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OPTIMALITY CONDITIONS FOR ISOPERIMETRIC CONTINUOUS-TIME OPTIMIZATION PROBLEMS

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Abstract: In this paper we deal with a nonsmooth case of isoperimetric convex continuous-time programming problem with inequality integral constraint and phase constraint, defined in $L_\infty([0,T],\mathbb{R}^n)$. In order to obtain necessary optimality conditions for this problem, we will use a theorem of the alternative from [15] as a main tool.

Keywords: Continuous-time programming, convex programming, isoperimetric problems, necessary optimality conditions, theorems of the alternative.

MSC: 90C30, 90C46.

1. INTRODUCTION

Let's consider the following isoperimetric problem

$$\begin{aligned}
&\text{maximize} \quad J(x(\cdot)) = \int_0^T f(t,x(t))dt \\
&\text{subject to} \int_0^T h_i(t,x(t))dt \geq 0, \quad i \in I = \{1,...,m\}, \\
&\qquad g_j(t,x(t)) \geq 0, \quad j \in J = \{1,...,k\} \quad \text{a.e. in} \quad [0,T], \\
&\qquad x(\cdot) \in L_\infty([0,T],\mathbb{R}^n),
\end{aligned} \tag{P}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R},\ h_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}, i \in I,\ g_j : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $j \in J$ are given functions. All integrals in this paper are in the Lebesgue sense.

Extremal problem which includes integral inequality constraint, but no phase constraint, is called Lyapunov-type problem and is presented in [1]. The authors obtained optimality conditions for that case without a convexity assumption. Our aim in this work is to provide necessary optimality conditions for convex continuous-time problems involving both integral and phase inequality constraints, without differentiability assumption.

Since 1953, when Bellman [2] introduced what we now consider a continuous-time linear problem for the first time, many authors have dealt with optimal conditions for wider classes of linear and nonlinear continuous-time programming problems, but also duality theory [3, 4, 5]. In some works a smooth case is investigated, for example [6, 7, 8, 9, 10, 11, 12]. In [6] authors do Monte and de Oliveira developed new Karush-Kuhn-Tucker necessary conditions first for problems with inequality constraints and then for problems with both equality and inequality constraints. On the other hand, our results will refer to the nonsmooth case, which was also treated by Brandão et al. in [13] or Outrata and Römisch in [14]. Unfortunately, some results from the literature are not valid, as authors pointed out in [15].

As a powerful tool in this manuscript we will use theorem of the alternative presented in [15] by A.V. Arutyunov, S.E. Zhukovskiy and B. Marinković. A theorem of the alternative asserts that for two alternative systems exactly one of them has a solution. The relevance of these theorems in extremal problems and some interesting applications are offered in [16].

The rest of this paper is organized as follows. In section 2 we give some preliminaries for the problem, state auxiliary results and define the regularity condition which is necessary for applying the theorem of the alternative. In section 3 we prove our main results and provide an illustrative example.

2. PRELIMINARIES

All vectors are column vectors, unless transposed. Notation $w \leq 0$ means that $w_i \leq 0$ for all i and $w < 0$ means that $w_i < 0$ for all i. In a Banach space $(E, \|\cdot\|)$ we denote by $\langle \varphi, x\rangle$ the value of $\varphi \in E^*$ at $x \in E$, but also the inner product of vectors $\varphi, x \in \mathbb{R}^n$.

In text below, $\partial_x g(t,x)$ stands for the subdifferential of $g(t,\cdot)$ at a point $x \in E$ in the sense of convex analysis.

In articles [17, 7] authors use Gordan's Theorem and Motzkin-type theorem of the alternative for system of convex inequalities for deriving optimality criteria for extremal problems. Some results, unfortunately, weren't valid, as noticed in [15]. Here we use the correct form presented by Arutyunov et al [15]. To apply this theorem a certain regularity condition has to be satisfied.

