. \tag{5}$$

Which is the required Lyapunov exponent formula. When we take periodic orbit of order p, then the Lyapunov exponent is as follows:

$$\Delta = \frac{1}{p}\sum_{i=1}^{p} log|E_h'(s_i,r)|. \tag{6}$$

Further, it is noticed that in case of irregular iterative orbits other than periodic or fixed the full length of the orbit is impossible to use, therefore, a finite length of the orbit is used in case of irregular orbits.

Example 1. *Let* $E_h(s,r) = (1+hr)s(1-s)$ *be the one-dimensional logistic map, where* $s \in [0,1]$. *Then, determine the maximum Lyapunov exponent value when* $h = 0.1$ *and* $r = 15$.

Solution: It is trivial from the dynamics of a Euler's logistic map for $h = 0.1$ and $r = 15$, there exists a stable fixed point $\frac{hr}{1+hr}$ in the system, that is, $\frac{hr}{1+hr} = 0.6$. Therefore, to determine the maximum Lyapunov exponent we solve the given equation (5). For this, first of all let us consider the Euler's system $E_h(s, r) = (1 + hr)s(1 - s)$, then, we have

$$E'_h(s, r) = (1 + hr) - 2(1 + hr)s.$$

Then, substituting the $h = 0.1$, $r = 15$ and $s = 0.6$, we determine

$$\begin{aligned}
E'_{0.1}(0.6, r) &= (1 + 0.1 \times 15) - 2 \times (1 + 0.1 \times 15) \times 0.6, \\
E'_{0.1}(0.6, r) &= 2.5 - 2 \times 2.5 \times 0.6, \\
E'_{0.1}(0.6, r) &= -0.5. \qquad (7)
\end{aligned}$$

Then, putting the value of (7) in (5), we obtain the required Lyapunov exponent

$$\Delta = ln| - 0.5| = -0.6931.$$

Thus, at $h = 0.1$ and $r = 15$, we get the required Lyapunov exponent value -0.3010. It is clear that for the fixed point $\frac{hr}{1+hr} = 0.6$ the maximum Lyapunov exponent is negative and hence by the definition of the Lyapunov exponent, negative value denotes stability of the orbit as well as fixed point.

Example 2. *Let $E_h(s, r) = (1 + hr)s(1 - s)$ be the one-dimensional logistic map, where $s \in [0, 1]$. Then, determine the maximum Lyapunov exponent value when $h = 0.1$ and $r = 22$.*

Solution: It is trivial from the dynamics of a Euler's logistic map for $h = 0.1$ and $r = 22$, there exists two periodic points $s_1 = 0.5130$ and $s_2 = 0.7995$. Therefore, to determine the maximum Lyapunov exponent we solve the given equation (5). For this, first of all let us consider the Euler's system $E_h(s, r) = (1 + hr)s(1 - s)$, then, we have

$$E'_h(s, r) = (1 + hr) - 2(1 + hr)s.$$

Then, substituting the $h = 0.1$, $r = 22$ and $s_1 = 0.5130$ and $s_2 = 0.7995$, we determine

$$\begin{aligned}
E'_{0.1}(0.5130, r) &= (1 + 0.1 \times 22) - 2 \times (1 + 0.1 \times 22) \times 0.5130, \\
E'_{0.1}(0.5130, r) &= -0.0832, \qquad (8) \\
E'_{0.1}(0.7995, r) &= (1 + 0.1 \times 22) - 2 \times (1 + 0.1 \times 22) \times 0.7995, \\
E'_{0.1}(0.7995, r) &= -1.9168. \qquad (9)
\end{aligned}$$

