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COMPUTING THE BOUNDS ON THE LOSS RATES

J.-M. FOURNEAU

PRiSM, Université de Versailles Saint-Quentin, France
{jmf,nih}@prism.uvsq.fr

L. MOKDAD

LAMSADE, Université de Paris Dauphine, France
{mokdad}@lamsade.dauphine.fr

N. PEKERGIN

CERMSEM, Université de Paris I Sorbonne, France

Abstract: We consider an example network where we compute the bounds on cell loss rates. The stochastic bounds for these loss rates using simple arguments lead to models easier to solve. We proved, using stochastic orders, that the loss rates of these easier models are really the bounds of our original model. For ill-balanced configurations these models give good estimates of loss rates.

Keywords: Discrete time Markov chains, tochastic bounds, ATM switch, loss rates.

1. INTRODUCTION

ATM (Asynchronous Transfer Mode) technology is intended to support a wide variety of services and applications and to satisfy a range of Quality-of-Service (QoS). The QoS is measured by a set of parameters intended to characterize the performance of the network. These performances depend generally on the switch performances. We are interested in computing the cell loss rates in a multistage ATM switch. Loss rates are very important because they may be part of the contract on the quality of service between the user and the network provider. Using a numerical method to compute loss rates is very difficult because of the size of the model. So, we propose a stochastic method to compute upper and lower bounds on the loss rates. To do this, we propose two simple systems which are easier to evaluate and which provide upper and lower bounds on the considered performance measure. We prove that the loss rates on the easier systems are really bounds on the original system. We make this proof, using stochastic method based on stochastic ordering and stochastic comparisons [8, 2]. The switch, we considered is decomposed into several queues with feed-forward routing. All queues are finite and the external arrivals always take place at the first stage. The variability of the input processes provokes losses in the queues. The topology of this switch leads us to use a decomposition to find loss rates stage by stage. To compute the loss rate in the first stage, several solutions can be considered according to the arrival process. If we assume i.i.d. batch arrivals of Markov modulated batch arrivals (MMBP), we can easily build a Markov chain of one buffer. Let B be the size of the buffer, iff we consider i.i.d. batch process, the chain has $B+1$ states. For a MMBP with n states for the modulation, the size of the chain is $n*(B+1)$. Thus, the numerical computation is always possible. If we restrict ourselves to less general processes, analytical solutions may also be obtained (see Beylot's PhD thesis for some results on Clos networks [1]). However, the second stage is much more difficult to analyze. Indeed, it is quite impossible to know exactly the arrival process into a buffer in the second stage even if we assume a simple i.i.d. batch arrival process at the first stage. The output process of the first stage is usually unknown due to the loss at the first stage and the superposition of such processes is unknown even if we assume independence. It may be possible that under some restricted assumptions, some asymptotic results may be established. We do not try to prove such a result here, but we hope that we will be able to combine asymptotic results and bounds in the near future.

The paper is organized as follows. Section 2 presents modelization of ATM switch by Stochastic Automata Network. In Section 3, we propose two models which provide stochastic bounds for the loss rates, while in Section 4, we present numerical results which show that for ill-balanced loads the results may be quite good.

2. MODELIZATION BY STOCHASTIC AUTOMATA NETWORKS

2.1. Stochastic Automata Networks

Markovian Models give tools to modelize sequential small systems. But Markovian models for parallel systems are not solved efficiently. Stochastic Automata Networks (SAN) have been introduced to allow us to modelize complex parallel systems. The SAN approach identifies in the system, the jobs that can be executed independetly except in specific points named synchronized points. In SAN, each job is represented by an automation. The size of the Markov chain of the system is the product of the size of each automation. An external event which communicates with a job is modelized by a transition from one state of the automation to another. This event can be a local event or a synchronized event.

- Local event assigns one job, so one automation. The rate of this transition can be fixed if it just depends on the corresponding automation or functional if the transition rate depends on the states of the other automata.
- Synchronized event assigns the states of several jobs. It represents state change of several automata simultaneously.

It has been proved in [6] that, if the states are in lexicographic order, then the generator matrix Q of the Markov chain associated to a continuous-time SAN is given by:

$$Q = \bigoplus_{i=1}^{N} F_i + \sum_{j=1}^{S} \left(\bigotimes_{i=1}^{N} S_{i,j} - \bigotimes_{i=1}^{N} R_{i,j} \right).$$

The transition matrix of the Markov chain associated to a discrete-time SAN is given by:

$$P = \bigotimes_{i=1}^{N} F_i + \sum_{j=1}^{S} \left(\bigotimes_{i=1}^{N} S_{i,j} - \bigotimes_{i=1}^{N} R_{i,j} \right)$$

where

- $\otimes$ and $\oplus$ are the tensor product and sum, respectively. (See Appendix for details.)
- N is the total number of automata in the network and S is the number of synchronizations.
- F_i is the transition matrix of the local transition. So it is the transition matrix of automaton i without synchronizations.
- $S_{i,j}$ is the transition matrix of automaton i due to synchronization j.
- $R_{i,j}$ is a matrix representing the normalization associated to the synchronization j on the automaton i.

The main advantage of this methodology is its ability to represent the Markov chain associated to the SAN model by a compact formula. This point is particularly important since it allows us to deal with systems which may have very large state spaces.

In the following section, we show how we modelize our system with the SAN methodology.