Let $X \subset E$ be closed and convex. Consider the system

$$\begin{cases} f_i(t,x) \leq 0, & i \in I_1 \\ f_i(t,x) < 0, & i \in I_2 \\ x \in X, \end{cases} \tag{1}$$

with given functions $f_i : [0,T] \times E \to \overline{\mathbb{R}}, i = 1, ..., k$ and I_1, I_2 are sets of indexes such that $I_1 \sqcup I_2 = \{1, ..., k\}$ ($\sqcup$ stands for disjoint union). A solution of system (1) is a function $x(\cdot) \in L_\infty([0,1], X)$ such that for a.e. $t \in [0,1]$ the following holds:

$$f_i(t, x(t)) \leq 0,\ i \in I_1,\quad f_i(t, x(t)) < 0,\ i \in I_2,\quad x(t) \in X.$$

As well, we consider that the following conditions are true:

1. functions $f_i(t, \cdot)$ are convex and continuous on X, and $X \subset int(dom(f_i(t, \cdot)))$ for a.e. $t \in [0,1], i = 1, ..., k$;
2. functions $f_i(\cdot, x)$ are Lebesgue measurable for all $x \in X, i = 1, ..., k$;
3. for each $K \geq 0$ there exists $M = M(K) \geq 0$ such that

$$|x| \leq K \Rightarrow |f_i(x,t)| \leq M \ \ a.e.\, t \in [0,1],\ \forall x \in X,\ i = 1, ..., k.$$

For more details on systems of convex inequalities the reader is referred to [18].

Definition 1. *We say that (1) is regular, if there exist a function $\bar{x}(\cdot) \in L_\infty([0,1], X)$, reals $R \geq 0$ and $\alpha > 0$ such that for a.e. $t \in [0,1]$ and for all $x \in X$ with $\|x - \bar{x}(t)\| \geq R$, there exists a vector $e = e(t,x) \in -T_X(x)$, $\|e\| = 1$, satisfying*

$$\langle x^*, e \rangle \geq \alpha \quad \forall x^* \in \partial_x f_i(t,x),\ i \in \mathcal{I}(t,x),$$

where $T_X(x)$ is a tangent cone to the set X at the point x and

$$\mathcal{I}(t,x) := \left\{ i : f_i(t,x) = \max_{j=1,\ldots,k} f_j(t,x) \right\},\ \ t \in [0,1],\ x \in X.$$

Theorem 2. ***(Theorem of the Alternative [15])*** *Assume that the Banach space E is separable, the system (1) is regular, and for a.e. $t \in [0,1]$ there exists a vector $u = u(t) \in X$ such that $f_i(t, u(t)) < 0$ for each $i \in I_1$. Then, one and only one of the following assertions is valid.*

i *There exists a solution $\chi(\cdot)$ for system (1);*

ii *There exists a nonzero function $\varphi(\cdot) = (\varphi_1(\cdot), ..., \varphi_k(\cdot)) \in L_\infty([0,1], \mathbb{R}^k_+)$ such that $\varphi_i(t) \not\equiv 0$ for some $i \in I_2$ and*

$$\sum_{i=1}^{k} f_i(t,x)\varphi_i(t) \geq 0 \quad \text{for a.e. } t \in [0,1], \quad \forall x \in X.$$

This theorem stays valid by changing interval $[0,1]$ to interval $[0,T]$.

3. NECESSARY OPTIMALITY CONDITIONS

For the posed problem (P), let

$$\mathbb{F} = \{x(\cdot) \in L_\infty([0,T], \mathbb{R}^n) : \int_0^T h_i(t, x(t))dt \leq 0,\ i \in I,\ g_j(t, x(t)) \leq 0,\ j \in J \ \text{a.e. in } [0,T]\}$$

be the set of all feasible solutions.

Definition 3. *A feasible solution $\hat{x}(\cdot)$ is said to be a global maximizer of (P) if*

$$J(\hat{x}(\cdot)) \geqslant J(x(\cdot)), \quad \forall x(\cdot) \in \mathbb{F}.$$

Let's assume that the functions $f(\cdot,x)$, $h_i(\cdot,x)$ and $g_j(\cdot,x)$ are Lebesgue measurable and integrabile for all $x \in \mathbb{R}^n$, $i \in I, j \in J$. Also functions $f(t,\cdot)$, $h_i(t,\cdot), i \in I$ and $g_j(t,\cdot), j \in J$ are convex for almost every $t \in [0,T]$ and for each $M \geq 0$, $N \geq 0$, $P \geq 0$ there exist, respectively, $L = L(M) \geq 0$, $K = K(N) \geq 0$ and $Q = Q(P) \geq 0$ such that

$$\begin{aligned}
&\|x\| \leq M \Rightarrow |f(t,x)| \leq L, \quad \forall x \in \mathbb{R}^n, \quad \text{a.e. in } [0,T],\\
&\|x\| \leq N \Rightarrow |h_i(t,x)| \leq K, \quad \forall x \in \mathbb{R}^n, \quad i \in I \quad \text{a.e. in } [0,T],\\
&\|x\| \leq P \Rightarrow |g_j(t,x)| \leq Q, \quad \forall x \in \mathbb{R}^n, \quad j \in J \quad \text{a.e. in } [0,T].
\end{aligned}$$