Then, putting the value of (8) and (9) in (5), we obtain

$$\Delta = \frac{1}{2}[ln| - 0.0832| + ln| - 1.9168|] = -0.7973.$$

Thus, at $h = 0.1$ and $r = 22$, we get the required Lyapunov exponent value -0.7973. Hence by the definition of the Lyapunov exponent, negative value denotes stability of the orbit as well as periodic points of order 2. Figure 1-4 shows the maximum Lyapunov exponent behavior for the particular values of fixed point, periodic point and chaotic regime. When $h = 0.1$ and $r = 4.5, 5, 5.5$ the Lyapunov trajectory approaches to a fixed negative Lyapunov value as shown in Figure 1 and Figure 2 shows that at $r = 21, 22,$ and 23 the Lyapunov trajectory for periodic regime also approaches to a negative maximum Lyapunov exponent value. Further, for the chaotic regime all the trajectories at $r = 26$, 28 and 29 always approaches to positive maximum Lyapunov exponent as shown in Figure 3. Similarly, Figure 4 shows the Lyapunov exponent trajectories at $h = 0.4$. From the diagram it is clear that for fixed point and periodic points the trajectories approach to negative Lyapunov value and for chaotic regime it approaches to positive Lyapunov exponent. Thus, proceeding in this way the maximum Lyapunov exponent values for some particular values of r and h also may be determined.

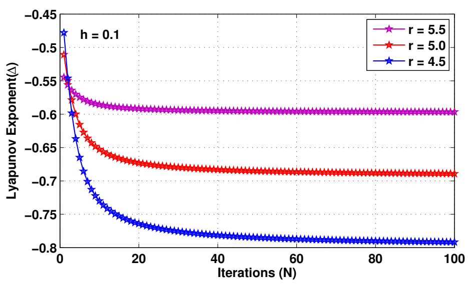

Figure 1: Lyapunov exponent for fixed orbits when $0 \leq r \leq 20$ and $h = 0.1$

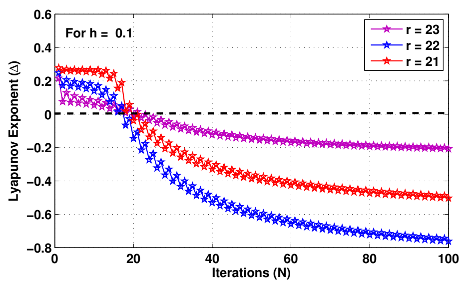

Figure 2: Lyapunov exponent for periodic orbits when $20 < r \leq 25.69$ and $h = 0.1$

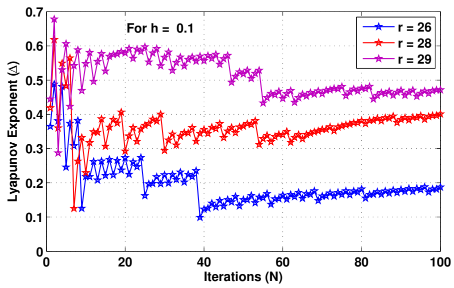

Figure 3: Lyapunov exponent for chaotic orbits when $r > 25.6996$ and $h = 0.1$

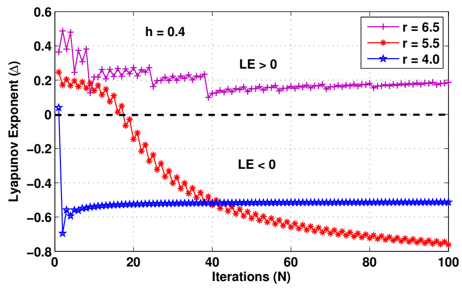

Figure 4: Lyapunov exponent for fixed, periodic and chaotic orbits for $0 < r \leq 7.5$

3. LYAPUNOV EXPONENT: AN EXPERIMENTAL STUDY

In the above section, we have described the Lyapunov exponent method for logistic map using Euler's Numerical Algorithm. Also, the time-series representation of maximum Lyapunov exponent for a particular value of fixed point, periodic point and chaotic regime is also determined using computational method in Matlab. Figure 1 and 2 describes the maximum value of Lyapunov exponent for a fixed value of Euler's parameter h and growth rate parameter r while the Figure 3 examines the chaotic regime. Nowadays, Lyapunov exponent is assumed another eminent property in nonlinear dynamical systems. Therefore, the section deals with the maximum range of the Lyapunov exponent for complete fixed, periodic and chaotic orbit of a one-dimensional system and a measure of sensitive dependence is measured. For the sake of simplicity, three cases for $h = 0.1, 0.4$ and 0.7 are taken in to consideration for the analysis.

