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THE EXACT DEFUZZIFICATION METHOD UNDER POLYNOMIAL APPROXIMATION OF VARIOUS FUZZY SETS

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Abstract: This article deals with the new approach of finding the defuzzification / ranking index of various types of fuzzy sets. Traditionally, in most of the articles on fuzzy decision making the defuzzification methods are not justified with respect to that of highest aspiration levels. This study highlights an efficient defuzzification (ranking) method which links between the gaps on the defuzzified values obtained using $\alpha - cuts$ and without $\alpha - cuts$ of fuzzy numbers. Moreover, for a given problem different membership grades are found by different researchers which are confusing and contradicts the conceptual uniqueness of fuzzy set itself. To resolve these issues, first of all, we have studied a polygonal fuzzy set by means of an interpolating polynomial function. However, in fuzzy set theory we usually seek the highest membership grade for ranking any kind of decision-making problem therefore, maximizing the polynomial function, we get the index value of the proposed fuzzy set. An artificial intelligence (AI) based solution algorithm has also been developed to find the exact defuzzified value. Indeed, considering two numerical examples we have compared these ranking values with some of the existing state of- arts under higher aspiration levels. Finally, some graphical illustrations have also been done to justify the proposed approach.

Keywords: Polynomial fuzzy set, traditional defuzzification methods, new defuzzification algorithm, application.

MSC: 2020 Classification Code: 03B52, 90C23, 90 C70.

1. INTRODUCTION

The concept of fuzzy set was studied first by Zadeh [1] incorporating the uncertainties in the real-world problems. In the definition of fuzzy set, if X be the universal set then we have the membership grade $\mu(x) \in [0,1]$ $for\ x \in X$. However, in fuzzy decision making we seek $x \in X$ such that $\mu(x) \geq \alpha$, where we want $Max\ \alpha \in (0,1)$, means $\mu(x) \to 1$. If $\alpha > 0.5$ then the uncertain system is called strong fuzzy system otherwise it is called weak fuzzy system. Most of the existing defuzzification methods did not satisfy $x \in X$ such that $\mu(x) \to 1$. But using $\alpha -$ level set, few significant values near 1 may appear. Thus, to assure $\mu(x) \to 1$ we should find a unified rule to achieve the same. Unfortunately, in the literature thousands of research papers have been studied by the eminent researchers where the condition $\mu(x) \to 1$ is not maintained. The basic job of this study is to develop an AI based algorithm for maximizing aspiration level of the proposed fuzzy sets and rank them to get the decision of a real-world problems accordingly. It is essential because, the concept of fuzzy deviations of a number x is solely associated with the "around x". More deviation could weaken the concept of fuzzy set which may be untrue in practice. Suppose we are going to fuzzify the objective function z subject to some constraints. Then as per literature survey concern, the problem becomes: $\begin{matrix} Maximize\ \ \alpha \\ Subject\ to\ \mu(\ z) \geq \alpha \end{matrix}$ where the function $\mu(\ z)$ represents the membership function of the fuzzy number $\tilde{z}$. Not a single article has been studied where we see the problem like $\begin{matrix} Minimize\ \ \alpha \\ Subject\ to\ \mu(\ z) \geq \alpha \end{matrix}$. Since we are seeking $Maximize\ \ \alpha$ in the interval $(0,1)$ but not in $[0,1]$(at 0 the number itself represents outside the proposed set and at 1 , the number itself converts in to crisp/ classical set) so the above problem reaches to $\begin{matrix} Maximize\ \ \alpha\ (\to 1) \\ Subject\ to\ \mu(\ z) \geq \alpha\ (\to 1) \end{matrix} \Rightarrow \mu(\ z) \longrightarrow 1 \Rightarrow \mu(\ x) \longrightarrow 1, \forall\ x \in \Re$. We also note that, optimum value obtained from the solution of a classical set has the membership grade 1, but in fuzzy set (sense) solution set having membership grade 1 corresponds a fuzzy number. Throughout several decades researchers did not write any complain/ able to distinguish between the solutions under fuzzy set and the fuzzy numbers (classical set). Rather they served results in hybrid mode (crisp and fuzzy) in the name of fuzzy set exclusively(!) in a more sophisticated way. Therefore, in this study, we are exclusively talking about fuzzy set whose membership value belongs to (0, 1) only without making any confusion.