Let

$$\begin{aligned}
&\phi_0(t,x) := -\int_0^T \langle \partial_x f(t,\hat{x}(t)), x - \hat{x}(t)\rangle\, dt < 0,\\
&\phi_i(t,x) := -\int_0^T \langle \partial_x h_i(t,\hat{x}(t)), x - \hat{x}(t)\rangle\, dt \leq 0, \;\; i \in I, \qquad \text{(S)}\\
&\phi_j(t,x) := -g_j(t,\hat{x}(t)) - \langle \partial_x g_j(t,\hat{x}(t)), x - \hat{x}(t)\rangle \leq 0, \;\; j \in J, \text{ a.e. in } [0,T],\\
&x \in \mathbb{R}^n,
\end{aligned}$$

be a system corresponding to the problem (P).

Lemma 4. *If there exists a global maximizer $\hat{x}(\cdot)$ for (P), then system (S) doesn't have a solution.*

Proof. Let $\overline{x}(\cdot)$ be a solution to the system (S). It follows that

$$\int_0^T \langle \partial_x f(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle\, dt > 0, \tag{2}$$

$$\int_0^T \langle \partial_x h_i(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle\, dt \geq 0, \;\; i \in I, \tag{3}$$

$$g_j(t,\hat{x}(t)) + \langle \partial_x g_j(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle \geq 0, \;\; j \in J, \;\; \text{a.e. in } [0,T]. \tag{4}$$

Then $\overline{x}(\cdot)$ is feasible solution to the problem (P), because

$$\int_0^T (h_i(t,\overline{x}(t)) - h_i(t,\hat{x}(t)))dt \geq \int_0^T \langle \partial_x h_i(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle\, dt, \;\; i \in I,$$

so $\int_0^T h_i(t,\overline{x}(t))dt \geq 0,\ i \in I$. Analogously we obtain $g_j(t,\overline{x}(t)) \geq 0,\ j \in J$ a.e. in $[0,T]$.

From the fact that

$$f(t,\overline{x}(t)) - f(t,\hat{x}(t)) \geq \langle \partial_x f(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t) \rangle \text{, a.e. in } [0,T],$$

by integrating on $[0,T]$ we have

$$\int_0^T (f(t,\overline{x}(t)) - f(t,\hat{x}(t)))dt \geq \int_0^T \langle \partial_x f(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t) \rangle \, dt > 0.$$

Hence

$$J(\overline{x}(\cdot)) = \int_0^T f(t,\overline{x}(t))dt > \int_0^T f(t,\hat{x}(t))dt = J(\hat{x}(\cdot)),$$

which means that $\hat{x}(\cdot)$ is not a global maximizer for (P), as we assumed. □

Definition 5. *We say that regularity condition for system (S) is satisfied, if there exist a function $\overline{x}(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, reals $R \geq 0$ and $\alpha > 0$ such that for a.e. $t \in [0,T]$ and for all $x \in \mathbb{R}^n$ with $\|x - \overline{x}(t)\| \geq R$, there exists $e = e(t,x) \in \mathbb{R}^n$, $\|e\| = 1$, which satisfies*

$$\langle \partial_x \phi_l(t,x), e \rangle \geq \alpha, \quad \forall l \in \mathcal{I}(t,x),$$

where

$$\mathcal{I}(t,x) := \left\{ l : \phi_l(t,x) = \max_{p \in \{0\} \cup I \cup J} \phi_p(t,x) \right\}, \ \ t \in [0,T],\ x \in \mathbb{R}^n$$

denotes the set of active indexes of the system (S).

Slater's constraint qualification is said to be satisfied if there exists $x = x(t) \in \mathbb{R}^n$ such that $h_i(t,x(t)) > 0,\ i \in I$ and $g_j(t,x(t)) > 0,\ j \in J$ a.e. in $[0,T]$.