Case 1: Maximum Lyapunov Exponent: when $h = 0.1$, and $0 \leq r \leq 30$

It is trivial that at $h = 0.1$ the one-dimensional Euler's logistic map shows complete dynamical behavior for the growth rate parameter range $0 \leq r \leq 30$. Also, it is true that for the parameter $0 \leq r \leq 20$ it gives fixed-point nature and system always remains in the fixed state and for the parameter $20 < r \leq 25.6996$ the system becomes periodic and always remains in the periodic states of order $2, 4, 8, 16, 32, \ldots$ and so on. Further, as the growth rate parameter r increases from 25.6996 the system approaches to chaotic regime. In Figure 5 and 6, the graphical plotting of maximum Lyapunov exponent $\triangle$ versus the Euler's parameter h and growth rate parameter r is drawn. In the diagram the positive islands for the chaotic nature and the negative islands shows for no irregular activity, that is, no chaos. Taking the initiator $s_0 \in [0, 1]$, and $h = 0.1$ the system $(1 + hr)s(1 - s)$ is iterated and a sequence of Lyapunov exponent values is generated for the prescribed range of parameter r, that is, $0 \leq r \leq 30$. The complete Lyapunov sequence is drawn on the graph as shown in Figure 5. The magnified version of the figure is shown in Figure 6 which describes the chaotic islands for the chaotic nature. Finally, the Lyapunov spectrum approaches to a maximum Lyapunov exponent value 0.6934. Moreover, Figure 7 shows the complete bifurcation plot and Figure 8 give the comparative analysis of Bifurcation plot versus Lyapunov exponent.

Case 2: Maximum Lyapunov Exponent: when $h = 0.4$, and $0 \leq r \leq 7.5$

At $h = 0.4$ the one-dimensional Euler's logistic map shows complete dynamical behavior for the growth rate parameter range $0 \leq r \leq 7.5$. For $0 \leq r \leq 5$ it gives fixed-point nature and system always remains in the fixed state and for the parameter $5 < r \leq 6.4299$ the system becomes periodic and always remains in the periodic states of order $2, 4, 8, 16, 32, \ldots$ and so on. Further, as the growth rate parameter r increases from 6.4299 the system approaches to chaotic regime. In Figure 9 and 10, the graphical plotting of maximum Lyapunov exponent $\triangle$ versus

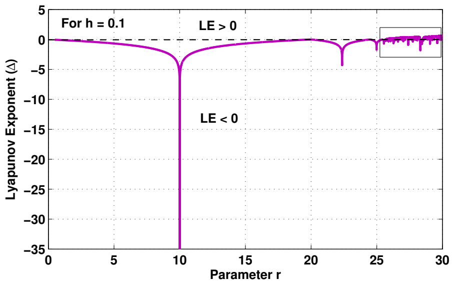

Figure 5: Lyapunov Exponent plot for the map $E_h(x, r)$ for $h = 0.1$ and $r \in [0, 30]$

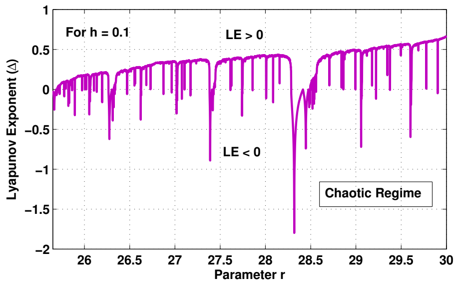

Figure 6: Magnified Chaotic Lyapunov regime for the map $E_h(x, r)$ for $h = 0.1$