2.2. Modelization

We show in this section how we modelize an ATM switch using SAN. In order to simplify this modelization, we consider a switch with two stages as shown in Fig. 1.

Each queue is modelized by an automaton. Therefore, there are three queues in the system: two in the first stage and one in the second stage. The routing probabilities from the first stage to the second one are β_1 and β_2. The size of Markov chain is $(B_0+1)\times(B_1+1)\times(B_2+1)$. The system behaviour will be described through four synchronizations and one function. First, let us define these synchronizations and the function. The description of each synchronization is given as follows:

- S_0 is a synchronization which indicates that there is no service in queue 1 and no service in queue 2.
- S_1 is a synchronization which indicates that there is a service in queue 1 and no service in queue 2.

- S_2 is a synchronization which indicates that there is no service in queue 1 (buffer empty) and there is a service in queue 2.
- S_{12} is a synchronization which indicates that there is a service in both queue 1 and queue 2.

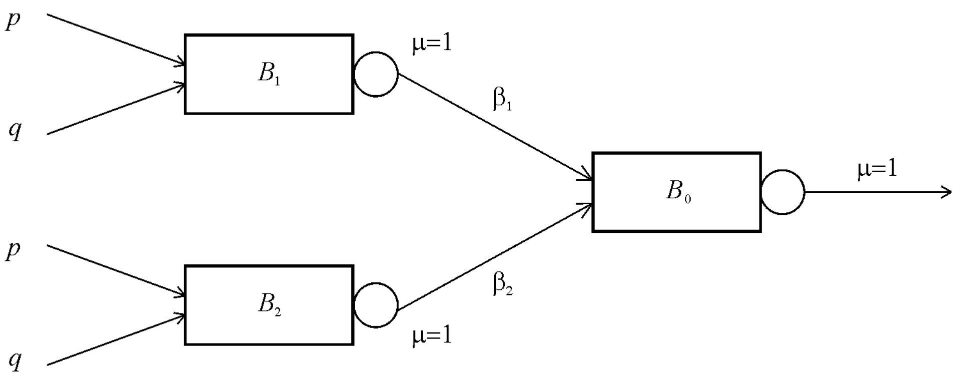

Figure 1: Switch with two stages

Let f be the function for geometric arrival process in the first stage where:

- $f(0)$ is the probability that there is no customer arrival with $f(0) = (1-p)(1-q)$
- $f(1)$ is the probability that there is one customer arrival with $f(1) = (1-p)q + p(1-q)$
- $f(2)$ is the probability that there are two customer arrivals with $f(2) = pq$.

We show in Figures 2, 3 and 4, the different automata. Fig. 2 shows the automaton corresponding to buffer 1 in the first stage. The automaton corresponding to buffer 2 in the first stage is given by Fig. 3. The Fig. 4 shows the automaton corresponding to buffer in the second stage.

3. MODELS, STOCHASTIC BOUNDS AND PROOFS

3.1. Stochastic Ordering

In this section, we give only the basic definitions and theorems of the strong (sample-path) ordering that will be used in this paper. We refer to the book of Stoyan [8] for an excellent survey of stochastic bounding technique applied in queuing theory.

First, let us give the definition of the sample path stochastic comparison of two random variables X and Y defined on a totally ordered space S, (a subset of R or N), since it is the most intuitive one.

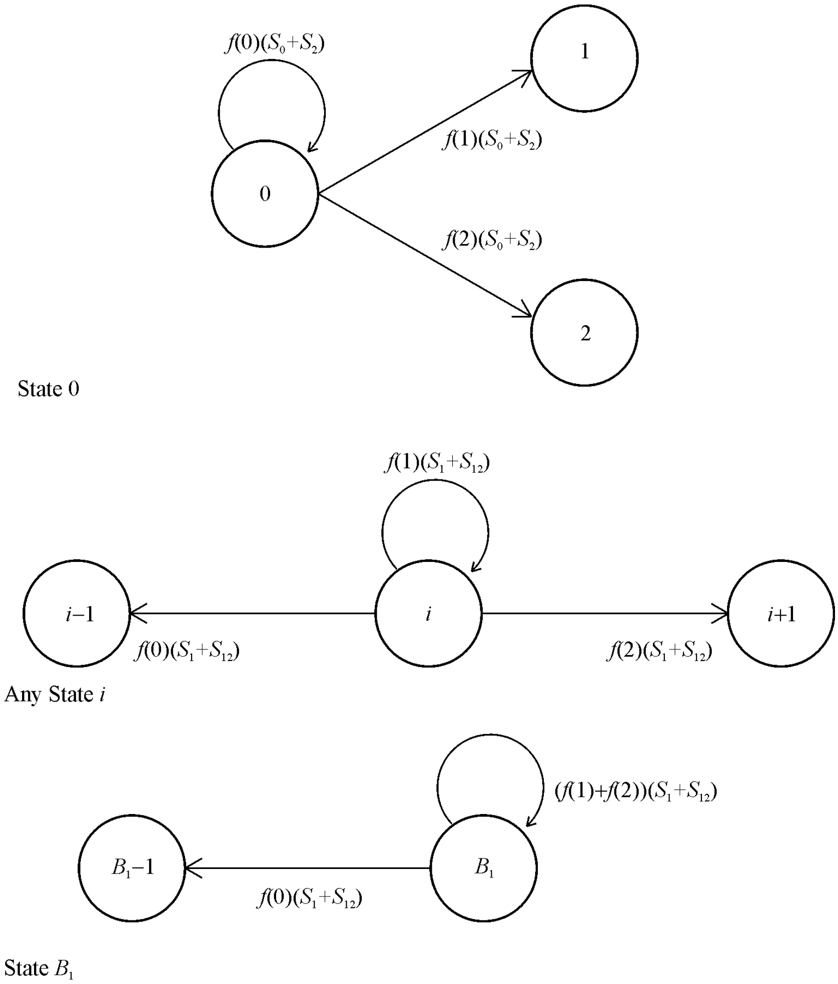

Figure 2: Automaton 1

Definition 1. *X is said to be less than Y in the sense of the sample-path (strong) ordering ($X \leq_{st} Y$) if and only if*

