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THE UNIQUE SOLVABILITY CONDITIONS FOR A NEW CLASS OF ABSOLUTE VALUE EQUATION

Shubham KUMAR
Pandit Dwarka Prasad Mishra - Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, Madhya Pradesh, India
shub.srma@gmail.com

DEEPMALA
Pandit Dwarka Prasad Mishra - Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, Madhya Pradesh, India
dmrai23@gmail.com

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Abstract: In this article, we investigate the solution of a new class of the absolute value equation (NCAVE) $A_1x - |B_1x - c| = d$. Based on spectral radius condition, singular value condition and row and column $\mathcal{W}$-property, some necessary and sufficient conditions for unique solvability for NCAVE are gained. Some new results for the unique solvability of the new generalized absolute value equation (NGAVE) $A_1x - |B_1x| = d$ are also obtained.

Keywords: Absolute value equation, unique solution, sufficient condition, vertical linear complementarity problem.

MSC: 15A18, 90C05, 90C30.

1. INTRODUCTION

The absolute value equations (AVEs) are an interesting topic for researchers in the optimization field. Firstly, Jiri Rohn [1] in 2004 considered the generalized absolute value equation (GAVE) $A_1x + B_1|x| = b$ and provided an alternative theorem for the solution of this equation. Many authors studied about GAVE and its particular case as $B_1 = I$ (see [2, 3, 4, 5] and references therein).

The importance of AVEs is due to their wide applications in many domains of mathematics. Absolute value equations have many applications in various fields of applied mathematics, like game theory, linear complementarity problems (LCP), optimization problems, linear interval systems, bi-matrix games, etc. The LCP is a general problem that unifies quadratic programs, linear programs, bi-matrix games, and absolute value equations. LCP can be written as an equivalent form of AVE and vice-versa, so the results of the linear complementarity problem are also applicable for the absolute value equations and conversely.

In this article, we are considering a new class of the absolute value equation (NCAVE)

$$A_1x - |B_1x - c| = d, \tag{1}$$

where $A_1, B_1 \in R^{n\times n}$ and $c, d \in R^n$ are given.

When we take $c = 0$ (zero vector) and $B_1 = I$ (Identity matrix) in (1), then we get a new generalized absolute value equation (NGAVE)

$$A_1x - |B_1x| = d, \tag{2}$$

and standard absolute value equation

$$A_1x - |x| = d, \tag{3}$$

respectively.

The general form of (3) is generalized absolute value equation (GAVE)

$$A_1x - B_1|x| = d. \tag{4}$$

In 2021, NGAVE (2) was first considered by Wu [6], and discussed its different conditions for a unique solution and indicated that the work of Wu [6] could be extended for the NCAVE (1). Based on our knowledge, no one has yet studied a new class of the AVE (1) in detail. So there are some gaps and void conditions for their unique solutions. As it has non-differentiable and non-linear terms, studying the NCAVE is exciting and challenging. The study of the absolute value equations is going in two directions: one is a theoretical analysis of AVEs (see [2, 4, 7, 8, 9, 10] and references therein). Another one is, based on theoretical analysis to develop some numerical methods (see [11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein), for the solution of AVEs. Solving and checking the unique solution of the AVEs is an NP-hard problem [3].

We will denote $D = diag(d_i)$ with $0 \le d_i \le 1$ is a diagonal matrix. $I_{n\times n}$, $O_{n\times n}$ are denotes identity matrix and zero matrix, respectively. $\sigma_{max}(.)$ (or $\sigma_1(.)$) and $\sigma_{min}(.)$ (or $\sigma_n(.)$) are denotes maximum and minimum singular value, respectively and $\rho(.)$ is use for the spectral radius of a matrix.

This article is arranged as, Section (2) contains some useful results for further uses in Section (3). In Section (3), we obtain the unique solution condition for NCAVE (1). We conclude our discussion in Section (4).

2. PRELIMINARIES

In this section, we recall some definitions, lemmas and theorems for further use.

Definition 1. *[20] The LCP(r,P) is defined as:*

$$0 \leq z \perp Pz + r \geq 0, \tag{5}$$

where $r \in R^n$, $P \in R^{n\times n}$ *and z is unknown.*

Definition 2. *[20] For unknown* $x \in R^n$*, vertical linear complementarity problem (VLCP) is defined as*

$$G_1x + p \geq 0, H_1x + q \geq 0, (G_1x + p)^T(H_1x + q) = 0, \tag{6}$$

where $p, q \in R^n$ *and* $G_1, H_1 \in R^{n\times n}$.

