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# NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR FRACTIONAL INTERVAL-VALUED VARIATIONAL PROBLEMS

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**Abstract:** In this paper a special kind of variational programming problem involving fractional interval-valued objective function is considered. For such type of problem, insights into LU optimal solutions have been discussed. Using the LU optimal concept, we established optimality conditions for the considered problem. Further, We formulated a Mond-Weir dual problem and discussed appropriate duality theorems for the relationship between dual and primal problems.

**Keywords:** Variational programming problem, interval-valued programming problem, LU optimal solution, optimality, duality.

**MSC:** 26A51, 49J35, 90C32.

## 1. INTRODUCTION

During the last three decades, calculus of variation has received great interest from the researchers in the areas of optimization theory. This is the consequence of the fact that in combination with the calculus of variation, optimization theory, has proved to be quite useful for many applications in pure and applied mathematics, such as Industrial process control, Aeronautical design and management of space structures, Impulsive control problems, Mechanical engineering problems, Economics and other diverse fields (see, for example, [1, 2, 3, 4]).

The calculus of variation together with optimization theory was first explored and extended by Hanson [5]. After the work of Hanson [5], many results to variational programming problems have been explored by different authors. As regards the consideration of variational programming problem, Mond and Hanson [6] established optimality conditions and several duality results under convexity in abstract space. Under generalized $\rho$-invexity assumptions, Bhatia and Kumar [7] established duality results in the sense of Wolfe and Mond-Weir for non-convex multi-objective control problems. Recently, Kumar and Dagar [8] an improved class of higher-order $B$-type-I functions has been introduced and discussed sufficient optimality conditions and duality results for multi-objective semi-infinite variational problem, in which there is an infinite set of inequality constraints. For more results related to the variational programming problems one can refer [9, 10, 11, 12, 13].

The importance of an inteval-valued programming problem is well known in optimization theory as they occur in large number of applications in different fields, such as decision making [14], medical diagnosis [15], portfolio optimization [16] and many more others. Bhurjee and Panda [17] provided an overview of an interval-valued optimization problem by developing a methodology to study an efficient solution to an interval-valued optimization problem. Using Clarke generalized gradient, Singh et al. [18] introdused LU-$(p, r)$-$[\rho^L, \rho^U]$-$(\eta,\ \theta)$-invex interval-valued functions and gave the sufficient optimality conditions for the inteval-valued optimization problem. Further, Singh et al. [18] derived Strong, weak and strict converse duality theorems for Mond-Wier and Wolf type duality problems. In [9], Ahmad et al. established optimal conditions and duality theorems for the interval-valued variational problem. Also, they have investigated the saddle point optimality conditions and presented a relationship between saddle point of the Lagrangian function and optimal solution of the primal. Over the last decade, several authors have been investigated Optimal conditions and duality results for the interval-valued optimization problems, for more information about its theory and applications, one can refer [19, 20, 21, 22, 23, 24, 25, 26, 27].

It is well known that fractional programming problems play an important role in almost all areas of mathematics as well as in other branches of science, management and economics (see, for example, [28, 29, 30, 31]. However, to the author's knowledge, there are no results for the variational programming problems with fractional interval-valued objective function in the optimization literature. Therefore, this paper focuses to study optimality conditions and Mond-Weir type duality results for such a class of variational programming problems with Interval-valued fractional objective function.

## 2. PRELIMINARIES

Let the interval $\Omega = [a_1, a_2]$ be a real valued interval. Let the functions $\Theta : \Omega \times R^l \times R^l \to R$, $\Upsilon : \Omega \times R^l \times R^l \to R$ and $\eta : \Omega \times R^l \times R^l \to R^k$ be continuously differentiable with respect to their arguments and $\pi : \Omega \to R^l$, $\dot{\pi}(\nu)$ denotes the derivative of $\pi$ with respect $\nu$. We denote the partial derivatives of $\Theta$ with respect to the arguments $\nu$, $\pi$, $\dot{\pi}$ by

$$\Theta_\nu, \Theta_\pi = \left(\frac{\partial\Theta}{\partial\pi_1}, \frac{\partial\Theta}{\partial\pi_2}, ..., \frac{\partial\Theta}{\partial\pi_n}\right)^T, \Theta_{\dot{\pi}} = \left(\frac{\partial\Theta}{\partial\dot{\pi}_1}, \frac{\partial\Theta}{\partial\dot{\pi}_2}, ..., \frac{\partial\Theta}{\partial\dot{\pi}_n}\right)^T,$$

respectively, where, $T$ represents the transpose operator. Similarly, $\eta_\pi$ and $\eta_{\dot{\pi}}$ represent $k \times l$ Jcobian matrices of $\eta$ with respect to $\pi$ and $\dot{\pi}$, respectively. Let $\Pi$ be the continuously differentiable space function $\Pi : \Omega \to R^l$ with the norm $\|\pi\| = \|\pi\|_\infty + \|D\pi\|_\infty$, where the differential operator $D$ is defined by

$$\mu = D\pi \Leftrightarrow \Pi(\nu) = \Pi(a_1) + \int_{a_1}^{\nu} \mu(\nu) d\nu,$$

here $\Pi(a_1)$ is a given boundary value. Therefore, $D = \frac{d}{d\nu}$ except at points of discontinuities.

In order to proceed further, we need the following fundamental concepts of interval mathematics:

Let $\frac{\mathbb{A}}{\mathbb{B}} = \left[\frac{\alpha_1^L}{\gamma_1^L}, \frac{\alpha_1^U}{\gamma_1^U}\right]$ and $\frac{\mathbb{C}}{\mathbb{D}} = \left[\frac{\alpha_2^L}{\gamma_2^L}, \frac{\alpha_2^U}{\gamma_2^U}\right]$ be two fractional closed intervals with $\frac{\alpha_1^L}{\gamma_1^L} \leq \frac{\alpha_1^U}{\gamma_1^U}$ and $\frac{\alpha_2^L}{\gamma_2^L} \leq \frac{\alpha_2^U}{\gamma_2^U}$, $\gamma_1^L, \gamma_1^U, \gamma_2^L, \gamma_2^U \neq 0$.

(i) $\frac{\mathbb{A}}{\mathbb{B}} + \frac{\mathbb{C}}{\mathbb{D}} = \left[\frac{\alpha_1^L}{\gamma_1^L} + \frac{\alpha_2^L}{\gamma_2^L}, \frac{\alpha_1^U}{\gamma_1^U} + \frac{\alpha_2^U}{\gamma_2^U}\right]$,

(ii) $\frac{-\mathbb{A}}{\mathbb{B}} = \left[\frac{-\alpha_1^U}{\gamma_1^U}, \frac{-\alpha_1^L}{\gamma_1^L}\right]$,

(iii) $\frac{\mathbb{A}}{\mathbb{B}} - \frac{\mathbb{C}}{\mathbb{D}} = \frac{\mathbb{A}}{\mathbb{B}} + \left(\frac{-\mathbb{C}}{\mathbb{D}}\right) = \left[\frac{\alpha_1^L}{\gamma_1^L} - \frac{\alpha_2^U}{\gamma_2^U}, \frac{\alpha_1^U}{\gamma_1^U} - \frac{\alpha_2^L}{\gamma_2^L}\right]$,

(iv) $\theta\left(\frac{\mathbb{A}}{\mathbb{B}}\right) = \begin{cases} \left[\frac{\alpha_1^L}{\gamma_1^L}, \frac{\alpha_1^U}{\gamma_1^U}\right], & \text{if } \theta \geq 0, \\ \left[\frac{\alpha_1^U}{\gamma_1^U}, \frac{\alpha_1^L}{\gamma_1^L}\right], & \text{if } \theta < 0. \end{cases}$

An order relation $\leq_{LU}$ defined Within two intervals $\frac{\mathbb{A}}{\mathbb{B}}$ and $\frac{\mathbb{C}}{\mathbb{D}}$ :

(i) $\frac{\mathbb{A}}{\mathbb{B}} \leq_{LU} \frac{\mathbb{C}}{\mathbb{D}} \Leftrightarrow \frac{\alpha_1^L}{\gamma_1^L} \leq \frac{\alpha_2^L}{\gamma_2^L}$ & $\frac{\alpha_1^U}{\gamma_1^U} \leq \frac{\alpha_2^U}{\gamma_2^U}$.