$$X \leq_{st} Y \Leftrightarrow \text{Prob}(x > a) \leq \text{Prob}(Y > a) \quad \forall a \in S .$$

In other terms, we compare the probability distribution functions of X and Y : it is more probable for Y to take larger values than for X . Moreover, $X =_{st} Y$ means that X and Y have the same distribution.

The state representation vectors of complex systems are generally multidimensional, thus the state spaces may not be totally ordered. In such cases, we must first choose the order relation on this space that must be reflexive and transitive but not necessarily anti-symmetric. In the sequel, we denote by $\preceq$ the pre-order or the partial order relation on the state space. The stochastic order associated with this vector ordering will be then denoted by $\preceq_{st}$.

The generic definition of a stochastic order is given by means of a class of functions. The strong stochastic ordering is associated with the increasing functions. We now give the generic definition in the general case: the random variables are defined on a space S, endowed with a relation order $\preceq$ (pre-order or partial order):

Definition 2.

$$X \preceq_{st} Y \Leftrightarrow Ef(X) \leq Ef(Y)$$

for every function $f: S \rightarrow R$ $\preceq$*-increasing, whenever the expectation exists.* f *is* $\preceq$*-increasing if and only if* $\forall x, y \in S,\ x \preceq y \rightarrow f(x) \leq f(y)$.

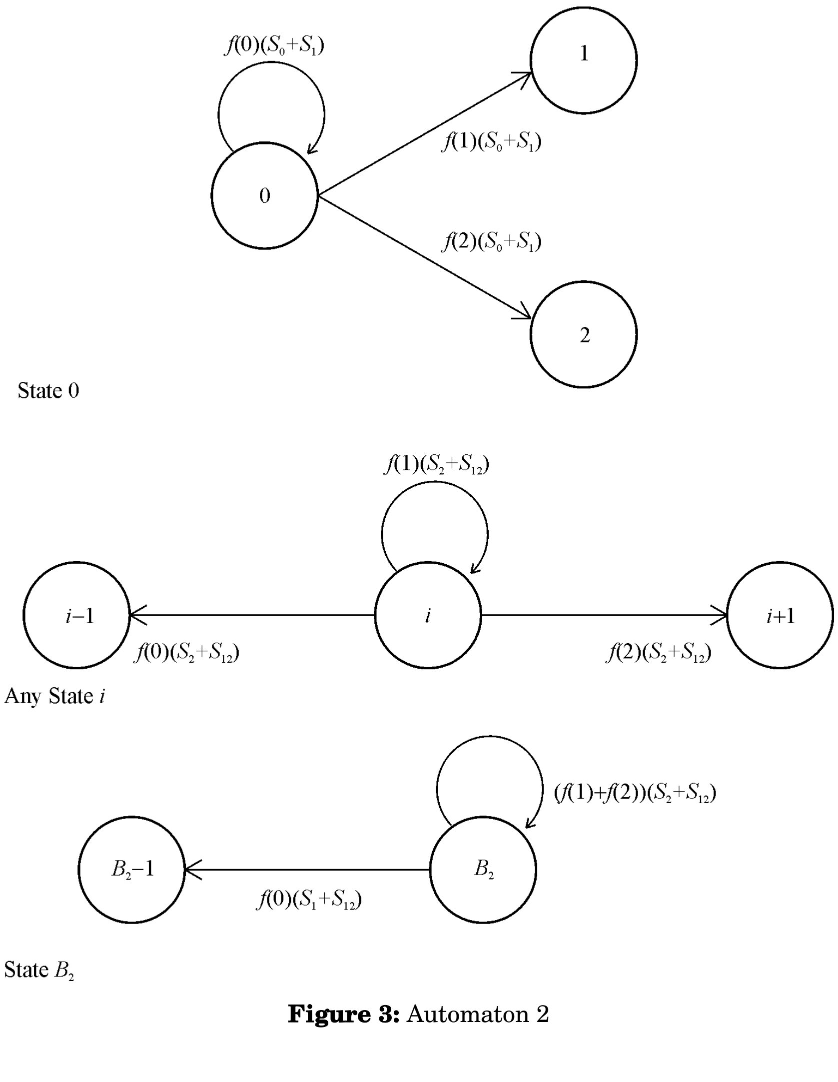

Figure 3: Automaton 2

We state only the sample-path properties of the strong stochastic ordering that will be applied to demonstrate the existence of stochastic comparison.

$X \preceq_{st} Y$, if and only if there exist random variables $\bar{X}$, $\bar{Y}$ defined on the same space, such that

- $\bar{X} =_{st} X$ and $\bar{Y} =_{st} Y$
- $\bar{X} \preceq \bar{Y}$ almost surely $(\text{Prob}(\bar{X} \preceq \bar{Y}) = 1)$.