Theorem 6. *Let $\hat{x}(t)$ be a global maximizer for (P). Assume that regularity condition for (S) and Slater's constraint qualification are satisfied. Then, there exist $\hat{u} \in \mathbb{R}^m$ and $\hat{v}(t) \in L_\infty([0,T],\mathbb{R}^k)$ satisfying the following conditions:*

1. $\hat{u} \geq 0, \quad \hat{v}(t) \geq 0$ a.e. in $[0,T]$,
2. $\hat{v}_j'(t) g_j(t,\hat{x}(t)) = 0,\ j \in J$ a.e. in $[0,T]$,
3. $f(t,x(t)) + \sum_{i=1}^m \hat{u}_i h_i(t,x(t)) + \sum_{j=1}^k \hat{v}_j(t) g_j(t,x(t)) \geq$ $f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i h_i(t,\hat{x}(t)) + \sum_{j=1}^k \hat{v}_j(t) g_j(t,\hat{x}(t))$,

   $\forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, a.e. in $[0,T]$.

Proof. Suppose that $\hat{x}(\cdot)$ is a global maximizer for (P). Then by Lemma 4 we conclude the system

$$-\int_0^T \langle \partial_x f(t,\hat{x}(t)), x-\hat{x}(t)\rangle\, dt < 0,$$

$$-\int_0^T \langle \partial_x h_i(t,\hat{x}(t)), x-\hat{x}(t)\rangle\, dt \le 0, \quad i \in I$$

$$-g_j(t,\hat{x}(t)) - \langle \partial_x g_j(t,\hat{x}(t)), x-\hat{x}(t)\rangle \le 0, \quad j \in J$$

is inconsistent a.e. in $[0,T]$. From Theorem 2 there exists a nonzero function $(\hat{\lambda}(\cdot), \hat{\varphi}_1(\cdot), ..., \hat{\varphi}_m(\cdot), \hat{\psi}_1(\cdot), ..., \hat{\psi}_k(\cdot)) \in L_\infty([0,T], \mathbb{R}_+^{m+k+1})$ with $\hat{\lambda}(t) \not\equiv 0$ a.e. in $[0,T]$, such that

$$\hat{\lambda}(t)\int_0^T \langle \partial_x f(s,\hat{x}(s)), x(s)-\hat{x}(s)\rangle\, ds + \sum_{i=1}^m \hat{\varphi}_i(t)\int_0^T \langle \partial_x h_i(t,\hat{x}(s)), x(s)-\hat{x}(s)\rangle\, ds +$$
$$\sum_{j=1}^k \hat{\psi}_j(t)(g_j(t,\hat{x}(t)) + \langle \partial_x g_j(t,\hat{x}(t)), x(t)-\hat{x}(t)\rangle) \le 0, \tag{5}$$

$\forall x(\cdot) \in L_\infty([0,T], \mathbb{R}^n)$, a.e. in $[0,T]$. For $x(\cdot) = \hat{x}(\cdot)$ we obtain

$$\sum_{j=1}^k \hat{\psi}_j(t) g_j(t,\hat{x}(t)) \le 0.$$

It holds that $\hat{\psi}_j(t) \ge 0$ and $g_j(t,\hat{x}(t)) \ge 0$ a.e. in $[0,T]$, $j \in J$, so it must be

$$\sum_{j=1}^k \hat{\psi}_j(t) g_j(t,\hat{x}(t)) = 0. \tag{6}$$

Now from (5) we have

$$\hat{\lambda}(t)\int_0^T \langle \partial_x f(s,\hat{x}(s)), x(s)-\hat{x}(s)\rangle\, ds + \sum_{i=1}^m \hat{\varphi}_i(t)\int_0^T \langle \partial_x h_i(t,\hat{x}(s)), x(s)-\hat{x}(s)\rangle\, ds +$$
$$\sum_{j=1}^k \hat{\psi}_j(t)\langle \partial_x g_j(t,\hat{x}(t)), x(t)-\hat{x}(t)\rangle \le 0, \quad \forall x(\cdot) \in L_\infty([0,T], \mathbb{R}^n), \text{ a.e. in } [0,T].$$

Integrating this inequality on $[0,T]$, we get

$$\int_0^T \hat{\mu}\,\langle \partial_x f(t,\hat{x}(t)), x(t)-\hat{x}(t)\rangle\, dt +$$

$$\int_0^T \left(\sum_{i=1}^{m} \hat{\eta}_i \left\langle \partial_x h_i(t,\hat{x}(t)), x(t)-\hat{x}(t) \right\rangle + \sum_{j=1}^{k} \hat{\psi}_j(t) \left\langle \partial_x g_j(t,\hat{x}(t)), x(t)-\hat{x}(t) \right\rangle \right) dt \le 0,$$