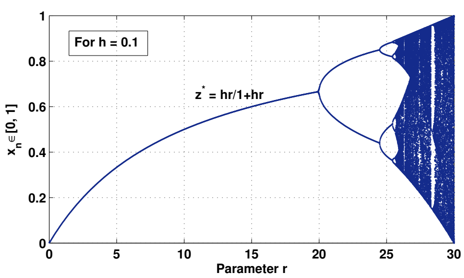

Figure 7: Period-doubling plot for the given map $E_h(x, r)$ when $h = 0.1$

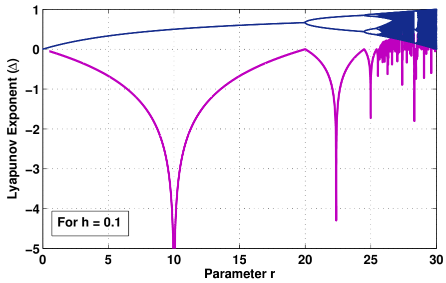

Figure 8: Lyapunov versus Bifurcation plot for the map $E_h(x, r)$ for $h = 0.1$

the Euler's parameter h and growth rate parameter r is drawn. In the diagram the positive islands for the chaotic nature and the negative islands shows for no irregular activity, that is, no chaos. Taking the initiator $s_0 \in [0, 1]$, and $h = 0.4$ the system $(1 + hr)s(1 - s)$ is iterated and a sequence of Lyapunov exponent values is generated for the prescribed range of parameter r, that is, $0 \leq r \leq 7.5$. The complete Lyapunov sequence is drawn on the graph as shown in Figure 9. The magnified version of the figure is shown in Figure 10 which describes the chaotic islands for the chaotic nature. Figure 11 shows the complete bifurcation plot and Figure 12 give the comparative analysis of Bifurcation plot versus Lyapunov exponent. Finally, the Lyapunov spectrum approaches to a maximum Lyapunov exponent value 0.6934.

Case 3: Maximum Lyapunov Exponent: for $h = 0.7$, and $0 \leq r \leq 4.25$

In this case, we present the Lyapunov exponent behavior of the Euler's type logistic map when $h = 0.7$ and the growth rate parameter approaches from 0 to 4.25. In the dynamics of Euler's logistic map, the system exhibits fixed point nature for the parameter $0 \leq r \leq 2.8339$ and the system admits periodic behavior of order

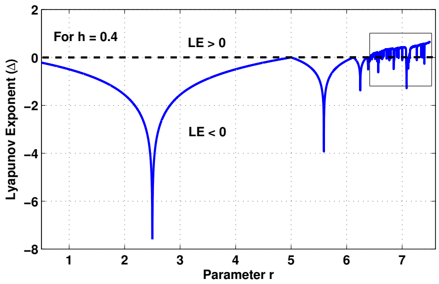

Figure 9: Lyapunov Exponent plot for the map $E_h(x, r)$ for $h = 0.4$ and $r \in [0, 7.5]$

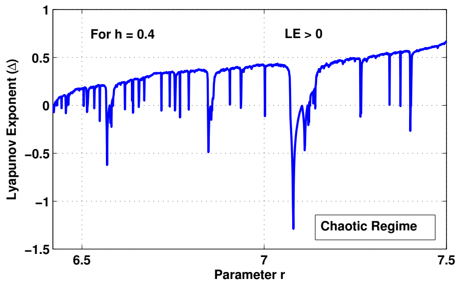

Figure 10: Magnified Chaotic Lyapunov regime for the map $E_h(x, r)$ for $h = 0.4$

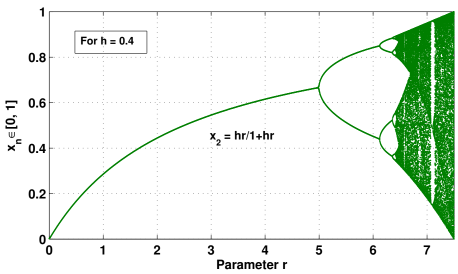

Figure 11: Period-doubling plot for the given map $E_h(x, r)$ when $h = 0.4$

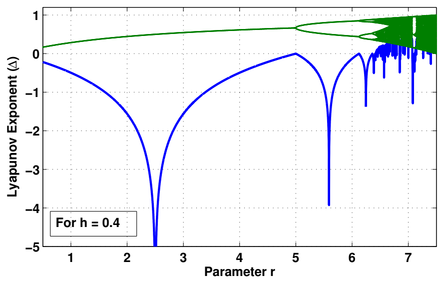

Figure 12: Lyapunov versus Bifurcation plot for the map $E_h(x, r)$ for $h = 0.4$

$2, 4, 8, 16, 32, \ldots$ and so on when the parameter r approaches from 2.8339 to 3.6696. But as the value of parameter r approaches through 3.6696 the system becomes fully chaotic. Here, in this case we determine the maximum Lyapunov exponent for the complete dynamics of the Euler's type logistic system, experimentally. The Figures 13 and 14 are plotted in which the Lyapunov exponent approaches to its maximum value for the growth rate parameter r. Figure 13 shows that as r approaches from 0 to 2.8339 the Lyapunov spectrum always remains in the negative cycle similarly as r approaches from 3.6696 the spectrum again admits negative regime on the graph, that means, in case of fixed and periodic states the maximum Lyapunov always remains in negative regime. Further, the positive regime of the Lyapunov spectrum is zoomed in the Figure 14. While the Figure 15 shows the complete bifurcation plot and Figure 16 give the comparative analysis of Bifurcation plot versus Lyapunov exponent. It is examined that as the range of the parameter r approaches through 3.6696 the Lyapunov exponent spectrum starts to approach in the positive quadrant. Finally, the maximum Lyapunov exponent for the Euler's type logistic map approaches to 0.6187.