In this work, we find bounding systems on a reduced state space, thus the state space of the considered system and the bounding ones are not same. Therefore we compare them on a common state space. To do this, we first project the underlying spaces into this common one, and then compare the images on this space. This type of comparison is called *comparison of images or comparison of state functions* [2]. In the sequel, since our main goal is comparing Markov chains, we assume that the considered state spaces are discrete.

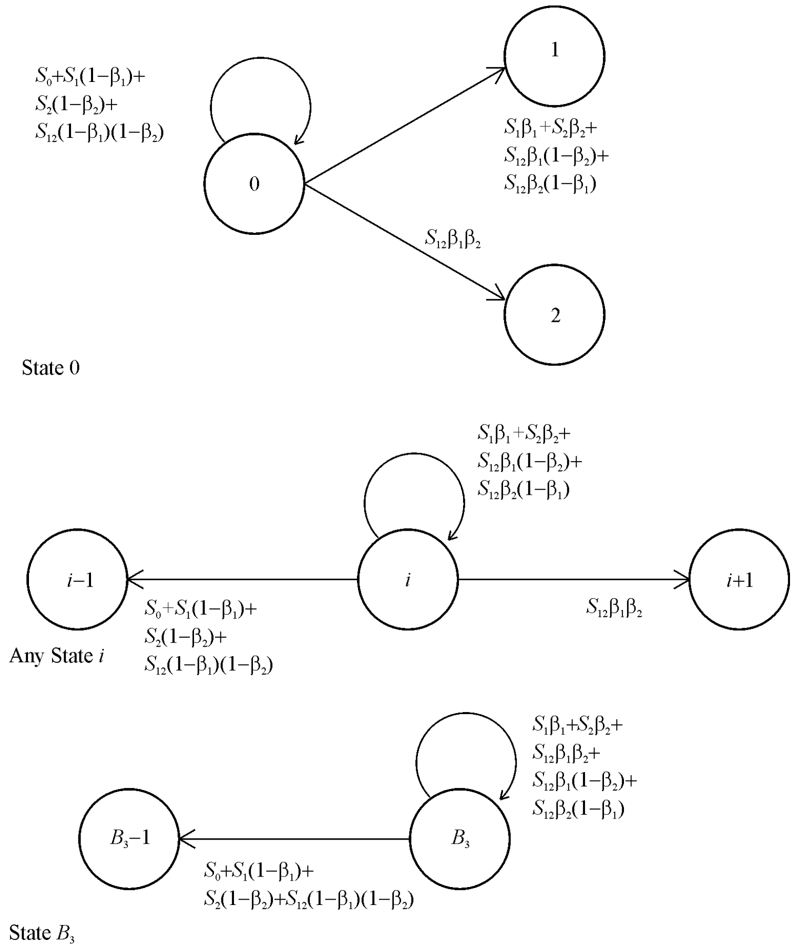

Figure 4: Automaton 3

Definition 3. *Let X (resp. Y) be a random variable which takes values on a discrete, countable space E (resp. F), and G be a discrete, countable state space endowed with a pre-order* $\preceq$; $\alpha : E \to G$ *(resp.* $\beta : F \to G$ *) be a many-to-one mapping. The image of X on G is less in the sense of* $\preceq_{st}$ *than the image of Y on G if and only if*

$$\alpha(X) \preceq_{st} \beta(Y) .$$

The comparison of the images may be defined more intuitively by representing the projection applications by matrices. Let M_α, M_β denote the matrices representing the underlying mappings, and the probability vectors p, q represent respectively the random variables X, Y . If

$$M_\alpha[i,j],\ i \in E \quad \text{and} \quad j \in G = \begin{cases} 1 & \text{if } \alpha(i) = 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\alpha(X) \preceq_{st} \beta(Y) \Leftrightarrow pM_\alpha \preceq_{st} qM_\beta . \tag{1}$$

Let us now assume that the state space comparison G be $\{1,\ldots,n\}$, then the comparison of images (equation 1) is defined by partial sums:

$$\forall i : n\ldots1 \quad \sum_{k=i}^{n} \sum_{j=1}^{n} p[j] \times M_\alpha[j,k] \le \sum_{k=i}^{n} \sum_{j=1}^{n} q[j] \times M_\beta[j,k] .$$

Obviously, the stochastic comparison of random variables is extended to the comparison of stochastic processes. There are two definitions, one of them corresponds to the comparison of one-dimensional increasing functionals, while the other is the comparison of the multidimensional functionals. We give both definitions in the context of Markov chains, nevertheless they are more general. Let $\{X(t),\ t \in T\}$ and $\{Y(t),\ t \in T\}$ be two Markov chains with discrete state space S (time parameter space may be discrete $T = N^+$ or continuous $T = R^+$).

Definition 4. $\{X(t),\ t \in T\}$ *is said to be less than* $\{Y(t),\ t \in T\}$ *with respect to* $\preceq_{st}$ ($\{X(t)\} \preceq_{st} \{Y(t)\}$) *if and only if*

$$X(t) \preceq_{st} Y(t) \quad \forall t \in T$$

which is equivalent to

$$Ef(X(t)) \le Ef(Y(t)) \quad \forall t \in T$$

for every $\preceq$*-increasing functional f, whenever the expectations exist.*

3.2. Sochastic Models

We consider here a network with several input buffers (see Fig. 5). In this section, we focus on the application of stochastic bounds to the second stage of the network. We assume that the arrivals follow an i.i.d. batch process. But, our methodology also applies to a Markov modulated batch arrival process and to the other stages of the network. We will show how to handle such cases in the conclusions.