(ii) $\frac{\mathbb{A}}{\mathbb{B}} <_{LU} \frac{\mathbb{C}}{\mathbb{D}} \Leftrightarrow \frac{\mathbb{A}}{\mathbb{B}} \leq_{LU} \frac{\mathbb{C}}{\mathbb{D}}$ & $\frac{\mathbb{A}}{\mathbb{B}} \neq \frac{\mathbb{C}}{\mathbb{D}}$, this gives as

$$\begin{cases} \frac{\alpha_1^L}{\gamma_1^L} < \frac{\alpha_2^L}{\gamma_2^L} \\ \frac{\alpha_1^U}{\gamma_1^U} \leq \frac{\alpha_2^U}{\gamma_2^U} \end{cases}, \text{ or } \begin{cases} \frac{\alpha_1^L}{\gamma_1^L} \leq \frac{\alpha_2^L}{\gamma_2^L} \\ \frac{\alpha_1^U}{\gamma_1^U} < \frac{\alpha_2^U}{\gamma_2^U} \end{cases}, \text{ or } \begin{cases} \frac{\alpha_1^L}{\gamma_1^L} < \frac{\alpha_2^L}{\gamma_2^L} \\ \frac{\alpha_1^U}{\gamma_1^U} < \frac{\alpha_2^U}{\gamma_2^U}. \end{cases}$$

Consider the following fractional interval-valued variational problem

$$\min \frac{\left[\int_{a_1}^{a_2} \Theta^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu, \int_{a_1}^{a_2} \Theta^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu\right]}{\left[\int_{a_1}^{a_2} \Upsilon^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu, \int_{a_1}^{a_2} \Upsilon^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu\right]}$$

subject to

$\eta(\nu,\pi(\nu),\dot{\pi}(\nu)) \leq 0, \ \nu \in \Omega,$

$\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2,$

which further reduces to the problem

$$\min \left[\frac{\int_{a_1}^{a_2} \Theta^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} \Upsilon^U(\nu,z(\nu),\dot{\pi}(\nu))d\nu}, \frac{\int_{a_1}^{a_2} \Upsilon^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} \Upsilon^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu,}\right]$$

subject to

$\eta(\nu,\pi(\nu),\dot{\pi}(\nu)) \leq 0, \ \nu \in \Omega,$

$\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2,$

here the functions $\Theta^L : \Omega \times R^l \times R^l \to R$, $\Theta^U : \Omega \times R^l \times R^l \to R$, $\Upsilon^L : \Omega \times R^l \times R^l \to R$, $\Upsilon^U : \Omega \times R^l \times R^l \to R$ and $\eta : \Omega \times R^l \times R^l \to R^k$ are continuously differentiable. Let $\int_{a_1}^{a_2} \Theta^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu, \int_{a_1}^{a_2} \Theta^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu \geq 0$ and
$\int_{a_1}^{a_2} \Upsilon^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu, \int_{a_1}^{a_2} \Upsilon^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu > 0.$
Set $\Theta^L = p^L, \Upsilon^U = q^L, \Theta^U = p^U, \Upsilon^L = q^U$, then, the above problem reduces to

$$\text{(IVVP)} \qquad \min \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}\right]$$

subject to

$$\eta(\nu,\pi(\nu),\dot{\pi}(\nu)) \leq 0, \ \nu \in \Omega, \tag{1}$$

$$\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2. \tag{2}$$

The region where the constraints are satisfied (feasibility region), $\mathbb{Z} = \{\pi \in \Pi : \pi(a_1) = \theta_1, \pi(a_2) = \theta_2$ and $\eta_i(\nu, \pi, \dot{\pi}) \leq 0, i \in I = \{1, , 2, ..., k\},\ \nu \in \Omega\}$.

**Definition 1.** *A feasible point $\bar{\pi}$ is said to be a LU optimal solution of the problem (IVVP), if there exists no feasible point $\pi \in \Pi$ such that*

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}\right]$$

$$<_{LU} \left[\frac{\int_{a_1}^{a_2} p^L(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}\right].$$

Hereafter, for convenience, we write $\Theta(\nu, \pi, \dot{\pi})$ in place of $\Theta(\nu, \pi(\nu), \dot{\pi}(\nu))$. Now, we define convexity, pseudo-convexity and quasi-convexity for variational problem as follows:

**Definition 2.** *The functional $\int_{a_1}^{a_2} \Theta(\nu, \pi, \dot{\pi})d\nu$ is said to be convex (strictly convex) at $\bar{\pi} \in \Pi$ on $\Pi$ if, for all $\pi \in \Pi$,*

$$\int_{a_1}^{a_2} \Theta(\nu, \pi, \dot{\pi})d\nu - \int_{a_1}^{a_2} \Theta(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu$$

$$\geq (>) \int_{a_1}^{a_2} \left\{(\pi - \bar{\pi})^T \Theta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) + (\dot{\pi} - \dot{\bar{\pi}})^T \Theta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})\right\} d\nu.$$

**Definition 3.** *The functional $\int_{a_1}^{a_2} \Theta(\nu, \pi, \dot{\pi})d\nu$ is said to be pseudo-convex at $\bar{\pi} \in \Pi$ on $\Pi$ if, for all $\pi \in \Pi$,*

$$\int_{a_1}^{a_2} \Theta(\nu, \pi, \dot{\pi})d\nu < \int_{a_1}^{a_2} \Theta(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu$$

$$\Rightarrow \int_{a_1}^{a_2} \left\{(\pi - \bar{\pi})^T \Theta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) + (\dot{\pi} - \dot{\bar{\pi}})^T \Theta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})\right\} d\nu < 0.$$

**Definition 4.** *The functional* $\int_{a_1}^{a_2} \Theta(\nu, \pi, \dot{\pi})d\nu$ *is said to be quasi connvex at* $\bar{\pi} \in \Pi$ *on* $\Pi$ *if, for all* $\pi \in \Pi$,

$$\int_{a_1}^{a_2} \Theta(\nu, \pi, \dot{\pi})d\nu \leq \int_{a_1}^{a_2} \Theta(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu$$

$$\Rightarrow \int_{a_1}^{a_2} \left\{(\pi - \bar{\pi})^T \Theta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) + (\dot{\pi} - \dot{\bar{\pi}})^T \Theta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})\right\} d\nu \leq 0.$$

## 3. OPTIMALITY CONDITIONS

For the given feasible solution $\bar{\pi}$, let us consider two fractional problems as outlined below:

$$\text{(FP1)} \ \min \ \phi^L(\pi) = \frac{\int_{a_1}^{a_2} p^L(\nu, \pi, \dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu, \pi, \dot{\pi})d\nu}$$

subject to

$$\eta(\nu, \pi, \dot{\pi}) \leq 0, \ \nu \in \Omega,$$

$$\frac{\int_{a_1}^{a_2} p^U(\nu, \pi, \dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu, \pi, \dot{\pi})d\nu} \leq \frac{\int_{a_1}^{a_2} p^U(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu},$$

$$\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2.$$

$$\text{(FP2)} \ \min \ \phi^U(\pi) = \frac{\int_{a_1}^{a_2} p^U(\nu, \pi, \dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu, \pi, \dot{\pi})d\nu}$$

subject to

$$\eta(\nu, \pi, \dot{\pi}) \leq 0, \ \nu \in \Omega,$$

$$\frac{\int_{a_1}^{a_2} p^L(\nu, \pi, \dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu, \pi, \dot{\pi})d\nu} \leq \frac{\int_{a_1}^{a_2} p^L(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu, \bar{\pi}, \dot{\bar{\pi}})d\nu},$$

$$\pi(a_1) = \theta_1, \quad \theta(a_2) = \theta_2.$$

The following lemma connects the problem (IVVP) to the fractional problems (FP1) and (FP2).