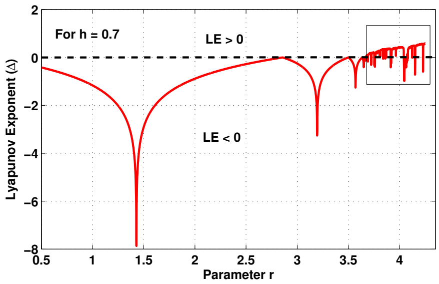

Figure 13: Lyapunov Exponent plot for the map $E_h(x, r)$ for $h = 0.7$ and $r \in [0, 4.25]$

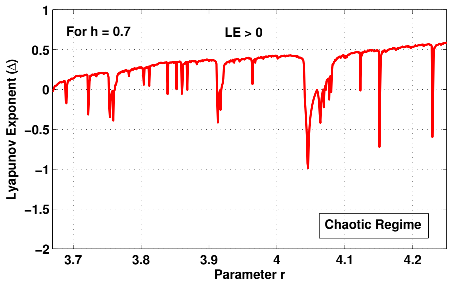

Figure 14: Magnified Chaotic Lyapunov regime for the map $E_h(x, r)$ for $h = 0.7$

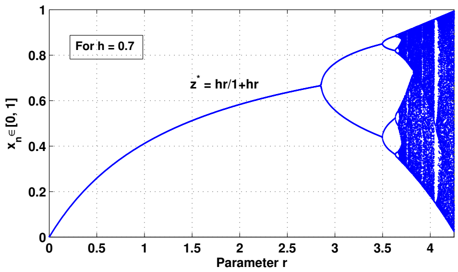

Figure 15: Period-doubling plot for the given map $E_h(x, r)$ when $h = 0.7$

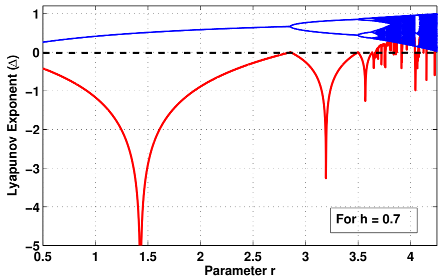

Figure 16: Lyapunov versus Bifurcation plot for the map $E_h(x, r)$ for $h = 0.7$

4. CONCLUSION

This article deals with the maximum Lyapunov exponent value of the logistic map using Euler's numerical algorithm system. In section 2, the Lyapunov exponent method is derived using logistic map as well as Euler's numerical algorithm followed by some examples which determines the maximum Lyapunov exponent for a particular value of fixed and periodic points. Further, in section 3 an experimental study to determine the Lyapunov exponent of Euler's type logistic map is carried out. Section is divided into three cases. In case-1 the maximum Lyapunov exponent for $h = 0.1$ and $0 \leq r \leq 30$ is shown in which the negative Lyapunov exponent shows the stability in fixed and periodic states and the negative Lyapunov exponent regime give the chaotic nature of the system. Similarly, in case 2 and 3 the maximum Lyapunov exponent is shown for the Euler's parameter $h = 0.4$ and 0.7, respectively followed by Lyapunov plot. Due to its higher stability range and higher chaotic regime the Euler's type logistic system may have various applications in Science and Engineering such as transportation problems, cryptography, in optimization, secure communication, etc.

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REFERENCES

[1] H. Poincare, *Les Methods Nouvells de la Mecanique Leleste*. Gauthier Villars, Paris, 1899.

[2] R. A. Holmgren, *A First Course in Discrete Dynamical Systems*. Springer Verlag, New York Inc., 1994.

[3] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, 1948.

[4] ——, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*. Addison-Wesley, 1992.