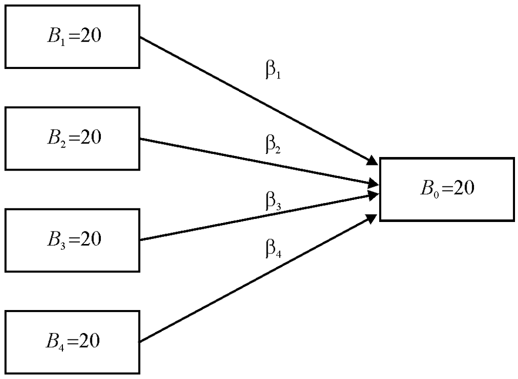

Figure 5: Exact model

Let m be the number of input buffers. Obviously, $(N_0(t), N_1(t), N_2(t),...,N_m(t)),\ t \geq 0$ is a discrete-time Markov chain. We now define two systems which are easier to evaluate and which provide upper and lower bounds on the considered performance measure (i.e. the cell loss rate at buffer 0). These systems have a smaller size, then it is not possible to compare directly the steady-state distribution on the same state space. This is a major difference with Truffet's approach which is based on the comparison of the same space with distributions which are obtained analytically . We first define the state space of comparison ε , and the pre-order $\preceq$ defined on this space.

We will use a limited representation for the input buffers in the space ε , while we represent explicitly the evolution in buffer 0. Let $s = (N_0, X_1, X_2,...,X_m) \in \varepsilon$, where

- N_0 is the exact number of cells at buffer 0.
- for the buffers of the first stage i.e., $1 \leq i \leq m$:
  - $X_i = 1$, if there are some cells at buffer i ,
  - $X_i = 0$, if there are no cells.

Since $N_0 \in \{0, B_0\}$ and $X_i \in \{0,1\}$ $1 \leq i \leq m$, the comparison state space is $\varepsilon = \{0 \cdots B_0\} \times \{0,1\} \times \cdots \times \{0,1\}$ where $\times$ is the cartesian product. We now define the pre-order $\preceq$ on ε : Let $x = (x_0, x_1,...,x_m)$, $y = (y_0, y_1,..., y_m) \in \varepsilon$.

$$\begin{cases} x \preceq y \ \text{ if } \ x_0 \leq y_0 \ \text{ and } \ x_1 = y_1 \cdots x_m = y_m \ \text{ and} \\ x = y \ \text{ if } \ x_i = y_i,\ 0 \leq i \leq m \end{cases}$$

It may be worthy to remark that this pre-order is chosen in order to compare the cell loss rates at buffer 0 (Eq. 5). Intuitively, when we compare the systems with the same capacity for buffer 0 if $x, y \in \varepsilon$ are two states such that $x \preceq y$, then the number of loss cells at state x will be less or equal to the number of lost cells at state y .

We compare the images of the considered systems on the state space ε in the sense of the stochastic order $\preceq_{st}$. The basic definitions and theorems for stochastic bounds are given in the Appendix, and more detailed information can be found in [2, 8, 9]. First, we define the following many-to-one mappings in order to project the state spaces of the compared systems into ε. Let $S^{\inf}$ be the state space of the system which provides the lower bound, while $S^{\sup}$ be the state space of the one associated to the upper bound

$$\varphi : S^{\inf} \to \varepsilon \quad \alpha : S \to \varepsilon \quad \beta : S^{\sup} \to \varepsilon .$$

Remember that the considered system can be modeled by a discrete-time Markov chain with rather general assumptions on the arrivals, and will be denoted by $\{s(t)\}_t$. Let bounding systems be discrete-time chains denoted by $\{s(t)^{\inf}\}_t$ and $\{s(t)^{\sup}\}_t$. The comparison of discrete-time Markov chains is defined as the conservation of the stochastic order on the initial distributions at each step (see Def. 5 in the Appendix). Then one must demonstrate the following stochastic order relations between the images of the chains:

$$\varphi(s^{\inf}(t)) \preceq_{st} \alpha(s(t)) \preceq_{st} \beta(s^{\sup}(t)) \quad \forall t \geq 0 \tag{2}$$

We now give an outline of the proof, using a sample-path approach (see Appendix):

1. In the first step we prove the existence of realizations verifying the following inequalities:

   $$\varphi(s^{\inf}(t)) \preceq \alpha(s(t)) \preceq \beta(s^{\sup}(t)) \quad \forall t \geq 0$$

   Because of the pre-order $\preceq$, one must build the realizations such that:

   for the lower bound:

   - for all input buffers, $1 \leq i \leq m$:

     if $X_i(t) = 0$, then $X_i^{\inf}(t) = 0,\ \forall t \geq 0$.

     This condition means that when no arrival may occur from buffer i to buffer 0 in the original system, then no arrivals may occur in the lower bounding system
   - and $N_0(t) \geq N_0^{\inf}(t),\ \forall t \geq 0$.

   for the upper bound:

   - for all input buffers, $1 \leq i \leq m$:

     if $X_i(t) = 1$, then $X_i^{\sup}(t) = 1,\ \forall t \geq 0$.