**Lemma 5.** *The solution $\bar{\pi}$ is an LU optimal solution of the problem (IVVP) iff $\bar{\pi}$ is also the optimal solution for the problems (FP1) and (FP2).*

*Proof.* Consider $\bar{\pi}$ as a LU optimal solution to the problem (IVVP). Then it results as, there is no $\pi \in \Pi$ such that

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu}\right] <_{LU} \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right].$$

Thus, we have

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a}^{b} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases}, \tag{3}$$

or

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a}^{b} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases}, \tag{4}$$

or

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a}^{b} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases}. \tag{5}$$

Suppose, contrary to the result, if $\bar{\pi}$ does not satisfy the constraints of (FP1), then there exists a point where $\pi$ meets the constraints of (FP1).

$$\eta(\nu,\pi,\dot{\pi}) \leq 0,\ \nu \in \Omega,\ \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} \leq \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu},$$
$$\text{and } \pi(a_1) = \theta_1, \pi(a_2) = \theta_2$$

such that

$$\frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}.$$

By $\eta(\nu,\pi,\dot{\pi}) \leq 0,\ \nu \in \Omega$ and $\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2$, it follows that $\pi$ is a feasible solution of (IVVP). Hence, the inequalities $\frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} \leq \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}$ and $\frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}$ contradicts (3), that is they contradict the assumption that $\bar{\pi}$ is a LU optimal solution of (IVVP). By applying the similar arguments as above, it can also be show that $\bar{\pi}$ provides an LU optimal solution for (FP2).

Conversely, suppose $\bar{\pi}$ is an optimal solution for the problems (FP1) and (FP2). On the contrary, suppose that $\bar{\pi}$ is not a LU optimal solution for the problem (IVVP). This means that there exist a feasible solution $\pi$, that is, By $\eta(\nu,\pi,\dot{\pi}) \leq 0,\ \nu \in \Omega$ and $\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2$ such that

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu}\right] <_{LU} \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right],$$

that is

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases}, \tag{6}$$

or

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases}, \tag{7}$$

or

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases}. \tag{8}$$

By $\eta(\nu,\pi,\dot{\pi}) \leq 0,\ \nu \in \Omega,\ \pi(a_1) = \theta_1, \pi(a_2) = \theta_2$ and the second inequality in (6), it follows that $\pi$ is a feasible solution to (FP1). Consequently, the first inequality in (6) contradicts the assumption that $\bar{\pi}$ is an optimal solution for the problem (FP1). Further, by $\eta(\nu,\pi,\dot{\pi}) \leq 0,\ \nu \in \Omega,\ \pi(a_1) = \theta_1, \pi(a_2) = \theta_2$ and the first inequality in (7), it follows that $\pi$ is a feasible solution to (FP2). Thus, the second inequality in (7) contradicts the assumption that $\bar{\pi}$ is an optimal solution to the problem (FP2). $\square$

Now we give an example that illustrates the Lemma 5.

**Example 6.** *Consider the following variational programming problem with fractional interval-valued objective function:*

$$\text{(IVVP1)} \qquad \min \quad \left[ \frac{\int_{a_1}^{a_2} p^L(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu, \pi(\nu), \dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu, \pi(\pi), \dot{\pi}(\nu))d\nu} \right]$$

subject to,

$$-3\pi(\nu) - 2\pi^3(\nu) \leq 0,$$

$$\pi(0) = 1, \pi(1) = 1, \nu \in [0, 1]$$

*where,* $p^L(\nu, \pi(\nu), \dot{\pi}(\nu)) = \pi^3(\nu) + 2\pi^2(\nu) + 2\nu$, $q^L(\nu, \pi(\nu), \dot{\pi}(\nu)) = \pi^2(\nu) + 2\pi(\nu) + \nu$, $p^U(\nu, \pi(\nu), \dot{\pi}(\nu)) = \pi(\nu) + 2\nu$, $q^U(\pi, \pi(\nu), \dot{\pi}(\nu)) = \pi^2(\nu) + \nu$, *therefore, the feasible set of the problem (IVVP1) is* $\mathbb{Z}_1 = \Big\{ \pi \in \Pi : \pi(0) = 1,\ \pi(1) = 1$ *and* $-3\pi(\nu) - 2\pi^3(\nu) \leq 0 \Big\}$. *We can see that the point* $\bar{\pi}(\nu) = 1$ *is feasible, so we have*

$$\frac{\int_{a_1}^{a_2} p^L(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu} = \frac{\int_0^1 (\pi^3(\nu) + 2\pi^2(\nu) + 2\nu)d\nu}{\int_0^1 (\pi^2(\nu) + 2\pi(\nu) + \nu)d\nu} = \frac{\int_0^1 (3 + 2\nu)d\nu}{\int_0^1 (3 + \nu)d\nu} \ (since, \pi(\nu) = 1) = \frac{8}{7}$$

$$\frac{\int_{a_1}^{a_2} p^U(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu} = \frac{\int_0^1 (\pi(\nu) + 2\nu)d\nu}{\int_0^1 (\pi^2(\nu) + \nu)d\nu} = \frac{\int_0^1 (1 + 2\nu)d\nu}{\int_0^1 (1 + \nu)d\nu} \ (since, \pi(\nu) = 1) = \frac{4}{3}.$$

*This implies that* $$\frac{\int_{a_1}^{a_2} p^L(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu} < \frac{\int_{a_1}^{a_2} p^U(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu, \bar{\pi}(\nu), \dot{\bar{\pi}}(\nu))d\nu}$$

*Next, we show that,* $\pi = 1$ *is LU optimal for (IVPP1):*

*we have*

$$\frac{\int_{a_1}^{a_2} p^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} - \frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}$$

$$= \frac{\int_0^1 (\pi^3(\nu)+2\pi^2(\nu)+\nu)d\nu}{\int_0^1 (\pi^2(\nu)+2\pi(\nu)+\nu)d\nu} - \frac{8}{7} > 0, \forall\ 1 \neq \pi \in \mathbb{Z}_1$$

*and*

$$\frac{\int_{a_1}^{a_2} p^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} - \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}$$

$$= \frac{\int_0^1 (\pi(\nu)+2\nu)d\nu}{\int_0^1 (\pi^2(\nu)+\nu)d\nu} - \frac{4}{3} > 0, \forall\ 1 \neq \pi \in \mathbb{Z}_1$$

*Thus, we see that there exists no* $\pi \in \Pi$ *such that*

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu} \\ \dfrac{\int_{a}^{b} p^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu} \end{cases},$$

*or*

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} < \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu} \end{cases},$$

*or*

$$\begin{cases} \dfrac{\int\limits_{a_1}^{a_2} p^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int\limits_{a_1}^{a_2} q^L(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} < \dfrac{\int\limits_{a_1}^{a_2} p^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int\limits_{a_1}^{a_2} q^L(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu} \\ \dfrac{\int\limits_{a_1}^{a_2} p^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu}{\int\limits_{a_1}^{a_2} q^U(\nu,\pi(\nu),\dot{\pi}(\nu))d\nu} < \dfrac{\int\limits_{a_1}^{a_2} p^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu}{\int\limits_{a_1}^{a_2} q^U(\nu,\bar{\pi}(\nu),\dot{\bar{\pi}}(\nu))d\nu} \end{cases}.$$