$\forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, where

$$\hat{\mu} = \int_0^T \hat{\lambda}(t)dt > 0 \quad \text{and} \quad \hat{\eta}_i = \int_0^T \hat{\varphi}_i(t)dt \ge 0, \quad i \in I. \tag{7}$$

Previous inequality becomes

$$\int_0^T \left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), x(t)-\hat{x}(t) \right\rangle dt \le 0, \tag{8}$$

$\forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$ by setting

$$\hat{u}_i = \frac{\hat{\eta}_i}{\hat{\mu}} \quad i \in I \quad \text{and} \quad \hat{v}_j(t) = \frac{\hat{\psi}_j(t)}{\hat{\mu}} \quad j \in J.$$

Let's prove that from (8) we have

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), x(t)-\hat{x}(t) \right\rangle = 0, \tag{9}$$

$\forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, a.e. in $[0,T]$.

Assume that there exists $\overline{x}(\cdot) \in L_\infty(A,\mathbb{R}^n)$ such that

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), \overline{x}(t)-\hat{x}(t) \right\rangle > 0 \quad \text{a.e. in } A,$$

where set $A \subset [0,T]$ doesn't have measure zero. Defining a function

$$y(t) = \begin{cases} \overline{x}(t), & t \in A, \\ 0, & t \notin A, \end{cases}$$

we obtain

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), y(t)-\hat{x}(t) \right\rangle > 0 \quad \text{a.e. in } [0,T],$$

and by integrating on $[0,T]$

$$\int_0^T \left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), y(t)-\hat{x}(t) \right\rangle dt > 0,$$

which is a contradiction to (8).

Conversely we'll assume that there exists $\overline{x}(\cdot) \in L_\infty(A, \mathbb{R}^n)$ such that

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t) \right\rangle < 0 \quad \text{a.e. in } A, \tag{10}$$

where set $A \subset [0,T]$ doesn't have measure zero. Let's define a function

$$y(t) = \begin{cases} 2\hat{x}(t) - \overline{x}(t), & t \in A, \\ 0, & t \notin A. \end{cases}$$

Now we have

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), y(t) - \hat{x}(t) \right\rangle =$$

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i \partial_x h_i(t,\hat{x}(t)) + \sum_{j=1}^{k} \hat{v}_j(t) \partial_x g_j(t,\hat{x}(t)), \hat{x}(t) - \overline{x}(t) \right\rangle > 0 \quad \text{a.e. in } [0,T],$$

which also brings us to contradiction to (8), so (9) must be valid.

Consequently

$$0 = \langle \partial_x f(t,\hat{x}(t)), x(t) - \hat{x}(t) \rangle +$$

$$\sum_{i=1}^{m} \hat{u}_i \langle \partial_x h_i(t,\hat{x}(t)), x(t) - \hat{x}(t) \rangle + \sum_{j=1}^{k} \hat{v}_j(t) \langle \partial_x g_j(t,\hat{x}(t)), x(t) - \hat{x}(t) \rangle \leq$$

$$f(t,x(t)) - f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i (h_i(t,x(t)) - h_i(t,\hat{x}(t))) + \sum_{j=1}^{k} \hat{v}_j(t)(g_j(t,x(t)) - g_j(t,\hat{x}(t))),$$

$\forall x(\cdot) \in L_\infty([0,T], \mathbb{R}^n)$, a.e. in $[0,T]$, so condition 3 holds. Condition 2 follows from (6) , while condition 1 is satisfied by construction. □