However, if we wish to go through the publications of contemporary research articles, few notable works may be discussed over here. Using left and right $\alpha - cuts$ of fuzzy number Yager [2] introduced a simplified form of defuzzification method. Runkler and Glesner [3] discussed some axioms on defuzzifying fuzzy sets. Filev and Yager [4] developed a level set based approach on defining new defuzzification methods. In the same year Yager and Filev[5] discovered new technique based on a fuzzy set selection and defuzzification. Hellendoorn and Thomas [6] proposed fuzzy controllers for defuzzification of fuzzy numbers. Klir and Yuan [7] discussed the basic concept of fuzzy logic and its applications in their book. Allahviranloo and Sancifard [8] and Deng [9] invented some defuzzification methods for ranking of fuzzy numbers based on center of gravity and got ideal solutions. Ezzati et *al.* [10] gave a novel approach for raking of fuzzy numbers. Wang et *al.* [11] proposed new technique of deviation degree for ranking L-R fuzzy number. Xu et *al.* [12] studied a note on ranking generalized fuzzy numbers. Zhang et *al.* [13] invented a new method for group decision making of ranking fuzzy numbers and they studied it to real life problems. Buckley and Chanas [14] proposed a fast method of ranking fuzzy alternatives. Cheng [15] discovered a new approach for ranking fuzzy numbers using distance method. Chu and Tsao [16] developed an area using centroid point and original point method for ranking fuzzy numbers. Bortolan and Degani [17] gave some methods for ranking fuzzy subsets. Wierman [18] discussed central values of various fuzzy numbers as their defuzzification. Abbasbandy and Asady [19] developed ranking of fuzzy numbers by using signed distance method. Kim and Park [20] proposed index of optimism for ranking fuzzy numbers. Liou and Wang [21] evaluate integral value for ranking fuzzy numbers. Vincent and Dat [22] improved ranking method for fuzzy numbers with integral values. Chen and Tang [23] developed ranking non-normal p-norm for trapezoidal fuzzy numbers with integral value. Abbasbandy and Hajjari [24] constructed a new approach for ranking of trapezoidal fuzzy numbers. Chutia et *al.* [25] proposed a new method for ranking parametric form of fuzzy numbers using value and ambiguity.

Shahsavari et *al.* [26] invented a novel method for ranking fuzzy numbers based on different areas of fuzzy number. Fuzzy critical path method based on ranking methods using hexagonal fuzzy numbers for decision making was developed by Samayan and Sengottaiyan [27]. For decision making problems Fahmi et *al.* [28] discovered expected values of aggregation operators on cubic trapezoidal fuzzy number and its application. Menaka [29] developed ranking of octagonal intuitionistic fuzzy numbers. Moreover, the two dimensional (2D) fuzzy numbers, based on learning theory was wisely invented by De and Mahata [30] described a new defuzzification method for cloudy and dense fuzzy sets. In its continuations De and Beg [31] discussed the novel defuzzification methods for triangular dense fuzzy sets. In the near past, De [32] developed a new defuzzification technique for the triangular dense fuzzy lock sets. De [33] introduced degree of fuzziness and fuzzy decision making. Bellman and Zadeh [34] gave a new method for decision making with fuzzy environment. Verdegay [35] worked on fuzzy mathematical programming. De et *al.* [36] invented first the concept of doubt fuzzy set and the corresponding ranking value for its defuzzification. Ontiveros-Robles et *al.* [37-38] discussed on α-plane aggregation/ integration rules for Type-2 fuzzy systems with the help of Newton–Cotes formula. Ontiveros-Robles et *al.* [39] discussed a new methodology based on a continuous root-finding karnik mendel algorithm. Gazi et *al.* [40] invented new ranking rules for decision making in restaurant locations.Pre-diagnosis of disease has been analysed with the help of generalized dual hesitant hexagonal fuzzy set [41]. Decision making with cloud service providers via intuitionistic fuzzy uncertainty has also been discussed [42]. A novel deteriorating fuzzy Marxian supply chain model has been solved which keeps a milestone in the fuzzy domain [43]. Use of reliability theory in the optimization problem has been analysed followed by hybrid genetic and particle swarm optimization algorithm [44]. In developing optimization problem some other researchers have used Euler's algorithm [45]. Optimization with saddle point problem has also been studied by Laha et *al*. [46]. A set valued optimization problem with global stability and their existence has been developed [47]. Although, fuzzy linear and non-linear decision making problems have been solved by several renowned researchers using the maximization of aspiration levels (Nasseri and Bavandi [48], Delgado and Verdegay [49], Safi et al [50], Werners[51] etc.) Moreover, to get a clear concept on the traditional defuzzification methods and our proposed new method we may follow the Table 1(a) given below.