*Hence, the point $\pi = 1$ is $LU$ optimal for (IVVP).*
*Let's construct the problem (FP1) and (FP2) in the following manner:*

$$\text{(FP1)}: \quad \min \left[ \frac{\int_0^1 (\pi^3(\nu) + 2\pi^2(\nu) + 2\nu)d\nu}{\int_0^1 (\pi^2(\nu) + 2\pi(\nu) + \nu)d\nu} \right]$$

*subject to,*

$$-3\pi(\nu) - 2\pi^3(\nu) \le 0,$$

$$\int_0^1 (3\pi(\nu) - 4\pi^2(\nu) + 2\nu)d\nu \le 0,$$

$$\pi(0) = 1, \pi(1) = 1.$$

$$\textit{(FP2)}: \quad \min \left[ \frac{\int\limits_0^1 (\pi(\nu) + 2\nu)d\nu}{\int\limits_0^1 (\pi^2(\nu) + \nu)d\nu} \right]$$

*subject to,*

$$-3\pi(\nu) - 2pi^3(\nu) \le 0,$$

$$\int_0^1 (7\pi^3(\nu) + 6\pi^2(\nu) - 16\pi(\nu) + 6\nu)d\nu \le 0,$$

$$\pi(0) = 1, \pi(1) = 1.$$

*The optimal solution to problems (FP1) and (FP2) is $\pi = 1$.*

On the lines of Lemma 4.1 of Bector et al. [32], the lemma that we state below, will be need in the sequel.

**Lemma 7.** *$\bar{\pi}$ is a LU optimum of the problem (IVVP) iff $\bar{\pi}$*

$$\text{minimizes} \left[ \frac{\int_{a_1}^{a_2} p^L(\nu, \pi, \dot{\pi}) d\nu}{\int_{a_1}^{a_2} q^L(\nu, \pi, \dot{\pi}) d\nu} \right], \text{ on the constraint set}$$

$$N = \left\{ \pi \in \Pi \middle| \frac{\int_{a_1}^{a_2} p^U(\nu, \pi, \dot{\pi}) d\nu}{\int_{a_1}^{a_2} q^U(\nu, \pi, \dot{\pi}) d\nu} \leq \frac{\int_{a_1}^{a_2} p^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) d\nu}{\int_{a_1}^{a_2} q^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) d\nu}, \eta(\nu, \pi, \dot{\pi}) \leq 0, \ t \in \Omega, \right.$$

$$\left. \pi(a_1) = \theta_1, \pi(a_2) = \theta_2 \right\}.$$

Consider the following single-objective fractional variational problem:

$$\text{(D)} \quad \min \phi(\pi) = \frac{\int_{a_1}^{a_2} p(\nu, \pi, \dot{\pi}) d\nu}{\int_{a_1}^{a_2} q(\nu, \pi, \dot{\pi}) d\nu}$$

subject to

$$\ell(\nu, \pi, \dot{\pi}) \leq 0, \ \nu \in \Omega,$$
$$\pi(a_1) = \theta_1, \quad \pi(a_2) = \theta_2$$
$$\pi \in \Pi,$$

here the functions $p : \Omega \times R^l \times R^l \to R$ and $q : \Omega \times R^l \times R^l \to R$ are continuously differentiable with respect to their arguments such that $\int_{a_1}^{a_2} p(\nu, \pi, \dot{\pi}) d\nu \geq 0$ and $\int_{a_1}^{a_2} q(\nu, \pi, \dot{\pi}) d\nu > 0$, $\forall\ \pi$ satisfying constraints of (D) and the function $\ell : \Omega \times R^l \times R^l \to R^k$ is continuously differentiable with respect to their arguments.

Now, we give an optimality result for (D) which is a special case of Theorem 3.3 of Zalmai [33].

**Theorem 8.** *Let $\bar{\pi}$ be an optimal solution to the variational programming problem (D) and Slater's constraint qualification (see, Theorem 2.1 of Chandra et al. [34]) be satisfied at $\bar{\pi}$ . Then, it is possible to have piecewise smooth function $\lambda : \Omega \to R^k$, $\lambda(\nu) \geq 0$ such that*

$$\Psi(\bar{\pi}) p_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi(\bar{\pi}) q_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) + (\lambda(\nu))^T \ell_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})$$

$$= \frac{d}{dt}[\Psi(\bar{\pi})p_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi(\bar{\pi})q_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T \ell_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})], \tag{9}$$

$$(\lambda(\nu))^T \ell(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0, \tag{10}$$

*In this case,* $\Phi(\bar{\pi})$ *represents the numerator of* $\phi(\bar{\pi})$ *while* $\Psi(\bar{\pi})$ *represents its denominator.*

In the following, we establish the necessary optimality conditions for the problem (IVVP).

**Theorem 9 (Necessary optimality conditions).** *Let* $\bar{\pi}$ *be a LU optimal solution of the problem (IVVP) and Slater's constraint qualification satisfied at* $\bar{\pi}$*. Then, it is possible to have piecewise smooth function* $\lambda : \Omega \to R^k$, $\lambda(t) \geq 0$ *such that*

$$\begin{aligned}
&\left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + \left[\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] \\
&+(\lambda(\nu))^T \eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) \\
&= \frac{d}{d\nu}\Big\{ \left[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + \Big[\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) \\
&\qquad -\Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\Big] + (\lambda(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) \Big\},
\end{aligned} \tag{11}$$

$$(\lambda(\nu))^T \eta(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0, \tag{12}$$

*where* $\Phi^L(\bar{\pi})$ *is equal to the numerator of* $\phi^L(\bar{\pi})$ *and* $\Psi^L(\bar{\pi})$ *to its denominator. Also,* $\Phi^U(\bar{\pi})$ *is equal to the numerator of* $\phi^U(\bar{\pi})$ *and* $\Psi^U(\bar{\pi})$ *to its denominator.*

*Proof.* By assumption, $\bar{\pi}$ is a LU optimal solution for the problem (IVVP) and at $\bar{\pi}$ the Slater's constraint is satisfied. If $\bar{\pi}$ is a LU optimal solution then by Lemma 5, $\bar{\pi}$ is also an optimal solution for the problems (FP1) and (FP2). Hence, by Lemma 7, at $\bar{\pi}$ the minimum value of $\dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu}$ is obtained on the constraint set

$$N_L = \left\{ \pi \in \Pi \middle| \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} \leq \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu}, \eta(\nu,\pi,\dot{\pi}) \leq 0,\ \nu \in \Omega, \right.$$
$$\left. \pi(a_1) = \theta_1, \pi(a_2) = \theta_2 \right\}$$

and the minimum value of $\frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu}$ is obtained at $\bar{\pi}$ on the constraint set

$$N_U = \left\{ \pi \in \Pi \middle| \frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} \le \frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}, \eta(\nu,\pi,\dot{\pi}) \le 0,\ \nu \in \Omega, \right.$$
$$\left. \pi(a_1) = \theta_1, \pi(a_2) = \theta_2 \right\}.$$

By Theorem 8, it follows that there is possible to have piecewise smooth function $\lambda^L : \Omega \to R^k$, $\lambda^L(\nu) \ge 0$ and $\lambda^U : \Omega \to R^k$, $\lambda^U(\nu) \ge 0$ such that

$$\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^{\pi}_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda^L(\nu))^T h_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})$$
$$= \frac{d}{d\nu}\left[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda^L(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right], \quad (13)$$
$$(\lambda^L(\nu))^T \eta(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0, \quad (14)$$

and

$$\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda^U(\nu))^T \eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})$$
$$= \frac{d}{d\nu}\left[\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda^U(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right], \quad (15)$$
$$(\lambda^U(\nu))^T \eta(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0. \quad (16)$$