Definition 3. *[21] A matrix* $M \in R^{n\times n}$ *is called a P-matrix if all its principal minors are positive and further, every positive definite (PD) matrices are P-matrices.*

Definition 4. *[22] Let* $\mathcal{M} = \{M_1, M_2\}$ *denote the set of matrices with* $M_1, M_2 \in R^{n\times n}$*. A matrix* $R \in R^{n\times n}$ *is called a row (or column) representative of* $\mathcal{M}$*, if* $R_{j.} \in \{(M_1)_{j.}, (M_2)_{j.}\}$ *(or* $R_{.j} \in \{(M_1)_{.j}, (M_2)_{.j}\}$*) j=1,2,...,n, where* $R_{j.}$, $(M_1)_{j.}$*, and* $(M_2)_{j.}$ *(or* $R_{.j}$*,* $(M_1)_{.j}$*, and* $(M_2)_{.j}$*) denote the* j^{th} *row (or column) of R,* M_1 *and* M_2*, respectively.*

Definition 5. *[22] The set* $\mathcal{M}$ *holds the row (or column)* $\mathcal{W}$*-property if the determinants of all row (or column) representative matrices of* $\mathcal{M}$ *are positive.*

Lemma 6. *[22] A matrix* $M \in R^{n\times n}$ *is a P-matrix if and only if the determinants of all row representative matrices of* $\{I, M\}$ *are positive.*

Lemma 7. *[23] A matrix* $M \in R^{n\times n}$ *is a P-matrix if and only if matrix* $M + D(I - M)$ *or* $I - D + DM$ *is non-singular for any* D*.*

Lemma 8. *[20] Let* a_1*,*$b_1 \in R$*. Then* $a_1, b_1 \geq 0$*,* $a_1.b_1 = 0$ *if and only if* $a_1 + b_1 = |a_1 - b_1|$*. This result is also applicable for vectors in* R^n*.*

Lemma 9. *[20] For real square matrix A and B, we have* $\sigma_i(A + B) \geq \sigma_i(A) - \sigma_1(B)$*,* $i = 1, 2, ..., n$*.*

Theorem 10. *[22] Following statements are equivalent, for set* $\{G_1, H_1\}$*:*
(i) The VLCP (6) has a unique solution;
(ii) $\{G_1, H_1\}$ *holds the row* $\mathcal{W}$*-property;*
(iii) G_1 *is invertible and* $\{I, H_1G_1^{-1}\}$ *holds the row* $\mathcal{W}$*-property.*

Theorem 11. *[8, 10] The following statements are identical:*
(i) the AVE (3) has exactly one solution for any d;
(ii) $\{A_1 - I, A_1 + I\}$ *holds the column* $\mathcal{W}$*-property;*
(iii) (A_1-I) *is invertible and* $\{I, (A_1-I)^{-1}(A_1+I)\}$ *holds the column* $\mathcal{W}$*-property;*
(iv) $(A_1 - I)$ *is invertible and* $(A_1 - I)^{-1}(A_1 + I)$ *is a P-matrix;*
(v) $(A_1 + (I - 2D))$ *is invertible for any* D *;*
(vi) $\{(A_1-I)F_1+(A_1+I)F_2\}$ *is invertible, where* $F_1, F_2 \in R^{n\times n}$ *are two arbitrary non-negative diagonal matrices with* $diag(F_1 + F_2) > 0$.

Theorem 12. *[8, 10] If* A_1 *is non-singular matrix in AVE* $A_1x - |x| = d$*, and satisfy the conditions*

$$\rho(A_1^{-1}(I - 2D)) < 1 \tag{7}$$

for any D*, or*

$$\sigma_{max}(A_1^{-1}) < 1, \tag{8}$$

or

$$\rho(|A_1^{-1}|) < 1, \tag{9}$$

then AVE (3) has a unique solution for any d.