Table 1(a): Major literature review on the various defuzzification methods

Author(s) with year	Methods	Triangular $\langle a, m, b\rangle$	Trapezoidal $\langle a, l, r, b\rangle$
Yager R. R. (1981)	Area	$(a+2m+b)/4$	$(a+2(l+r)+b)/6$
Wierman (1997)	Mass	$\frac{b-a}{2}$	$\frac{b+r-l-a}{2}$
	COG	$\frac{a+m+b}{3}$	$\frac{b^2+rb+r^2-l^2-al-a^2}{3(b+r-l-a)}$
	MOG	$\max\left(m, \frac{b}{2}\right)$ $\alpha' = \frac{1}{3}$	$\max\left(r, \frac{b}{2}\right)$ $\alpha' = \frac{2r+b-2l-a}{3b+3r-3l-3a}$
	EAM	$\nu = \frac{a+m+b}{3}$	$\nu = \frac{1}{2}[\alpha'(l-a)+a+\alpha'(b-r)+b]$
	MOM	m	$\frac{l+r}{2}$
	Mode	m	$[l, r]$
Abbasbandy and Asady (2006)]	SD	$\frac{a+m+b}{3}$	$\frac{1}{4}(a+l+r+b)$
De and Mahata (2016)	Den	$\langle a\left(1-\frac{\rho}{1+n}\right), a, \left(1+\frac{\sigma}{1+n}\right)\rangle$	$\langle a\left(1-\frac{\rho}{1+n}\right), a, b, b\left(1+\frac{\sigma}{1+n}\right)\rangle$
De and Beg (2018)	Cld	$\langle a\left(1-\frac{\rho}{1+t}\right), a, \left(1+\frac{\sigma}{1+t}\right)\rangle$	$\langle a\left(1-\frac{\rho}{1+t}\right), a, b, b\left(1+\frac{\sigma}{1+t}\right)\rangle$
De (2020)	CM	$a\sqrt{\left[1+\frac{\sigma-\rho}{3}\right]^2+\frac{1}{9a^2}}$	$a\sqrt{\left[1+\frac{(\sigma_2-\sigma_1)(\sigma_2+2\sigma_1)-(\rho_2-\rho_1)(\rho_2+2\rho_1)}{3(\rho_2+\sigma_2-\rho_1-\sigma_1)}\right]^2+\frac{1}{9a^2}}$
	DOF	$\langle a\left(\frac{2}{\pi}\right)^{\frac{1}{k+1}}\rangle$ or $\langle a\left(2-\frac{2}{\pi}\right)^{\frac{1}{k+1}}\rangle$	$\langle\left[2-\frac{2}{\pi}\left(1+\frac{\rho_1+\sigma_1}{\rho_2+\sigma_2}\right)\right]^{\frac{1}{k+1}}\rangle$ or $\langle a\left[\frac{2}{\pi}\left(1+\frac{\rho_1+\sigma_1}{\rho_2+\sigma_2}\right)\right]^{\frac{1}{k+1}}\rangle$
This article	PA	$\mu(x) = Ax^2+Bx+C$, The positive root of $\mu'(x)=0$ is the ranking value of the fuzzy set $\widetilde{A}$	$\mu(x) = Ax^3+Bx^2+Cx+D$ The positive root of $\mu'(x)=0$ which gives maximum value of $\mu(x)$ is the ranking value of the fuzzy set $\widetilde{A}$.

Note1: COG: Center of gravity, MOG: Max of gravity, EAM: Expected alpha mean, MOM: Mean of maxima, SD: Signed Distance, Den: Dense, Cld: Cloudy, CM: Center of mass, DOF: Degree of fuzziness, PA: Polynomial approximation

However, we may list some authors who have used maximizing level α in their decision-making problems (shown in Table 1(b)).

Table 1(b): Some major literature study over aspiration level α in optimization problems

Method of Maximizing aspiration level α	
Author(s) with year	Area of Application
Vardegays (1982) , Zimmermann(1996) etc.	Linear and non-linear programming problem
Bellman and Zadeh (1970), Shams et *al*.(2012)	Linear programming problem (LPP)

From the above study it is seen that none of the researchers have studied the fuzzy set by means of polynomial functions to describe the ranking rules. Therefore, this article develops a new defuzzification / ranking method where all polynomial functions have been optimized. Moreover, this article organizes as follows. Section 2 includes preliminaries about fuzzy sets and polynomial approximations, polynomial function over triangular and trapezoidal fuzzy set. Section 3 develops new generalized defuzzification rules using algorithm and flow chart and measure of degree of fuzziness. section 4 studied numerical illustrations with two examples, section 5 discusses graphical illustrations and finally section 6 gives a concluding remark, limitation and scope of future work.

2. PRELIMINARIES

In this section, we discuss some essential definitions and results to develop the new approach.

2.1. *Definition 1: Fuzzy Set ([1]) Let X be the universal set. Then a fuzzy set on X is given by* $\widetilde{A} =< x, \mu(x) >$ *for all* $x \in X$ *and* $\mu : X \to [0,1]$. *For normal fuzzy set (fuzzy number) we have* $\mu(x) = 1$ *and if* $\mu(x) = 0$ *then x does not belong to fuzzy set. Thus, in true sense we always have* $\mu \in (0,1)$

Definition 2: Support of a Fuzzy Set. Let $\widetilde{A} =< x, \mu(x) >$ *be the fuzzy set defined on a universal set* X. *Then the support of* $\widetilde{A}$ *is given by* $S = \{x: \mu(x) > 0, x \in X\}$. *Alternatively, according to the notion of scaffolding, the lower range or the minimum support or "support" is defined by* $\epsilon = \{Min\ \mu(x) > 0 : \forall\ x \in X\}$

2.2. Lagrange Interpolating Polynomial Function

Let we have the data set (x_i, y_i) for $i = 0,1,2, \cdots, m$ obtained at an observation. Now to get a best fitted function $y = \eta(x)$ that satisfies more closeness to the values $y_0, y_1, \cdots, y_m$ at the points $x_0, x_1, \cdots, x_m$ respectively. We take $\omega(x)$ as the Lagrange interpolating polynomial. Since there are $(m + 1)$ data points (x_i, y_i), we can represent the function $\omega(x)$ by a polynomial of degree m and it is given by

$$\omega(x) = \delta_m x^m + \delta_{m-1} x^{m-1} + \cdots + \delta_1 x + \delta_0$$

where $\delta_0, \delta_1, \cdots, \delta_n$ are constants. Here we assume $\eta(x_i) = y_i$, $i = 0,1,2, \cdots, m$ and the function $\omega(x)$ passes through (x_i, y_i), representing a compact form as:

$\omega(x) = \sum_{j=0}^{m} V_i(x)\eta(x_i)$ where $V_k(x) = \prod_{j=0, k\neq j}^{m} \frac{x - x_j}{x_k - x_j}$ is called the Lagrangian polynomial, satisfying the condition $V_k(x) = \delta_{kj}$, the Kronecker delta.