From (13) to (16), we have

$$\left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + \left[\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$
$$\left. - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + (\lambda^L(\nu) + \lambda^U(\nu))^T \eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})$$
$$= \frac{d}{d\nu}\Big\{ \left[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + \left[\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$
$$\left. - \Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + (\lambda^L(t) + \lambda^U(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) \Big\}, \quad (17)$$
$$(\lambda^L(\nu) + \lambda^U(\nu))^T \eta(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0. \quad (18)$$

Let us denote $\lambda^L(\nu) + \lambda^U(\nu) = \lambda(\nu)$. Thus, from (20)-(21), we get

$$\left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + \left[\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$
$$\left. - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right] + (\lambda(\nu))^T \eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})$$

$$= \frac{d}{d\nu}\Big\{ \big[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\big] + \big[\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})$$
$$-\Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\big] + (\lambda(\nu))^T\eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\Big\},$$

$$(\lambda(\nu))^T\eta(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0.$$

Hence, the proof has been completed. □

**Theorem 10 (Sufficient optimality conditions).** *Suppose that $\bar{\pi}$ is a feasible solution of (IVVP) and there is a piecewise smooth function $\lambda : \Omega \to R^k$, $\lambda(\nu) \geq 0$ such that (11) and (12)are satisfied at $\bar{\pi}$. Also, assume that*

*(i) the functionals $\int_{a_1}^{a_2}\{\Psi^L(\bar{\pi})p^L(\nu,\pi,\dot{\pi}) - \Phi^L(\bar{\pi})q^L(\nu,\pi,\dot{\pi})\}d\nu$ and $\int_{a_1}^{a_2}\{\Psi^U(\bar{\pi})p^U(\nu,\pi,\dot{\pi}) - \Phi^U(\bar{\pi})q^U(\nu,\pi,\dot{\pi})\}d\nu$ are convex at $\bar{\pi}$ on $\Pi$,*

*(ii) the functional $\int_{a_1}^{a_2}\lambda(\nu)\eta(\nu,\pi,\dot{\pi})d\nu$ is convex at $\bar{\pi}$ on $\Pi$, then $\bar{\pi}$ is a LU optimal solution for (IVVP).*

*Proof.* Suppose $\bar{\pi}$ is not a LU optimal solution for (IVVP), then by Definition 1 there exists another feasible solution $\pi$ for (IVVP), such that

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu}\right]$$
$$<_{LU} \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right],$$

that is

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} \leq \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases},$$

or

$$\begin{cases} \dfrac{\int\limits_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int\limits_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} \le \dfrac{\int\limits_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int\limits_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int\limits_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int\limits_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int\limits_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int\limits_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases},$$

or

$$\begin{cases} \dfrac{\int\limits_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int\limits_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int\limits_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int\limits_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \\ \dfrac{\int\limits_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int\limits_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})d\nu} < \dfrac{\int\limits_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int\limits_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} \end{cases},$$

which implies

$$\int\limits_{a_1}^{a_2} \left\{\Psi^L(\bar{\pi})p^L(\nu,\pi,\dot{\pi}) - \Phi^L(\bar{\pi})q^L(\nu,\pi,\dot{\pi})\right\} d\nu$$
$$< \int\limits_{a_1}^{a_2} \left\{\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right\} d\nu,$$
$$\int\limits_{a_1}^{a_2} \left\{\Psi^U(\bar{\pi})p^U(\nu,\pi,\dot{\pi}) - \Phi^U(\bar{\pi})q^U(\nu,\pi,\dot{\pi})\right\} d\nu$$
$$\le \int\limits_{a_1}^{a_2} \left\{\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right\} d\nu,$$

or

$$\int_{a_1}^{a_2} \left\{ \Psi^L(\bar{\pi}) p^L(\nu, \pi, \dot{\pi}) - \Phi^L(\bar{\pi}) q^L(\nu, \pi, \dot{\pi}) \right\} d\nu$$

$$\leq \int_{a_1}^{a_2} \left\{ \Psi^L(\bar{\pi}) p^L(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} d\nu,$$

$$\int_{a_1}^{a_2} \left\{ \Psi^U(\bar{\pi}) p^U(\nu, \pi, \dot{\pi}) - \Phi^U(\bar{\pi}) q^U(\nu, \pi, \dot{\pi}) \right\} d\nu$$

$$< \int_{a_1}^{a_2} \left\{ \Psi^U(\bar{\pi}) p^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} d\nu,$$

or

$$\int_{a_1}^{a_2} \left\{ \Psi^L(\bar{\pi}) p^L(\nu, \pi, \dot{\pi}) - \Phi^L(\bar{\pi}) q^L(\nu, \pi, \dot{\pi}) \right\} d\nu$$

$$< \int_{a_1}^{a_2} \left\{ \Psi^L(\bar{\pi}) p^L(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} d\nu,$$

$$\int_{a_1}^{a_2} \left\{ \Psi^U(\bar{\pi}) p^U(\nu, \pi, \dot{\pi}) - \Phi^U(\bar{\pi}) q^U(\nu, \pi, \dot{\pi}) \right\} d\nu$$

$$< \int_{a_1}^{a_2} \left\{ \Psi^U(\bar{\pi}) p^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} d\nu,$$

which by hypothesis $(i)$, we have

$$\int_{a_1}^{a_2} \Bigg\{ (\pi - \bar{\pi})^T \left\{ \Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\}$$

$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \left\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} \Bigg\} d\nu < 0,$$

$$\int_{a_1}^{a_2} \Bigg\{ (\pi - \bar{\pi})^T \left\{ \Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(t, \bar{\pi}, \dot{\bar{\pi}}) \right\}$$

$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \left\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} \Bigg\} d\nu \leq 0,$$

or

$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \left\{ \Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\}$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \left\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} \Big\} d\nu \leq 0,$$
$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \left\{ \Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(t, \bar{\pi}, \dot{\bar{\pi}}) \right\}$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \left\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} \Big\} d\nu < 0,$$

or

$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \left\{ \Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\}$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \left\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} \Big\} d\nu < 0,$$
$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \left\{ \Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(t, \bar{\pi}, \dot{\bar{\pi}}) \right\}$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \left\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right\} \Big\} d\nu < 0,$$

From the above inequalities, we get

$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \left\{ [\Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})] \right.$$
$$\left. + [\Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})] \right\}$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \{ \left[ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right]$$
$$+ \left[ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right] \} \Big\} d\nu < 0. \tag{19}$$

On the other hand, from (11), we have

$$\int_{a_1}^{a_2} (\pi - \bar{\pi})^T \Big[ \big\{ \Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\}$$

$$+ \big\{ \Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + (\lambda(\nu))^T \eta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big] d\nu$$

$$= \int_{a_1}^{a_2} (\pi - \bar{\pi})^T \Big[ \frac{d}{d\nu} \big[ \big\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} +$$

$$\big\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + (\lambda(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big] \Big] d\nu$$

$$= \Big[ (\pi - \bar{\pi})^T \Big[ \big\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + \big\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})$$

$$- \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + (\lambda(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big] \Big] \Big|_{a_1}^{a_2}$$

$$- \int_{a_1}^{a_2} (\dot{\pi} - \dot{\bar{\pi}})^T \Big[ \big\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + \big\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})$$

$$- \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + (\lambda(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big] d\nu,$$

which by using (2), we obtain

$$\int_{a_1}^{a_2} (\pi - \bar{\pi})^T \Big[ \big\{ \Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + \Big\{ \Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})$$

$$- \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big\} + (\lambda(\nu))^T \eta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big] d\nu$$

$$= - \int_{a_1}^{a_2} (\dot{\pi} - \dot{\bar{\pi}})^T \Big[ \big\{ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + \big\{ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})$$

$$- \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} + (\lambda(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big] d\nu,$$

that is

$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \Big[ \big[ \Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big]$$
$$+ \big[ \Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big] + (\lambda(\nu))^T \eta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big]$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \Big[ \big[ \Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big]$$
$$+ \big[ \Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big] + (\lambda(\nu))^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \Big] \Big\} d\nu = 0. \tag{20}$$