[5] K. T. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems*. Springer Verlag, New York Inc., 1996.

[6] M. Ausloos and M. Dirickx, *The logistic Map and the Route to Chaos, From the Beginnings to Modern Applications*. Springer, New York, 2006.

[7] C. Robinson, *Dynamical systems: Stability, Symbolic Dynamics and Chaos*. CRC Press, Boca Raton, 1995.

[8] M. Andrecut, "Logistic map as a random number generator," *International Journal of Modern Physics B*, vol. 12, no. 09, pp. 921–930, 1998. doi: 10.1142/S021797929800051X

[9] H. Kocal and K. J. Palmer, "Lyapunov exponents and sensitive dependence," *Journal of dynamics and differential equations*, vol. 22, no. 3, pp. 381–398, 2010. doi: 10.1007/s10884-010-9169-y

[10] B. G. Abraham, C. and B. Cadre, "On lyapunov exponent and sensitivity," *Journal of mathematical analysis and applications*, vol. 290, no. 2, p. 395–404, 2004. doi: 10.1016/j.jmaa.2003.10.029

[11] H. Shao and Y. Shi, "Lyapunov exponents, sensitivity, and stability for non-autonomous discrete systems," *International Journal of Bifurcation and Chaos*, vol. 28, no. 7, pp. 1 850 088–95, 2018. doi: 10.1142/S0218127418500888

[12] B. Demir and S. Kocak, "A note on positive lyapunov exponent and sensitive dependence on initial conditions," *Chaos, Solitons and Fractals*, vol. 12, no. 11, pp. 2119–2121, 2001. doi: 10.1016/S0960-0779(00)00160-0

[13] M. A. A. Alyaseen, "Lyapunov exponent and sensitive dependence on initial conditions of duffing map," *International Journal of Mathematical Science and Engineering Applications*, vol. 8, no. 1, pp. 393–399, 2014. doi: 10.1007/s40815-018-0594-0

[14] J. Urias, R. Rechtman, and A. Enciso, "Sensitive dependence on initial conditions for cellular automata," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 7, no. 4, pp. 688–693, 1997. doi: 10.1063/1.166266

[15] M. Van Opstall, "Quantifying chaos in dynamical systems with lyapunov exponents," *Electronic Journal of Undergraduate Mathematics*, vol. 4, no. 1, pp. 1–8, 1998. doi: 10.1007/s40815-018-0594-0

[16] Ashish, J. Cao, and R. Chugh, "Chaotic behavior of logistic map in superior orbit and an improved chaos-based traffic control model," *Nonlinear Dynamics*, vol. 94, no. 2, pp. 959–975, 2018. doi: 10.1007/s11071-018-4403-y

[17] J. Ashish, Cao and R. Chugh, "Discrete chaotification in modulated logistic system," *International Journal of Bifurcation and Chaos*, vol. 31, no. 05, pp. 2 150 065–70, 2021. doi: 10.1142/S0218127421500656

[18] Ashish and J. Cao, "A novel fixed-point feedback approaches studying the dynamical behavior of standard logistic map," *International Journal of Bifurcation and Chaos*, vol. 29, no. 1, pp. 1–16, 2019. doi: 10.1142/S021812741950010X

[19] Ashish, J. Cao, and R. Chugh, "Controlling chaos using superior feedback technique with applications in discrete traffic models," *International Journal of Fuzzy Systems*, vol. 21, no. 5, pp. 1467–1479, 2019. doi: 10.1007/s40815-019-00636-8

[20] Ashish, J. Cao, F. Alsaadi, and A. K. Malik, "Discrete superior hyperbolicity in chaotic maps," *Chaos Theory and Applications*, vol. 3, pp. 34–42, 2021. doi: 10.51537/chaos.936679

[21] Ashish, J. Cao, and F. Alsaadi, "Chaotic evolution of difference equations in mann orbit," *Journal of Applied Analysis and Computation*, vol. 11, pp. 3063–3082, 2021. doi: 10.11948/20210164

[22] Renu, Ashish, and R. Chugh, "On the dynamics of a discrete difference map in mann orbit," *Computational and Applied Mathematics*, vol. 41, pp. 1–19, 2022. doi: 10.1007/s40314-022-01938-8