     This condition means that when an arrival may occur from buffer i to buffer 0 in the upper bounding system, then an arrival may occur in the original one.
   - and $N_0(t) \leq N_0^{\sup}(t),\ \forall t \geq 0$.
2. Then, the stochastic ordering $\preceq_{st}$ between the images (Eq. 2) follows from the first step as a consequence of the sample-path property (Eq. 1). Moreover, if there are steady-state distributions of the chains, then

$$\varphi(\Pi^{\inf}) \preceq_{st} \alpha(\Pi) \preceq_{st} \beta(\Pi^{\sup}) \quad (3)$$

where Π denotes the steady-state distribution.

3. The last step consists of the proof of the inequalities between the rewards on the steady-state distributions of the chains:

$$R^{\inf} \leq R \leq R^{\sup} \quad (4)$$

First we rewrite the reward function on the steady-state distribution defining cell loss rate (Eq. 5):

$$R = \sum_{s \in S} \pi(s) f(s) \quad \text{where} \quad f(s) = \sum_{j=1}^{m} p[j,s]((n_0 - 1)^+ + j - B_0)^+ \quad (5)$$

Remember that the arrival probabilities $p[j,s]$ for a state $s = (n_0, x_1, x_2, ..., x_m)$, are computed from the values of $x_i,\ 1 \leq i \leq m$. Then it is easy to see that if $s_1 \leq s_2$, then $f(s_1) \leq f(s_2)$, so $f(s)$ is a $\preceq$-increasing function.

Since the stochastic order has been proved between steady-state distributions (3), and the pre-order is chosen such that the functions defining the performance measure are $\preceq$-increasing, then the inequalities (Eq. 4) are a direct consequence of the stochastic order $\preceq_{st}$ (see definition with class of functions in the Appendix).

3.3. Lower Bound

We now propose systems providing lower bounds by considering the same topology for the network with smaller input buffers (see Fig. 6). Remember that the bounding system must be easer to evaluate than the original one. So, one must consider sufficiently small capacities to get a tractable numerical solution. Hence

$$B_i^{\inf} \leq B_i,\ 1 \leq i \leq m \quad \text{and} \quad B_0^{\inf} = B_0$$

Obviously, at least one of these inequalities must be strict.

We only give the demonstration of the first step.

- If $N_i^{\inf}(0) \leq N_i(0)$, since the external arrivals to the input buffers are the same, we have:

  $$N_i^{\inf}(t) \leq N_i(t), \quad 1 \leq i \leq m \ \ \forall t \geq 0$$

  Then the first condition is established:

  if $X_i(t) = 0$, then $X_i^{\inf}(t) = 0,\ \forall t \geq 0$

- We now consider the evolution of the cell number at buffer 0. A cell arrival to this buffer may occur if a service has been completed in the input buffers. As a result of the former step, one may build realizations of compared systems such that, if there is an arrival in the bounding system, then there is also an arrival in the original one. Therefore if $N_0^{\inf}(0) \le N_0(0)$, we may have

$N_0^{\inf}(t) \le N_0(t) \quad \forall t > 0$.

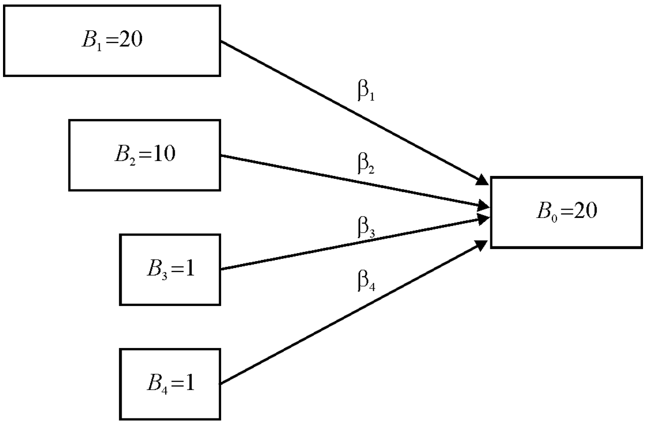

Figure 6: Model for lower bound

We do not prove the other steps. Since the stochastic order relation between the images of the steady-state distributions exists and the pre-order $\preceq$ is chosen such that the reward functions on these distributions are $\preceq$*-increasing*, we have the inequality (Eq. 4).

3.4. Upper bound

We simplify the original system by deleting some of the input buffers and replacing them by sources (see Fig. 7). An equivalent view is that these buffers are never empty. The resolution of the bounding system will be easier since we do not consider the evolution of the cell numbers at these input buffers. Let E be the set of the deleted input buffers, then

$X_j(t) = 1, \; \forall t \ge 0 \;\; j \in E$.

The buffer capacities for the other buffers are not changed:

$B_i^{\sup} = B_i$, if $i \not\in E$ and $B_0^{\inf} = B_0$.

Again, we only prove here the first step for the upper bound.