For feasibility of $\pi$ in the problem (IVVP), we have $\eta(\nu, \pi, \dot{\pi}) \leq 0,\ \nu \in \Omega$, which by using the fact $\lambda(\nu) \in R^k$, $\lambda(\nu) \geq 0$ and (9), we have

$$\int_{a_1}^{a_2} \lambda(\nu)^T \eta(\nu, \pi, \dot{\pi}) d\nu \leq \int_{a_1}^{a_2} \lambda(\nu)^T \eta(\nu, \bar{\pi}, \dot{\bar{\pi}}) d\nu.$$

By using the hypothesis $(ii)$, above inequality yields

$$\int_{a_1}^{a_2} \big\{ (\pi - \bar{\pi})^T [\lambda(\nu)^T \eta_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})] + (\dot{\pi} - \dot{\bar{\pi}})^T [\lambda(\nu)^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})] \big\} d\nu \leq 0. \tag{21}$$

On adding (20) and (21), we get

$$\int_{a_1}^{a_2} \Big\{ (\pi - \bar{\pi})^T \big\{ [\Psi^L(\bar{\pi}) p^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})]$$
$$+ [\Psi^U(\bar{\pi}) p^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\bar{\pi}}(\nu, \bar{\pi}, \dot{\bar{\pi}})] + \lambda(\nu)^T \eta_{\bar{\nu}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\}$$
$$+ (\dot{\pi} - \dot{\bar{\pi}})^T \big\{ [\Psi^L(\bar{\pi}) p^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})]$$
$$+ [\Psi^U(\bar{\pi}) p^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}})] + \lambda(\nu)^T \eta_{\dot{\bar{\pi}}}(\nu, \bar{\pi}, \dot{\bar{\pi}}) \big\} \Big\} d\nu < 0,$$

which is a contradiction to (20). This completes the proof of the theorem. □

**Theorem 11 (Sufficient optimality conditions).** *Suppose that $\bar{\pi}$ is a feasible solution of (IVVP) and there exists a piecewise smooth function $\lambda : \Omega \to R^k$, $\lambda(\nu) \geq 0$ such that (11) and (12) are satisfied at $\bar{\pi}$. Also, assume that*

(*i*) *the functional $\int_{a_1}^{a_2} \big[ \{ \Psi^L(\bar{\pi}) p^L(\nu, \pi, \dot{\pi}) - \Phi^L(\bar{\pi}) q^L(\nu, \pi, \dot{\pi}) \} + \{ \Psi^U(\bar{\pi}) p^U(\nu, \pi, \dot{\pi}) - \Phi^U(\bar{\pi}) q^U(\nu, \pi, \dot{\pi}) \} \big] d\nu$ is pseudoconvex at $\bar{\pi}$ on $\Pi$,* *(ii) the functional* $\int_{a_1}^{a_2} \lambda(\nu)\eta(\nu, \pi, \dot{\pi})d\nu$ *is quasi convex at* $\bar{\pi}$ *on* $\Pi$,

*then* $\bar{\pi}$ *is a LU optimal solution for (IVVP).*

*Proof.* Similar to the proof of Theorem 10. □

## 4. MOND WEIR-TYPE DUALITY

Now, we formulate the following dual of Mond and Weir for the considered fractional interval-valued variational programming problem (IVVP),

$$\text{(IVVMD)} \qquad \max \quad \left[ \frac{\int_{a_1}^{a_2} p^L(\nu, \mu, \dot{\mu})d\nu}{\int_{a_1}^{a_2} q^L(\nu, \mu, \dot{\mu})d\nu}, \frac{\int_{a_1}^{a_2} p^U(t, \mu, \dot{\mu})d\nu}{\int_{a_1}^{a_2} q^U(\nu, \mu, \dot{\mu})d\nu} \right]$$

subject to

$$\mu(a_1) = \theta_1, \quad \mu(a_2) = \theta_2,$$

$$\begin{aligned}
&\left[\Psi^L(\bar{\mu})p^L_{\bar{\mu}}(\nu, \bar{\mu}, \dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\bar{\mu}}(t, \bar{\mu}, \dot{\bar{\mu}})\right] + \left[\Psi^U(\bar{\mu})p^U_{\bar{\mu}}(\nu, \bar{\mu}, \dot{\bar{\mu}})\right. \\
&\qquad \left. -\Phi^U(\bar{\mu})q^U_{\bar{\mu}}(\nu, \bar{\mu}, \dot{\bar{\mu}})\right] + (\lambda(\nu))^T \eta_{\bar{\mu}}(\nu, \bar{\mu}, \dot{\bar{\mu}}) \\
&\quad = \frac{d}{d\nu}\Big\{ \left[\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu, \bar{\mu}, \dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu, \bar{\mu}, \dot{\bar{\mu}})\right] + \left[\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}((\nu, \bar{\mu}, \dot{\bar{\mu}})\right. \\
&\qquad\qquad \left. -\Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu, \bar{\mu}, \dot{\bar{\mu}})\right] + (\lambda(\nu))^T \eta_{\dot{\bar{\mu}}}(\nu, \bar{\mu}, \dot{\bar{\mu}}) \Big\},
\end{aligned} \tag{22}$$

$$(\lambda(\nu))^T \eta(\nu, \bar{\mu}, \dot{\bar{\mu}}) \geq 0. \tag{23}$$

Let $\mathcal{F}(\bar{\mu}) = \{(\lambda, \bar{\mu}) : \lambda \in \mathbb{R}^k, \bar{\mu} \in \Pi$: satisfying the constraints of (IVVMD),$\forall\ \nu \in \Omega\}$ be the set of all feasible points to (IVVMD) and $\Psi = \mathbb{Z} \cup pr_{\Pi}\mathcal{F}$.

**Definition 12.** *A feasible point* $(\bar{\lambda}, \bar{\mu})$ *is said to be an optimal point of a maximum type for (IVVMD), if there exists no feasible point* $(\lambda, \mu)$ *such that*

$$\frac{\int_{a_1}^{a_2} p^L(\nu, \mu, \dot{\mu})d\nu}{\int_{a_1}^{a_2} q^L(\nu, \mu, \dot{\mu})d\nu} <_{LU} \frac{\int_{a_1}^{a_2} p^U(\nu, \mu, \dot{\mu})d\nu}{\int_{a_1}^{a_2} q^U(\nu, \mu, \dot{\mu})d\nu}.$$

As a consequence, we prove several duality theorems that link primal (IVVP) and dual (IVVMD) problems under the assumption of convexity

**Theorem 13 (Weak duality).** *Let* $\bar{\pi}$ *be the feasible point for (IVVP) and* $(\bar{\lambda}, \bar{\mu})$ *be the feasible point for (IVVMD). Assume that,* (*i*) *the functions* $\int_{a_1}^{a_2}\{\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}})-\Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\}d\nu$ *and* $\int_{a_1}^{a_2}\{\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}})-\Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\}d\nu$ *are convex at* $\bar{\mu}$ *on* $\Pi$,

(*ii*) *the functional* $\int_{a_1}^{a_2}(\lambda(\nu))^T\eta(\nu,\mu,\dot{\mu})d\nu$ *is convex at* $\bar{\mu}$ *on* $\Pi$, *then*

$$\left[\frac{\int_{a_1}^{a_2}p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu},\frac{\int_{a_1}^{a_2}p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right]\nless_{LU}\left[\frac{\int_{a_1}^{a_2}p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu},\frac{\int_{a_1}^{a_2}p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\right].$$