Theorem 13. *[6]The following statements are equivalent:*
(i) the AVE (3) has a unique solution for any d;
(ii) $\{A_1 + I, A_1 - I\}$ *has the row* $\mathcal{W}$*-property;*
(iii) $(A_1 + I)$ *is invertible and* $\{I, (A_1 - I)(A_1 + I)^{-1}\}$ *has the row* $\mathcal{W}$*-property;*
(iv) $(A_1 + I)$ *is invertible and* $(A_1 - I)(A_1 + I)^{-1}$ *is a P-matrix;*
(v) $\{F_1(A_1+I)+F_2(A_1-I)\}$ *is invertible, where* $F_1, F_2 \in R^{n\times n}$ *are two arbitrary non-negative diagonal matrices with* $diag(F_1 + F_2) > 0$.

Theorem 14. *[8] If all diagonal entries of* $A_1 + I$ *have the same sign as the corresponding entries of* $A_1 - I$*, then AVE* $A_1x - |x| = d$ *has exactly one solution for any d, if any one of the following conditions is true:*
(i) $A_1 - I$ *and* $A_1 + I$ *are strictly diagonally dominant by columns;*
(ii) $A_1 - I$*,* $A_1 + I$ *and all their column representative matrices are irreducibly diagonally dominant by columns.*

Theorem 15. *[24] AVE* $A_1x - |x| = d$ *has exactly one solution for any d, if the interval matrix [25]* $[A_1 - I, A_1 + I]$ *is regular.*

3. MAIN RESULTS

In this section, we obtained sufficient and necessary conditions for the unique solution of NCAVE (1).

In the following proposition, NCAVE is written in equivalent AVE under the non-singularity condition on B_1.

Proposition 16. *If matrix B_1 is non-singular, then NCAVE (1) is expressed as the following AVE form*

$$A_1 B_1^{-1} y - |y| = f_1, \tag{10}$$

where $y = B_1 x - c$ and $f_1 = d - A_1 B_1^{-1} c$.

By the help of Proposition (16) with Theorems (11), (12) and (13), we obtained the following results, see Theorem (17), Theorem (18) and Theorem (19) respectively.

Theorem 17. *If $det(B_1) \neq 0$, then the following assertions are equivalent:*
(i) the NCAVE (1) has exactly one solution for any d;
(ii) $\{A_1B_1^{-1} - I, A_1B_1^{-1} + I\}$ holds the column $\mathcal{W}$-property;
(iii) $(A_1B_1^{-1} - I)$ is invertible and $\{I, (A_1B_1^{-1} - I)^{-1}(A_1B_1^{-1} + I)\}$ holds the column $\mathcal{W}$-property;
(iv) $(A_1B_1^{-1} - I)$ is invertible and $(A_1B_1^{-1} - I)^{-1}(A_1B_1^{-1} + I)$ is a P-matrix;
(v) $(A_1B_1^{-1} + (I - 2D))$ is invertible for any D ;
(vi) $\{(A_1B_1^{-1} - I)F_1 + (A_1B_1^{-1} + I)F_2\}$ is invertible, where $F_1, F_2 \in R^{n \times n}$ are two arbitrary non-negative diagonal matrices with $diag(F_1 + F_2) > 0$.

Theorem 18. *If A_1 is non-singular matrix and satisfies the conditions*

$$\rho(B_1 A_1^{-1}(I - 2D)) < 1 \tag{11}$$

for any D, or

$$\sigma_{max}(B_1 A_1^{-1}) < 1, \tag{12}$$

or

$$\rho(|B_1 A_1^{-1}|) < 1, \tag{13}$$

then the NCAVE (1) has a unique solution.

Theorem 19. *If $det(B_1) \neq 0$, then the following assertions are equivalent:*
(i) the NCAVE (1) has a unique solution;
(ii) $\{A_1B_1^{-1} + I, A_1B_1^{-1} - I\}$ has the row $\mathcal{W}$-property;
(iii) $(A_1B_1^{-1} + I)$ is invertible and $\{I, (A_1B_1^{-1} - I)(A_1B_1^{-1} + I)^{-1}\}$ has the row $\mathcal{W}$-property;
(iv) $(A_1B_1^{-1} + I)$ is invertible and $(A_1B_1^{-1} - I)(A_1B_1^{-1} + I)^{-1}$ is a P-matrix;
(v) $\{F_1(A_1B_1^{-1} + I) + F_2(A_1B_1^{-1} - I)\}$ is invertible, where $F_1, F_2 \in R^{n \times n}$ are two arbitrary non-negative diagonal matrices with $diag(F_1 + F_2) > 0$.