Example 7. *Let's take a look at the problem below.*

$$\text{Maximize} \qquad J(x(\cdot)) = \int_0^1 (1 - 2x_1(t) + x_2(t)) dt$$

$$\text{subject to} \qquad \int_0^1 (|x_1(t) - 1| + 3x_1(t) - 3) dt \geq 0$$

$$x_1(t) - 2x_2(t) + 1 \geq 0 \qquad \text{a.e. in } [0,1],$$

$$x(\cdot) \in L_\infty([0,1], \mathbb{R}^2).$$

Global maximizer is $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t)) = (1,1)$ *and* $|x_1(t) - 1| + 3x_1(t) - 3 > 0$, $x_1(t) - 2x_2(t) + 1 > 0$, *is satisfied for* $x(t) = (2,0)$. *Given* $x = (x_1, x_2) \in \mathbb{R}^2$, *for almost everywhere in* $[0,1]$, *we have*

$$\begin{aligned} &\phi_0(t,x) = 2x_1 - x_2 - 1, \\ &\phi_1(t,x) = -\left\{ \begin{array}{ll} \langle (2,0), (x_1 - 1, x_2 - 1) \rangle, & x_1 < 1, \\ \langle ([2,4] \times 0), (x_1 - 1, x_2 - 1) \rangle, & x_1 = 1, \\ \langle (4,0), (x_1 - 1, x_2 - 1) \rangle, & x_1 > 1, \end{array} \right. \\ &\phi_2(t,x) = -x_1 + 2x_2 - 1, \\ &x \in \mathbb{R}^2 \end{aligned} \tag{11}$$

Define
$R_0 = \{(x_1, x_2) \in \mathbb{R}^2 : (\theta + 2)x_1 - x_2 \geq \theta + 1,\, 3x_1 - 3x_2 \geq 0,\, \theta \in [2,4]\}$,
$R_1 = \{(x_1, x_2) \in \mathbb{R}^2 : (\theta + 2)x_1 - x_2 \leq \theta + 1,\, (\theta - 1)x_1 + 2x_2 \leq \theta + 1,\, \theta \in [2,4]\}$,
$R_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 3x_1 - 3x_2 \leq 0,\, (\theta - 1)x_1 + 2x_2 \geq \theta + 1,\, \theta \in [2,4]\}$.

One can easily check that

$$\bigcup_{i=0}^{2} R_i = \mathbb{R}^2.$$

For $x \in intR_0, \phi_0(t,x) = \max\{\phi_0(t,x), \phi_1(t,x), \phi_2(t,x)\}$, *a.e. in [0,1], i.e.,* $\mathcal{I}(t,x) = \{0\}$. *For* $x \in intR_1, \phi_1(t,x) = \max\{\phi_0(t,x), \phi_1(t,x), \phi_2(t,x)\}$, *a.e. in [0,1], i.e.,* $\mathcal{I}(t,x) = \{1\}$. *For* $x \in intR_2$, $\mathcal{I}(t,x) = \{2\}$ *and* $\{0,1\} \notin \mathcal{I}(t,x)$ *for* $x \in R_2$. *For* $x \in int(R_0 \cup R_1)$, $\mathcal{I}(t,x) = \{0,1\}$ *a.e. in [0,1]. For* $x \in int(R_1 \cup R_2), \mathcal{I}(t,x) = \{1,2\}$ *and finally for* $x \in int(R_0 \cup R_2), \mathcal{I}(t,x) = \{0,2\}$ *a.e. in [0,1].*

We can conclude that system

$$\begin{aligned} &\phi_0(t,x) = 2x_1 - x_2 - 1 < 0, \\ &\phi_1(t,x) = \theta(1 - x_1) \leq 0, \\ &\phi_2(t,x) = -x_1 + 2x_2 - 1 \leq 0, \\ &x \in \mathbb{R}^2, \theta \in [2,4], \end{aligned}$$

is regular with $\hat{x}(t) = (1,1)$, $R = \frac{1}{100}$, $\alpha = \frac{1}{\sqrt{17}}$ *and for a.e. in [0,1],* $e(t,x) = (\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}})$, *for* $x \in R_0$ *and* $e(t,x) = (-\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$, *for* $x \in int(R_1 \cup R_2)$. *Conditions 1-3 from Theorem 6 are satisfied for* $\hat{u} = \frac{3}{8}, \hat{v}(t) = \frac{1}{2}$ *when* $x_1 \geq 1$ *and for* $\hat{u} = \frac{3}{4}, \hat{v}(t) = \frac{1}{2}$ *when* $x_1 < 1$.

4. CONCLUSION

This paper addressed an isoperimetric continuous-time programming problem involving both integral and phase inequality constraints, without differentiability. Necessary optimality conditions were obtained under a suitable regularity assumption and using new theorem of the alternative in infinite-dimensional spaces. It would be interesting to see if it's possible to examine optimality conditions for isoperimetric continuous-time programming problems without convexity assumption, by following a similar approach.

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