2.3. Construction of Polynomial Functions on Triangular and Trapezoidal Fuzzy Set

a) Let us consider a triangular fuzzy number $\widetilde{A_1} =< a_1, a_2, a_3 >$ whose membership function is given by

$$\mu(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & for\ \ a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2} & for\ \ a_2 \le x \le a_3 \\ 0 & otherwise \end{cases} \quad (1)$$

and the graphical representation is shown in Figure 1(a). Now consider the fuzzy set (1) into the convex set whose vertices have the co-ordinates $A(a_1, \epsilon)$, $B(a_2, 1)$ and $C(a_3, \epsilon)$ with support ϵ respectively (shown in Figure 1(b)). Now utilizing the formula of Lagrange Interpolating polynomial function (subsection 2.2) we construct a polynomial approximation of (1) through the points A, B and C and get (2)

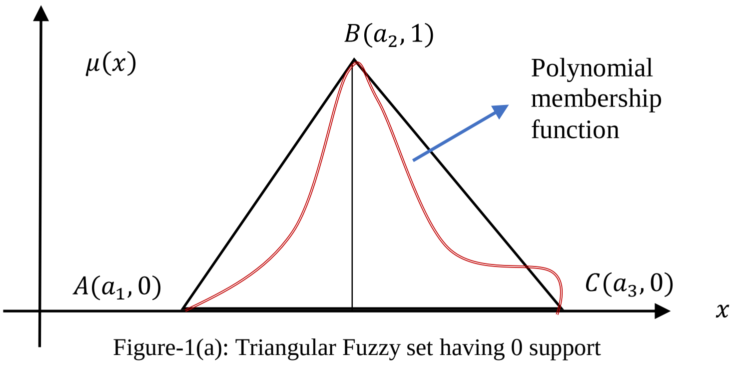

Figure-1(a): Triangular Fuzzy set having 0 support

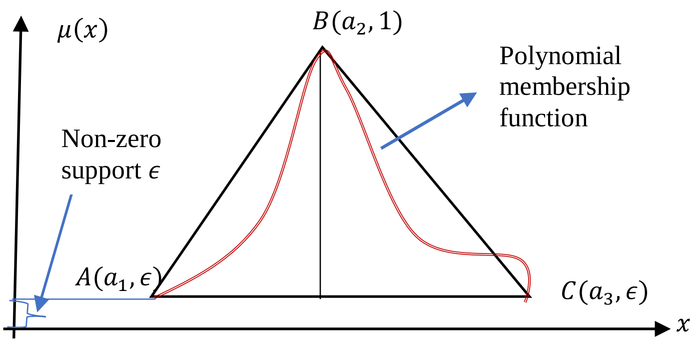

Figure-1(b): Triangular Fuzzy set having support ϵ

$$u(x) = -Ax^2 + Bx - C \quad (2)$$

where

$$\begin{cases} A = \left[\frac{\epsilon}{(a_2-a_1)(a_3-a_1)} - \frac{1}{(a_2-a_1)(a_3-a_2)} + \frac{\epsilon}{(a_3-a_1)(a_3-a_2)}\right] \\ B = \left[\frac{(a_1+a_3)}{(a_2-a_1)(a_3-a_2)} - \frac{(a_2+a_3)\epsilon}{(a_2-a_1)(a_3-a_1)} - \frac{(a_1+a_2)\epsilon}{(a_3-a_1)(a_3-a_2)}\right] \\ \text{and } C = \frac{a_1 a_3}{(a_2-a_1)(a_3-a_2)} - \frac{a_2 a_3 \epsilon}{(a_2-a_1)(a_3-a_1)} - \frac{a_1 a_2 \epsilon}{(a_3-a_1)(a_3-a_2)} \end{cases} \quad (3)$$

b) Let us consider a Trapezoidal fuzzy number $\widetilde{A_2} = \langle a_1, a_2, a_3, a_4 \rangle$ whose membership function is given by

$$\gamma(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & for\ a_1 \le x \le a_2 \\ 1 & for\ a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3} & for\ a_3 \le x \le a_4 \\ 0 & otherwise \end{cases} \quad (4)$$

And it is shown in Figure 2(a). Now consider (4) as a convex fuzzy set whose vertices are $P(a_1, \epsilon), Q(a_2, 1), R(a_3, 1)$ and $S(a_4, \epsilon)$ with support ϵ respectively(shown in Figure 2(b)