Based on Theorem (14) and Theorem (15), we can obtain the following results for NCAVE (1), see Theorem (20) and Theorem (21).

Theorem 20. *Let all diagonal entry of $A_1B_1^{-1} + I$ have the same sign as the corresponding entries of $A_1B_1^{-1} - I$. Then NCAVE (1) has exactly one solution for any d if any one of the following conditions is true:-*
(i) $A_1B_1^{-1} - I$ and $A_1B_1^{-1} + I$ are strictly diagonally dominant by columns;
(ii) $A_1B_1^{-1} - I$, $A_1B_1^{-1} + I$ and all their column representative matrices are irreducibly diagonally dominant by columns.

Theorem 21. *If matrix B_1 is non-singular, then NCAVE (1) has exactly one solution for any d, if the interval matrix $[A_1B_1^{-1} - I, A_1B_1^{-1} + I]$ is regular.*

The approach of Theorem (14) and Theorem (15) can apply to the AVE $A_1x - B_1|C_1x| = d$, related results are skipped here. For more about AVE $A_1x - B_1|C_1x| = d$ one may refer [26].

Now, based on Lemma (8), we get a relation between VLCP and NCAVE, see Lemma (22).

Lemma 22. *The NCAVE (1) is identical to the following VLCP*

$$\begin{aligned} &(A_1 + B_1)x - (d + c) \geq 0, (A_1 - B_1)x - (d - c) \geq 0 \\ &\{(A_1 + B_1)x - (d + c)\}^T.\{(A_1 - B_1)x - (d - c)\} = 0 \end{aligned} \tag{14}$$

Proof. The NCAVE $A_1x - d = |B_1x - c|$ is equal to $a_1 + b_1 = |a_1 - b_1|$, where $a_1 = \frac{(A_1+B_1)x}{2}$ - $\frac{(d+c)}{2}$, $b_1 = \frac{(A_1-B_1)x}{2}$ - $\frac{(d-c)}{2}$.
Then by using Lemma (8), we get $a_1 \geq 0$, $b_1 \geq 0$ and $a_1b_1 = 0$.
So our result holds. □

Based on Lemma (22), we get the following conditions for the unique solution of NCAVE (1), which are also given in [6] for the NGAVE (2) and results remain same for the NCAVE (1).

Theorem 23. *The following assertions are identical:*
(i) For any d, the NCAVE (1) has a unique solution;
(ii) $\{A_1 + B_1, A_1 - B_1\}$ holds the row $\mathcal{W}$-property;
(iii) A_1+B_1 is invertible and $\{I, (A_1-B_1)(A_1+B_1)^{-1}\}$ holds the row $\mathcal{W}$-property.

Proof. By simple observations of Theorem (10) and Lemma (22), our result of Theorem (23) is hold. □

We have the following result based on the Theorem (23) and Lemma (6).

Theorem 24. *Let $A_1 + B_1$ be non-singular. Then the NCAVE (1) has a unique solution if and only if matrix $(A_1 - B_1)(A_1 + B_1)^{-1}$ is a P-matrix.*

We get the following result based on Lemma (7).

Theorem 25. *The NCAVE (1) has exactly one solution if and only if matrix $A_1 + B_1 - 2DB_1$ is non-singular for any D.*

Proof. Since matrix $A_1 + B_1 - 2DB_1$ is non-singular for any D, so $(A_1 + B_1)$ is non-singular.
Now by simple calculations, we have

$$\begin{aligned} & I - D + D[(A_1 - B_1)(A_1 + B_1)^{-1}] \\ = & (I - D)(A_1 + B_1)(A_1 + B_1)^{-1} + D[(A_1 - B_1)(A_1 + B_1)^{-1}] \\ = & [A_1 + B_1 - 2DB_1](A_1 + B_1)^{-1}. \end{aligned}$$

This implies that matrix $I - D + D[(A_1 - B_1)(A_1 + B_1)^{-1}]$ is non-singular, this implies $(A_1 - B_1)(A_1 + B_1)^{-1}$ is a P-matrix. Then by Theorem (24), our result holds. □

Remark 26. *By taking $c = 0$ in Theorem 23, Theorem 24, Theorem 25 and conditions (11), (12), (13) of Theorem 18, we get the main results of the paper of Wu [6].*

Theorem (25) can be written in the following way when B_1 is an invertible matrix.