*Proof.* On the contrary to the result, suppose that

$$\left[\frac{\int_{a_1}^{a_2}p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu},\frac{\int_{a_1}^{a_2}p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right]<_{LU}\left[\frac{\int_{a_1}^{a_2}p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu},\frac{\int_{a_1}^{a_2}p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\right],$$

which means that,

$$\begin{cases}\dfrac{\int_{a_1}^{a_2}p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}<\dfrac{\int_{a_1}^{a_2}p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\\ \dfrac{\int_{a_1}^{a_2}p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\leq\dfrac{\int_{a_1}^{a_2}p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\end{cases},$$

or

$$\begin{cases}\dfrac{\int_{a_1}^{a_2}p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\leq\dfrac{\int_{a_1}^{a_2}p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\\ \dfrac{\int_{a_1}^{a_2}p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}<\dfrac{\int_{a_1}^{a_2}p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2}q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\end{cases},$$

or

$$\begin{cases} \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} < \dfrac{\int_{a_1}^{a_2} p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu} \\ \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} < \dfrac{\int_{a_1}^{a_2} p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu} \end{cases},$$

which implies that,

$$\begin{cases} \int_{a_1}^{a_2} \left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu \\ \qquad < \int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \\ \int_{a_1}^{a_2} \left[\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu \\ \qquad \leq \int_{a_1}^{a_2} \left[\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \end{cases}$$

or

$$\begin{cases} \int_{a_1}^{a_2} \left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu \\ \qquad \leq \int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \\ \int_{a_1}^{a_2} \left[\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu \\ \qquad < \int_{a_1}^{a_2} \left[\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \end{cases}$$

or

$$\begin{cases} \int_{a_1}^{a_2} \left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu \\ \qquad < \int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \\ \int_{a_1}^{a_2} \left[\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu \\ \qquad < \int_{a_1}^{a_2} \left[\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \end{cases}$$

which in accordance of hypothesis $(i)$, we have

$$\begin{cases} \int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\} \right. \\ \qquad \left. +(\mu-\bar{\mu})^T\{\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu < 0, \\ \int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{u})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\} \right. \\ \qquad \left. +(\mu-\bar{\mu})^T\{\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{u})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu \leq 0, \end{cases}$$

or

$$\begin{cases} \int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\} \right. \\ \qquad \left. +(\mu-\bar{\mu})^T\{\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu \leq 0, \\ \int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{u})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\} \right. \\ \qquad \left. +(\mu-\bar{\mu})^T\{\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{u})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu < 0, \end{cases}$$

or

$$\begin{cases} \int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\} \right. \\ \qquad \left. +(\mu-\bar{\mu})^T\{\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu < 0, \\ \int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{u})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\} \right. \\ \qquad \left. +(\mu-\bar{\mu})^T\{\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{u})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu < 0. \end{cases}$$

From the above inequalities, we have

$$\begin{aligned} &\int\limits_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) \right. \\ &\left. +\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\}\right] d\nu \\ &\qquad + \int\limits_{a_1}^{a_2} \left[\frac{d}{d\nu}(\mu-\bar{\mu})^T\{\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) \right. \\ &\qquad \left. +\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right] d\nu < 0. \end{aligned} \tag{24}$$

From the other side of (22), we have

$$\int\limits_{a_1}^{a_2} (\pi-\bar{\pi})^T \left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$

$$+\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\big]\,d\nu$$

$$= \int_{a_1}^{a_2}(\mu-\bar{\mu})^T\Bigg[\frac{d}{d\nu}\{\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})$$

$$+\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})-\Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})+(y(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\}\Bigg]d\nu.$$

$$\Rightarrow \int_{a_1}^{a_2}(\pi-\bar{\pi})^T\left[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$

$$\left.+\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{z})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]d\nu$$

$$= (\mu-\bar{\mu})^T\Big[\{\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})$$

$$+\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\}\Big]_{a_1}^{a_2}.$$

$$-\int_{a_1}^{a_2}\frac{d}{d\nu}(\mu-\bar{\mu})^T\Big[\{\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})$$

$$+\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\}\Big]d\nu.$$

By using $\pi(a_1) = \theta_1, \pi(a_2) = \theta_2$ and $\mu(a_1) = \theta_1, \mu(a_2) = \theta_2$ We can write the above equation as,

$$\int_{a_1}^{a_2}\Big\{(\pi-\bar{\pi})^T\left[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$

$$\left.+\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]$$

$$+\frac{d}{d\nu}(\mu-\bar{\mu})^T\Big[\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})$$

$$+\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\Big]\Big\}\,d\nu = 0. \tag{25}$$

Furthermore, the feasibility of $\bar{\pi}$ for (IVVP) in conjunction with $(\bar{\lambda},\bar{\mu})$ for (IVVMD) suggests that

$$\int_{a_1}^{a_2}(\lambda(\nu))^T\eta(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu \leq \int_{a_1}^{a_2}(\lambda(\nu))^T\eta(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu,$$

In view of hypothesis (ii),

$$\int_{a_1}^{a_2} \left[(\pi-\bar{\pi})^T\{(\lambda(\nu))^T\eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\}+(\mu-\bar{\mu})^T\{(\lambda(\nu))^T h_{\bar{\mu}}(\nu,\bar{\mu},\dot{\bar{\mu}})\}\right]d\nu \leq 0. \quad (26)$$

On adding (24) and (26), we get,

$$\begin{aligned}\int_{a_1}^{a_2} \Big\{ &(\pi-\bar{\pi})^T \left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.\\ &\left.+\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T\eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]\\ &+\frac{d}{d\nu}(\mu-\bar{\mu})^T\Big[\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) + \Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\\ &-\Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\Big]\Big\}\, d\nu < 0. \end{aligned} \quad (27)$$

which contradicts (25) and hence the theorem. □

**Theorem 14 (Strong duality).** *Consider $\bar{\pi}$ is a LU optimal for (IVVP), and let Slater's constraint qualification be satisfied by $\bar{\pi}$. Then there exist piece wise smooth functions $\bar{\lambda}:\Omega\to\mathbb{R}^k$, $\bar{\mu}(\nu)\in\Pi$, such that $(\bar{\lambda},\bar{\mu})$ is a feasible point for (IVVMD) and the two objective values are equal at $\bar{\pi}$ and $(\bar{\lambda}(\nu),\bar{\mu}(\nu))$ for(IVVP), (IVVMD) respectively. Further more, if the weak duality Theorem 13 holds between (IVVP) and (IVVMD), then $(\bar{\pi},\bar{\mu})$ is an LU-optimality for (IVVMD).*

*Proof.* According to the hypothesis, $\bar{\pi}$ is a LU optimal solution for (IVVP); therefore, by Theorem 9, there exist a piecewise smooth functions $\bar{\lambda}:\Omega\to\mathbb{R}^k$, $\mu(\nu)\in\Pi$ such that

$$\begin{aligned}&\left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]\\ &\qquad + \left[\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]+(\lambda(\nu))^T\eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\\ &= \frac{d}{d\nu}\Big\{\left[\Psi^L(\bar{\pi})p^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]\\ &\qquad + \left[\Psi^U(\bar{\pi})p^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]+(\lambda(\nu))^T\eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}})\Big\}\\ &(\lambda(\nu))^T\eta_{\dot{\bar{\pi}}}(\nu,\bar{\pi},\dot{\bar{\pi}}) = 0, \nu\in\Omega.\end{aligned}$$

Consequently, $(\bar{\lambda},\bar{\mu})$ is a feasible solution for (IVVMD) and the objective values of (IVVP) and (IVVMD) are equal. According to the weak duality Theorem 13, $(\bar{\lambda},\bar{\mu})$ is optimal for (IVVMD). □

**Theorem 15 (Strict converse duality).** *The feasible point for (IVVP) and (IVVMD) can be calculated as* $\bar{\pi}$ *and* $(\bar{\lambda}, \bar{\mu})$*, respectively, such that,,*