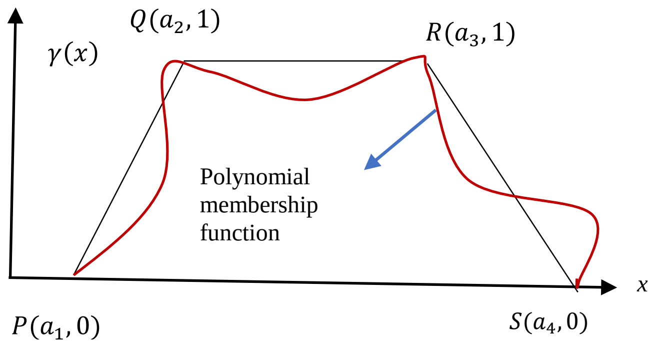

Figure 2(a): Trapezoidal fuzzy number having 0 support

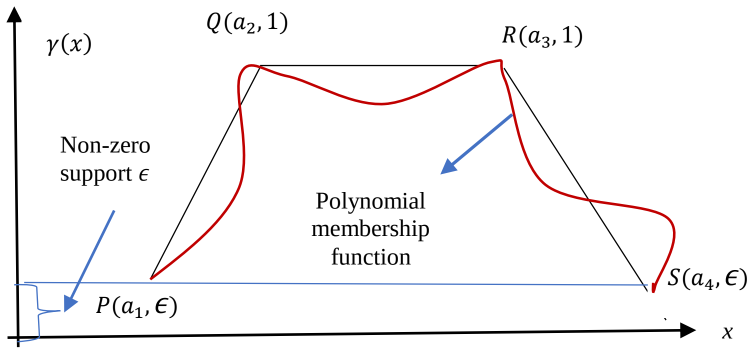

Figure 2(b): Trapezoidal fuzzy number having support ϵ

Thus, using subsection 2.2, the interpolating polynomial function can be obtained as

$$v(x) = Dx^3 - Ex^2 + Fx - G \tag{5}$$

where

$$\begin{cases} D = \dfrac{[a_1^2 - a_4^2 + (a_2+a_3)(a_4-a_1)]\epsilon}{(a_2-a_1)(a_4-a_1)(a_3-a_1)(a_4-a_3)(a_4-a_2)} \\ \quad + \dfrac{[a_2^2 - a_3^2 + (a_1+a_4)(a_3-a_2)]}{(a_2-a_1)(a_4-a_2)(a_3-a_1)(a_4-a_3)(a_3-a_2)} \\ E = \dfrac{[a_1^3 - a_4^3 + a_1^2 a_2 - a_1 a_3^2 + a_2^2 a_4 + a_3^2 a_4 + a_2 a_3 (a_4-a_1)]\epsilon}{(a_2-a_1)(a_4-a_1)(a_3-a_1)(a_4-a_3)(a_4-a_2)} \\ \quad + \dfrac{[a_2^3 - a_3^3 + a_1^2 a_3 + a_3 a_4^2 - a_2 a_4^2 - a_1^2 a_2 + a_1 a_4 (a_3-a_2)]}{(a_2-a_1)(a_4-a_2)(a_3-a_1)(a_4-a_3)(a_3-a_2)} \\ F = \dfrac{[a_2^2 a_4^2 + a_3^2 a_4^3 - a_1^2 a_3^2 - a_1^2 a_2^2 + a_1^3 a_2 + a_1^3 a_3 - a_2 a_4^3 - a_3 a_4^3 + a_2 a_3 (a_4^2 - a_1^2)]\epsilon}{(a_2-a_1)(a_4-a_1)(a_3-a_1)(a_4-a_3)(a_4-a_2)} \\ \quad + \dfrac{[a_3^2 a_4^2 + a_1^2 a_3^2 - a_2^2 a_4^2 - a_1^2 a_2^2 + a_1 a_2^3 - a_1 a_3^3 - a_3^3 a_4 + a_4 a_2^3 + a_1 a_4 (a_3^2 - a_2^2)]}{(a_2-a_1)(a_4-a_2)(a_3-a_1)(a_4-a_3)(a_3-a_2)} \\ G = \dfrac{[a_1 a_2^2 a_3^2 + a_3 a_2^2 a_4^3 + a_2 a_3^2 a_4^2 + a_2 a_1^3 a_3 - a_2 a_1^2 a_3^2 - a_2^2 a_1^3 a_3 - a_1^2 a_3^2 a_4 - a_2 a_3 a_4^3]\epsilon}{(a_2-a_1)(a_4-a_1)(a_3-a_1)(a_4-a_3)(a_4-a_2)} \\ \quad + \dfrac{[a_1 a_4^2 a_3^2 + a_4 a_1^2 a_3^2 + a_2 a_1^2 a_4^2 + a_1 a_2^3 a_4 - a_3 a_1^2 a_4^2 - a_2^2 a_4^2 a_1 - a_1^2 a_2^2 a_4 - a_1 a_4 a_3^3]}{(a_2-a_1)(a_4-a_2)(a_3-a_1)(a_4-a_3)(a_3-a_2)} \end{cases}$$