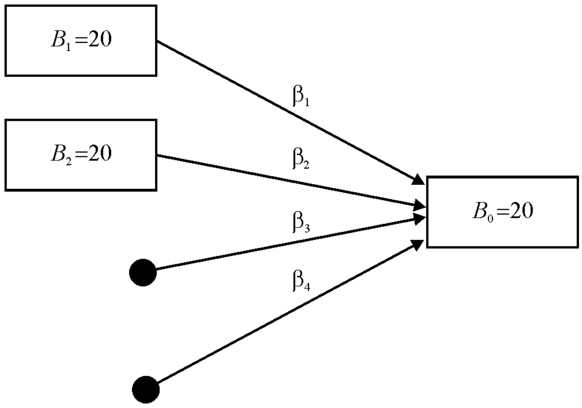

Figure 7: Model for upper bound

- Obviously, the cell numbers at the input buffers which are not deleted change in the same manner. Then if $N_i(0) \le N_i^{\sup}(0)$, we have:

  $N_i(t) \le N_i^{\sup}(t), \quad \forall t > 0 \;\; i \not\in E$

  Then the first condition is established for all input buffers, $1 \le i \le m$:

  If $X_i(t) = 1$ then $X_i^{\sup}(t) = 1 \;\; \forall t \ge 0$

- Now we consider the evolution at buffer 0. Since, if one cell arrival may occur in the original system, then it may also occur in the upper bounding one, then if $N_0(0) \le N_0^{\sup}(0)$, we may have:

  $N_0(t) \le N_0^{\sup}(t) \quad \forall t > 0$.

So we prove stochastic comparison between the images of the considered Markov chains. Since the same stochastic order relation exists between the steady-state distributions and the reward functions are $\preceq$*-increasing*, we have the inequality (Eq. 4).

4. NUMERICAL COMPUTATIONS

We apply this method to several topologies, several batch distributions of arrivals and several routing probabilities. We present here some typical results. We consider a system with 4 input buffers with the same size. Two cases are presented: buffers of size 10 and 20. The exact model is associated to a Markov chain of size $(B+1)^5$.

The upper bound is obtained with a model of two input buffers and two sources. Thus the chain size is only $(B+1)^3$. To compute the lower bounds, we keep two buffers unchanged and we change the size of the two others to only 2 cells. This leads to a chain of size $9(B+1)^3$. Clearly the upper bound is much easier to compute than the lower bound.

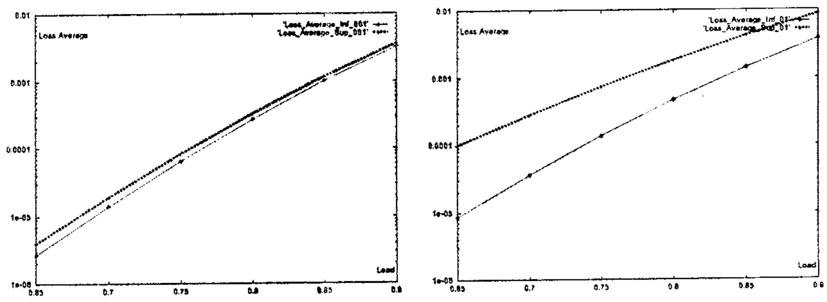

Figure 8: Buffer of size 10, $q = 0.01$ and $q = 0.1$

The best results are obtained when the flows of arrivals from the input buffers are unbalanced. For instance, in Figure 8 and 9, we present the bounds for buffer of sizes 10 and 20. We assume that the external arrivals batch is the superposition of 2 independent Bernoulli processes with probability p. So, the load in queues of the first stage is $2p$. The probabilities β_i are defined as $(0.4-q, 0.6-q, q, q)$.

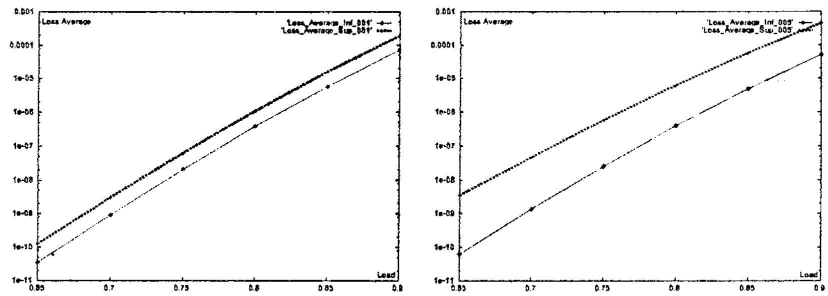

Figure 9: Buffer of size 20, $q = 0.01$ and $q = 0.05$

The second example is a system with buffers of size 20. The lower bound is computed using the following sizes for the 4 input buffers (20, 10, 1, 1). More accurate lower bounds may be found with more computation time.

We can compute several bounds using our results. For the upper bounds the number of buffers replaced by sources is arbitrary. For the lower bounds, all buffers may be shortened. Clearly, this gives a hierarchy of bounds with a tradeoff between accuracy and computation times.

Furthermore, even if we keep the state space constant, the lower bounds can be obtained by several strategies. For instance, we may consider a model with two buffers of size $\sqrt{A}$ or a model with a buffer of size $A/3-1$ and a buffer of size 2. These two configurations have roughly the same number of states. A natural question is to find some heuristics to change the buffer size and provide good lower bounds with approximately the same number of states as the model for upper bounds. These heuristics will probably be based on the output process intensity.

5. CONCLUSION

In this work, we present a method to estimate the cell loss rates in a second stage buffer of an ATM switch. Obviously, the considered system is a discrete time Markov chain, the lower numerical resolution is only tractable for very little buffer sizes. We propose to build bounding models of smaller sizes which are comparable in the sample-path stochastic ordering sense with the exact model.