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right] = \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\right]. \quad (28)$$

*Further, assume that*

(*i*) $\mu(\nu) \in \Pi$, $\mu(\nu) \geq 0$ *and the functions* $\int_{a_1}^{a_2}\{\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\}d\nu$ *and* $\int_{a_1}^{a_2}\{(\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\}d\nu$ *are strictly convex at* $\bar{\mu}$ *on* $\mathbb{Z}$

(*ii*) *The function* $\int_{a_1}^{a_2}(\lambda(\nu))^T\eta_{\bar{\mu}}(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu$ *is convex at* $\bar{\mu}$ *on* $\Pi$*. Then,* $\bar{\pi} = \bar{\mu}$ *and* $\bar{\mu}$ *is an LU-optimal point for (IVVP).*

*Proof.* To the contrary, suppose that, $\bar{\pi} \neq \bar{\mu}$. From (28), we have

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}\right] = \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\right]$$

That is,

$$\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} = \frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu},$$

$$\frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu} = \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu},$$

this implies

$$\int_{a_1}^{a_2} \left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu$$

$$= \int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \quad (29)$$

$$\int_{a_1}^{a_2} \left[\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu$$

$$= \int_{a_1}^{a_2} \left[\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu. \tag{30}$$

On adding (29) and (30), we get

$$\int_{a_1}^{a_2} \left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu$$

$$+ \int_{a_1}^{a_2} \left[\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu$$

$$= \int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu$$

$$+ \int_{a_1}^{a_2} \left[\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu. \tag{31}$$

However, for the feasibility of $(\bar{\lambda}, \bar{\mu})$ for (IVVMD), we have

$$\int_{a_1}^{a_2} \Big\{ (\pi-\bar{\pi})^T \left[\Psi^L(\bar{\pi})p^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right.$$

$$\left. +\Psi^U(\bar{\pi})p^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T\eta_{\bar{\pi}}(\nu,\bar{\pi},\dot{\bar{\pi}})\right]$$

$$+\frac{d}{d\nu}(\mu-\bar{\mu})^T \Big[\Psi^L(\bar{\mu})p^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})$$

$$+\Psi^U(\bar{\mu})p^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) + (\lambda(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}})\Big] \Big\} d\nu = 0. \tag{32}$$

Further, the feasibility of $\bar{\pi}$ to (IVVP) and $(\bar{\lambda}, \bar{\mu})$ to (IVVMD) gives

$$\int_{a_1}^{a_2} (\lambda(\nu))^T\eta(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu \leq \int_{a_1}^{a_2} (\lambda(\nu))^T\eta(\nu,\bar{\pi},\dot{\bar{\pi}})d\nu.$$

By the hypothesis (ii),the above inequality yields

$$\int_{a_1}^{a_2} \Big\{ (\bar{\pi}-\bar{\mu})^T(\lambda(\nu))^T\eta(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\frac{d}{d\nu}(\bar{\pi}-\bar{\mu})^T)(\lambda(\nu))^T\eta_{\dot{\bar{\mu}}}(\nu,\bar{\mu},\dot{\bar{\mu}}) \Big\} d\nu < 0,$$

which by (32) gives

$$\begin{aligned}\int_{a_1}^{a_2} \Big\{ &(\bar{\pi}-\bar{\mu})^T \left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right.\\ &\left. +\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}}) + (\lambda(\nu))^T\eta(\nu,\bar{\pi},\dot{\bar{\pi}})\right]\\ &+(\lambda-\bar{\lambda})^T\left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right.\\ &\left.+\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] + (\lambda(\nu))^T h(\nu,\bar{\mu},\dot{\bar{\mu}}) \Big\} d\nu > 0.\end{aligned}$$

By the hypothesis $(i)$, it follows from the above inequality

$$\begin{aligned}\int_{a_1}^{a_2} &\left[\Psi^L(\bar{\pi})p^L(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^L(\bar{\pi})q^L(\nu,\bar{\pi},\dot{\bar{\pi}})\right.\\ &\left.+\Psi^U(\bar{\pi})p^U(\nu,\bar{\pi},\dot{\bar{\pi}}) - \Phi^U(\bar{\pi})q^U(\nu,\bar{\pi},\dot{\bar{\pi}})\right] d\nu\\ &> \int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right.\\ &\left.+\Psi^U(\bar{\mu})p^U(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^U(\bar{\mu})q^U(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu, \qquad (33)\end{aligned}$$

which contradicts (31) and hence $\bar{\pi} = \bar{\mu}$. Moreover, if $\bar{\mu}$ is not a LU-optimal point for (IVVP), then there exists another feasible point $\pi$ for (IVVP) for the reason that

$$\left[\frac{\int_{a_1}^{a_2} p^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\bar{\mu},\dot{\bar{\mu}})d\nu}\right] \prec_{LU} \left[\frac{\int_{a_1}^{a_2} p^L(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^L(\nu,\pi,\dot{\pi})d\nu}, \frac{\int_{a_1}^{a_2} p^U(\nu,\pi,\dot{\pi})d\nu}{\int_{a_1}^{a_2} q^U(\nu,\pi,\dot{\pi})dt}\right]$$

That is, consider the case

$$\int_{a_1}^{a_2} \left[\Psi^L(\bar{\mu})p^L(\nu,\bar{\mu},\dot{\bar{\mu}}) - \Phi^L(\bar{\mu})q^L(\nu,\bar{\mu},\dot{\bar{\mu}})\right] d\nu$$

$$< \int_{a_1}^{a_2} \left[ \Psi^L(\pi) p^L(\nu, \pi, \dot{\pi}) - \Phi^L(\pi) q^L(\nu, \pi, \dot{\pi}) \right] d\nu,$$

$$\int_{a_1}^{a_2} \left[ \Psi^U(\bar{\mu}) p^U(\nu, \bar{\mu}, \dot{\bar{\mu}}) - \Phi^U(\bar{\mu}) q^U(\nu, \bar{\mu}, \dot{\bar{\mu}}) \right] d\nu$$

$$< \int_{a_1}^{a_2} \left[ \Psi^U(\pi) p^U(\nu, \pi, \dot{\pi}) - \Phi^U(\pi) q^U(\nu, \pi, \dot{\pi}) \right] d\nu, \tag{34}$$

In contrast, since $\bar{\pi} \neq \bar{\mu}$ derive in the same way as (33), we have,

$$\int_{a_1}^{a_2} \left[ \Psi^L(\bar{\pi}) p^L(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^L(\bar{\pi}) q^L(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right] d\nu$$

$$> \int_{a_1}^{a_2} \left[ \Psi^L(\bar{\mu}) p^L(\nu, \bar{\mu}, \dot{\bar{\mu}}) - \Phi^L(\bar{\mu}) q^L(\nu, \bar{\mu}, \dot{\bar{\mu}}) \right] d\nu,$$

$$\int_{a_1}^{a_2} \left[ \Psi^U(\bar{\pi}) p^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) - \Phi^U(\bar{\pi}) q^U(\nu, \bar{\pi}, \dot{\bar{\pi}}) \right] d\nu$$

$$> \int_{a_1}^{a_2} \left[ \Psi^U(\bar{\mu}) p^U(\nu, \bar{\mu}, \dot{\bar{\mu}}) - \Phi^U(\bar{\mu}) q^U(\nu, \bar{\mu}, \dot{\bar{\mu}}) \right] d\nu,$$

which contradicts (34) and hence leads to the result. This compltes the proof. □

## 5. CONCLUSION

Under the concept of LU optimality, generalized convexity extended to derive KKT necessary and sufficient conditions for optimality, for a larger class of Interval-valued fractional variational programming problems. The results developed in this paper can be generalized to the non-smooth interval-valued variational programming problem. The authors use this approach as an orientation for their future work.

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