After taking limit $\epsilon \to 0$ we get

$$\text{where } \begin{cases} D = \dfrac{1}{(a_2-a_1)(a_3-a_2)(a_4-a_2)} - \dfrac{1}{(a_3-a_1)(a_3-a_2)(a_4-a_3)} \\ E = \dfrac{(a_2^2 + a_3^2 - a_1^2 - a_4^2 - a_1 a_4 + a_2 a_3)}{(a_2-a_1)(a_3-a_1)(a_4-a_2)(a_4-a_3)} \\ F = \dfrac{(a_3+a_2)(a_1^2 + a_1 a_4 + a_4^2) - (a_1+a_4)(a_3^2 + a_2 a_3 + a_3^2)}{(a_2-a_1)(a_3-a_1)(a_4-a_2)(a_4-a_3)} \\ G = \dfrac{a_1 a_3 a_4}{(a_2-a_1)(a_3-a_2)(a_4-a_2)} - \dfrac{a_1 a_2 a_4}{(a_3-a_1)(a_3-a_2)(a_4-a_3)} \end{cases}$$

3. GENERALIZED RULE OF FINDING NEW DEFUZZIFICATION METHOD

Let the (n-1) -th order polynomial function be the membership function of the n-tuple fuzzy number $\widetilde{A} =< a_1, a_2, a_3, \dots \dots \dots a_n >$ can be represented as

$$\mu(x) = c_0 x^{n-1} + c_1 x^{n-2} + c_2 x^{n-2} + \cdots . c_{n-2} x + c_{n-1} \quad (6)$$

Now, the first order derivative of (6) gives

$$\mu'^{(x)} = (n-1)c_0 x^{n-2} + (n-2)c_1 x^{n-3} + (n-3)c_2 x^{n-4} + \cdots . + c_{n-2} \quad (7)$$

We also consider that all the roots of (7) are real. Otherwise, we discard the complex roots and only real roots are to be taken into considerations.

Then, for optimality the equation (7) can be replaced as

$$(x - \sigma_1)(x - \sigma_2)(x - \sigma_3) \dots \dots \dots \dots \dots . (x - \sigma_{n-2}) = 0 \quad (8)$$

having the roots $\sigma_1, \sigma_2, \sigma_3, \dots \dots \dots . ., \sigma_{n-2}$

We also let for some of the roots σ_j the values of the $\mu''(\sigma_j) < 0$ for $1 \le j \le n - 2$ and we define the solution set $B = \{\sigma_1, \sigma_2, \sigma_3, \dots \sigma_{j-2}, \sigma_{j-1}, \sigma_j\}$ say where the roots may or may not be in the hierarchical order. Therefore, the ranking value of the given fuzzy set $\widetilde{A} =< a_1, a_2, a_3, \dots \dots \dots a_n >$ is denoted by

$$\sigma_r = \left\{\sigma_k : 0 < \underset{\sigma_{k \in B}}{\text{Max}} \mu(\sigma_k) < 1, \ for\ 1 \le k \le j\right\} \quad (9)$$

Note2: The polynomial function $\mu(x)$ is a function of bounded variation.

Proof: *We know that the function $\mu(x)$ is bounded in [0, 1] and from the construction of membership function through the polynomial approximation it is also monotone. Then from the subject of real analysis the membership function, being a monotone function is a function of bounded variation. Let $[a, b]$ be the support of the fuzzy set $\mu(x)$ and we consider a partition $P = \{a < \sigma_1, \sigma_2, \sigma_3, \dots \dots \dots . . \sigma_{n-1} < b\}$ for the fuzzy set $\widetilde{A} =< a_1, a_2, a_3, \dots \dots \dots a_n >$ which are different from $\{a_1, a_2, a_3, \dots \dots \dots a_n\}$. Again, we assume that $c \in P$ such that, the function $\mu(x)$ is monotonic increasing in $[a, c]$ and decreasing in $[c, b]$. Thus, the total variation of $\mu(x)$ is given by*

$$V(\mu, a, b) = V(\mu, a, c) + V(\mu, c, b) \quad (10)$$

Where $V(\mu, a, c) = \sum_{i=1}^{l} |\mu(\sigma_i) - \mu(\sigma_{i-1})|$ and $V(\mu, c, b) = \sum_{i=l+1}^{m} |\mu(\sigma_i) - \mu(\sigma_{i-1})|$

With the partitions $P = P_1 \cup P_2$ for $P_1 = \{a < \sigma_1, \sigma_2, \dots . \sigma_l = c\}$

and $P_2 = \{c = \sigma_{l+1}, \sigma_{l+2}, \dots . \sigma_m < b\}$ respectively. Now we recall the total variation (10) as the total *degree of fuzziness* of the fuzzy set $\mu(x)$.

3.1. Algorithm for finding ranking index of a fuzzy set

Step 1 : Solve the crisp problem and set it as x_*.
Step 2: Set *n* =3.

Step 3: Construct a (n-1)th degree polynomial function $\mu(x)$ using Lagrange method over the n-gonal fuzzy set $\tilde{A} = \langle a_1, a_2, \cdots, a_n \rangle$ whose component wise membership grade points are $(w_1, w_2, \cdots, w_n)$ respectively.

Step 4 : Find the roots of the polynomial function $\mu'(x) = 0$. Let the roots of $\mu'(x) = 0$ be $\sigma_1, \sigma_2, \cdots, \sigma_{n-2}$.

Step 5 : Compute $\mu(x)|_{x=\sigma_i}, i = 1,2, \cdots, (n-2)$ and set $\mu_i = \{\mu(x)|_{x=\sigma_i}, i = 1,2, \cdots, (n-2)\}$.