Our model could be used to analyze rewards which are not decreasing functions of the steady-state distribution such as the losses or the delay. And it may be applied to all systems where the routing allows the decomposition and the analysis stage by stage for networks with independent flows of cells as feed-forward networks. Indeed, the same argument gives upper bound for the third stage (see Fig. 10). Some buffers are replaced by deterministic sources of cells with rate equal to 1. Then, these output processes follow the independent Bernoulli routing and are superposed with the other output processes which join at the third stage queue.

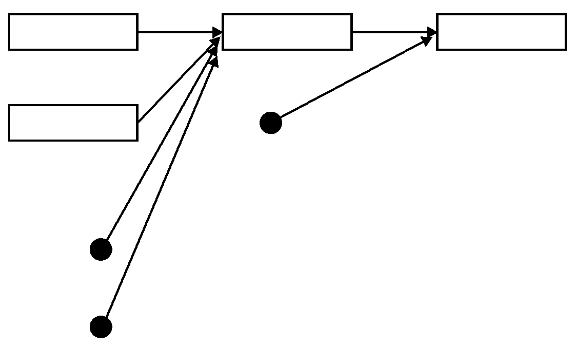

Figure 10: Upper bound for the third stage

Similarly, this method can be applied to networks with Markov modulated batch processes for the external arrivals. Deterministic sources will replace buffers to obtain the upper bound, while the model for lower bound will include the modulating chain to describe the external arrivals. Some interconnection networks exhibit dependence between the flows of cells after some stage. For instance, in the third stage of Clos networks, input processes are correlated because arrival processes into queues of the second stage are negatively correlated. It may be possible that upper bound be obtained using our technique even with such a negative correlation of input processes.

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APPENDIX 1

Definition 1. *Let* $A=[a_{ij}]$ *be a matrix of order* $n\times n$ *, and* $B=[b_{ij}]$ *a matrix of order* $p\times p$ *. The tensor product of A and B is a matrix C of order* $np\times np$ *such that C may be decomposed into* n^2 *blocks of size* p *.*

$$C=A\otimes B=\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

Definition 2. *Let* $A=[a_{ij}]$ *be a matrix of order* $n\times n$ *, and* $B=[b_{ij}]$ *a matrix of order* $p\times p$ *. The tensor sum of A and B is a matrix D defined by:*

$$D=A\oplus B=A\otimes I_p+I_n\otimes B$$

where I_p *and* I_n *represent the identity matrix of order* $p\times p$ *and* $n\times n$ *respectively.*

Example:

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$C=A\otimes B=\begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

$$D=A\oplus B=\begin{bmatrix} a_{11}+b_{11} & b_{12} & a_{12} & 0 \\ b_{21} & a_{11}+b_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22}+b_{11} & b_{12} \\ 0 & a_{21} & b_{21} & a_{22}+b_{22} \end{bmatrix}$$

APPENDIX 2

Let $\preceq$ be preorder (reflexive, transitive but not necessarily anti-symmetric) on a discrete, countable space ε. We consider two random variables X and Y defined respectively on discrete, countable spaces E and F, and their probability measures are given respectively by the probability vectors p and q where $p[i] = \text{Prob}(X = i),\ \forall i \in E$ (resp. $q[i] = \text{Prob}(Y = i),\ \forall i \in F$).

We define two many-to-one mappings $\alpha : E \to \varepsilon$ and $\beta : F \to \varepsilon$ to project the states of E and F into ε. First, we give the following proposition for the comparison of the images of X and Y on the space ε in the sense of $\preceq_{st}$ $(\alpha(X) \preceq_{st} \beta(Y))$:

Proposition 1. *The following propositions are equivalent*

- $\alpha(X) \preceq_{st} \beta(Y)$
- **definitions with class of functions:**

  $$\sum_{s\in\varepsilon} f(s) \sum_{n\in E|\alpha(n)=s} p[n] \le \sum_{s\in\varepsilon} f(s) \sum_{m\in F|\beta(m)=s} q[m] \qquad \forall f \preceq\text{-increasing}$$

  f is $\preceq$-increasing if $\forall x, y \in \varepsilon,\ x \preceq y \to f(x) \le f(y)$
- **definition with increasing sets:**

  $$\sum_{n\in E|\alpha(n)\in\Gamma} p[n] \le \sum_{m\in F|\beta(m)\in\Gamma} q[m] \ \text{ for all increasing sets } \Gamma$$

  Γ is an increasing set if $\forall x, y \in \varepsilon,\ x \preceq y$ *and* $x \in \Gamma \to y \in \Gamma$
- **sample-path property:**

  There exist random variables $\tilde{X}$ and $\tilde{Y}$ defined respectively on E and F, having the same probability measure as X and Y such that:

  $\alpha(\tilde{X}) \preceq \beta(\tilde{Y})$ *almost surely*

We now give the definition of the stochastic ordering between the images of discrete-time Markov chains.

Definition 5. *Let $\{X(i)\}_i$ (resp $\{Y(i)\}_i$) be discrete-time Markov chains in E (resp. F), we say the image of $\{X(i)\}_i$ on ε ($\{\alpha(X(i))\}_i$) is less than the image of the $\{Y(i)\}_i$, on ε ($\{\beta(Y(i))\}_i$) in the sense of $\preceq_{st}$ if*

$$\alpha(X(0)) \preceq_{st} \beta(Y(0)) \to \alpha(X(i)) \preceq_{st} \beta(Y(i)) \quad \forall i > 0 .$$