Theorem 27. *The NCAVE (1) has exactly one solution if and only if matrix $A_1B_1^{-1} + I - 2D$ is non-singular for any D.*

Based on Theorem (27) and Lemma (9), we get the following Theorems (28) and (29), respectively.

Theorem 28. *The NCAVE (1) has unique solution for any d if $\sigma_{min}(A_1B_1^{-1}) > 1$.*

Proof. By Lemma (9), we have
$\sigma_{min}(A_1B_1^{-1} + I - 2D) \geq \sigma_{min}(A_1B_1^{-1}) - \sigma_{max}(I - 2D)$, for any D.
If $\sigma_{min}(A_1B_1^{-1}) > 1$ then $\sigma_{min}(A_1B_1^{-1} + I - 2D) > 0$, this implies $(A_1B_1^{-1} + I - 2D)$ is non-singular, then by Theorem (27) our result is complete. □

Theorem 29. *The NCAVE (1) has unique solution for any d if $\sigma_{max}(B_1) < \sigma_{min}(A_1)$.*

Proof. By Lemma (9), we have
$\sigma_{min}(A_1 + B_1 - 2DB_1) \geq \sigma_{min}(A_1) - \sigma_{max}(B_1 - 2DB_1)$, for any D.
Since $\sigma_{max}(B_1 - 2DB_1) \leq \sigma_{max}(I - 2D)\sigma_{max}(B_1) \leq \sigma_{max}(B_1)$, as $\sigma_{max}(I - 2D) \leq 1$.
Then if $\sigma_{max}(B_1) < \sigma_{min}(A_1)$ holds then $\sigma_{min}(A_1 + B_1 - 2DB_1) > 0$, this implies matrix $A_1 + B_1 - 2DB_1$ is non-singular and by Theorem (25) our result is hold. □

Based on Lemma (7) and Theorem (24), we get the following result.

Theorem 30. *The NCAVE (1) has exactly one solution if and only if $det(A_1 + B_1) \neq 0$ and for any D, matrix $A_1 - B_1 + 2DB_1$ is non-singular.*

Proof. By simple calculations, we have

$$\begin{aligned} &(A_1 - B_1)(A_1 + B_1)^{-1} + D[I - (A_1 - B_1)(A_1 + B_1)^{-1}] \\ &= (A_1 - B_1)(A_1 + B_1)^{-1} + D(A_1 + B_1)(A_1 + B_1)^{-1} - D(A_1 - B_1)(A_1 + B_1)^{-1} \\ &= [A_1 - B_1 + 2DB_1](A_1 + B_1)^{-1}. \end{aligned}$$

This implies that matrix $(A_1 - B_1)(A_1 + B_1)^{-1} + D[I - (A_1 - B_1)(A_1 + B_1)^{-1}]$ is non-singular, so matrix $(A_1 - B_1)(A_1 + B_1)^{-1}$ is a P-matrix. Then by Theorem (24), our result holds. □

When we put $B_1 = I$ in Theorem (25) and Theorem (30), we get following important results for AVE (3).

Corollary 31. *Matrix $A_1 - I + 2D$ is non-singular for any D if and only if AVE $A_1x - |x| = d$ has exactly one solution.*

Corollary 32. *For non-singular matrix A_1, AVE $A_1x - |x| = d$ has exactly one solution if and only if matrix $A_1 + I - 2D$ is non-singular for any D.*

Remark 33. *Corollary (31) and Corollary (32) are the main results of [9] and Theorem (25) and Theorem (30) will become "The basic theorem of the linear system $A_1x = d$ for any d" by taking $B_1 = 0$ and $c = 0$. Further, by taking $c = 0$ in Theorem (17), Theorem (19), Theorem (20), Theorem (21), Theorem (27), Theorem (28), Theorem (29), and Theorem (30), we get the new results for the unique solvability of the NGAVE (2). These results are not covered in the paper of Wu [6].*

4. CONCLUSION

In this paper, we consider a new class of the AVE $A_1x - |B_1x - c| = d$ which is generalized form of the NGAVE $A_1x - |B_1x| = d$ and $A_1x - |x| = d$. Some necessary and sufficient results for a unique solution for NCAVE (1) are obtained. Earlier work in [6] and [9] are generalized for the appropriate choice of B and c. In Theorem(25) and Theorem(30), we got the basic theorem for linear system $A_1x = d$. Future discussions on the numerical solution of the NCAVE look to be interesting.

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