Step 6 : Assign $\mu(x^*) = \underset{0 < \mu(\sigma_i) \neq 1}{Max} \{\mu(\sigma_i), i = 1,2, \cdots, (n-2)\}$.

Step 7 : Check whether $x^* < x_*$ for cost variable x , otherwise $x^* > x_*$ for profit variable x .

Step 8: Find the optimum defuzzified value $x^*$of the fuzzy set $\tilde{A}$ with highest aspiration level $\mu(x^*)$. Go to Step 10.

Step 9: $n = n$+1. Go to Step 2.

Step 10 : Stop

3.2. Flow-chart of the proposed Algorithm

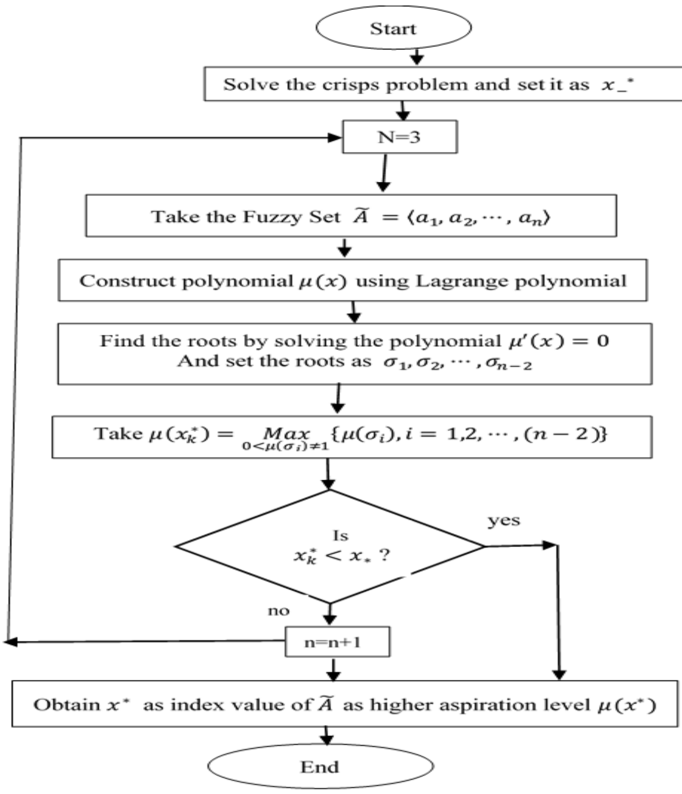

Figure 3: Flow Chart of new defuzzification method

4. NUMERICAL ILLUSTRATION

In this section we shall illustrate two examples that show the novelty of the proposed study.

4.1. *Numerical Example 1:* Let us assume the triangular fuzzy number $\langle \boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3} \rangle = \langle \mathbf{10}, \mathbf{15}, \mathbf{18} \rangle$ and that for trapezoidal fuzzy number we assume $\langle \boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{a_4} \rangle = \langle \mathbf{10}, \mathbf{15}, \mathbf{17}, \mathbf{18} \rangle$. The defuzzified value of the proposed fuzzy numbers under some methods are given in Table 2.

Table 2: Ranking index under various defuzzification methods

Fuzzy number	Triangular $\langle a,m,b\rangle$		Trapezoidal $\langle a,l,r,b\rangle$	
Methods	Ranking value	Aspiration level	Ranking value	Aspiration level
Area	14.5	0.9	15.33	1#
Mass	$\frac{b-a}{2} = 4$	0#	$\frac{b+r-l-a}{2} = 5$	0#
COG	$\frac{a+m+b}{3} = \frac{43}{3} \approx 14.33$	0.866	$\frac{b^2+rb+r^2-l^2-al-a^2}{3(b+r-l-a)} = \frac{74}{5} = 14.8$	0.96
MOG	$\max\left(m, \frac{b}{2}\right) = 15$	1#	$\max\left(r, \frac{b}{2}\right) = \max(17,9) = 17$	1#
EAM	$\alpha' = \frac{1}{3}$, $\nu = \frac{a+m+b}{3} \approx 14.33$	0.866	$\alpha' = \frac{2r+b-2l-a}{3b+3r-3l-3a} = \frac{2}{5}$, $\nu = \frac{1}{2}[\alpha'(l-a)+a+\alpha'(b-r)+b] = 15.2$	1#
MOM	$m = 15$	1#	$\frac{l+r}{2} = 16$	1#
MODE	$m = 15$	1#	$[l,r] = [15,17]$	1#
C M	15.5036564	0.832	17.7697934	0.770
Sig	14.5	0.90	15	1#
Dense fuzzy (2D)	14.8125 (N=1) 14.859(N=2) 14.8854167(N=3)	0.9625 0.9718 0.9771	14.075(N=1) 14.11875(N=2) 14.140556(N=3)	0.815 0.82375 0.8281
Cloudy fuzzy (2D)	14.9805, T=10	0.9961	14.2317756, T=10	0.8464
Degree of fuzziness (2D)	14.3961, (k=10, cost) 15.4290, (k=10, profit)	0.8792 0.857	13.7156 (k=10, cost) 14.2362 (k=10, profit)	0.74312 0.8472
Polynomial Approximation	14.625	0.925	17.01	0.99

Note 3: 0# and 1# indicates the non-belongingness in the fuzzy set and belongingness in the classical set respectively. However, the interpolating polynomial with respect to Triangular fuzzy number having components $a_1 = 10, a_2 = 15, a_3 = 18, \epsilon = 0.1$ is

$$f(x) = -\frac{7.2}{120}x^2 + \frac{210.6}{120}x - \frac{1284}{120} \tag{11}$$

and the interpolating polynomial with respect to the Trapezoidal fuzzy number having the components $a_1 = 10, a_2 = 15, a_3 = 17, a_4 = 18$ is given by

$$g(x) = -\frac{8}{210}x^3 + \frac{330}{210}x^2 - \frac{4408}{210}x + \frac{19080}{210} \quad (12)$$

Table 2 reveals the defuzzified values of the proposed numerical examples of the triangular fuzzy number < 10,15, 18 > and the Trapezoidal fuzzy number < 10, 15, 17, 18 >. The table values shows that they are not unique but exclusively depends upon the uses of formula. Moreover, it is observed that, the membership grade or the aspiration levels for these defuzzified values did not guarantee the aspiration levels to become a maximum. The methods based on two dimensional fuzzy set (2D) give significant values of the aspiration levels ($\alpha - level$). But if we wish to talk about the single valued membership function then we see the maximum aspiration level 0.90 comes from traditional method, namely from the signed distance method but the present study gives the highest aspiration level 0.925 among all the methods associated with the single valued membership function.Thus, studying with polynomial approximations we always keep the guarantee that aspiration levels are always become maximum.

4.2. *Numerical Example 2:* Let the objective function to be optimized is

$$Min\, z = ax + \frac{b}{x} \;\; for\; x \in \mathbb{R}^+ \quad (13)$$

such that a and b are triangular fuzzy numbers. We assume $\tilde{a} =< a_1,\ a_2, a_3 >=< 10, 15, 18 >$ and

$\widetilde{b} =< b_1,\ b_2, b_3 > = < 5, 8, 12 >$. Then we define

$$\tilde{z} =< z_1,\ z_2, z_3 >=< 10x + \frac{5}{x}, 15x + \frac{8}{x}, 18x + \frac{12}{x} > \quad (14)$$

Such that, the membership function
$$\mu_{\tilde{z}}(z) = \begin{cases} \frac{z - z_1}{z_2 - z_1} & for\;\; z_1 \le z \le z_2 \\ \frac{z_3 - z}{z_3 - z_2} & for\; z_2 \le z \le z_3 \\ 0 & elsewhere \end{cases} \quad (15)$$

then using $\alpha - cuts$ we have
$$\begin{cases} z \ge z_1 + (z_2 - z_1)\alpha \\ and\;\; z \le z_3 - (z_3 - z_2)\alpha \end{cases} \quad (16)$$

Now, utilizing signed distance method in (14) we write

$$I(\widetilde{z}) = \frac{1}{2}\int_0^1 \{z_1 + (z_2 - z_1)\alpha + z_3 - (z_3 - z_2)\alpha\}\, d\alpha = \frac{z_1 + 2z_2 + z_3}{4} = 14.5x + \frac{8.25}{x} \quad (17)$$

However, as per our numerical study, utilizing ranking rule developed at subsection 2.3, the required polynomial function (case of triangular fuzzy number) with support ϵ is given by

$$u(z) = -Az^2 + Bz - C \quad (18)$$

$$\text{where}\begin{cases} A = \left[\frac{1}{(z_2-z_1)(z_3-z_2)} - \frac{\epsilon}{(z_2-z_1)(z_3-z_1)} - \frac{\epsilon}{(z_3-z_1)(z_3-z_2)}\right] \\ B = \left[\frac{(z_1+z_3)}{(z_2-z_1)(z_3-z_2)} - \frac{(z_2+z_3)\epsilon}{(z_2-z_1)(z_3-z_1)} - \frac{(z_1+z_2)\epsilon}{(z_3-z_1)(z_3-z_2)}\right] \\ C = \left[\frac{z_1 z_3}{(z_2-z_1)(z_3-z_2)} - \frac{z_2 z_3 \epsilon}{(z_2-z_1)(z_3-z_1)} - \frac{z_1 z_2 \epsilon}{(z_3-z_1)(z_3-z_2)}\right] \end{cases} \tag{19}$$

Now, if we wish to consider trapezoidal fuzzy number, then assume $\widetilde{a} =< a_1,\ a_2, a_3, a_4 >=< 10, 15, 18, 20 >$ and $\widetilde{b} =< b_1,\ b_2, b_3, b_4 > = < 5, 8, 12, 15 >$. Then we define $\widetilde{z} =< z_1,\ z_2, z_3, z_4 >=< 10x + \frac{5}{x}\ , 15x + \frac{8}{x}\ , 18x + \frac{12}{x}, 20x + \frac{15}{x} >$ having membership function

$$\mu_{\widetilde{z}}(z) = \begin{cases} \frac{z-z_1}{z_2-z_1} & for\ \ z_1 \le z \le z_2 \\ 1 & for\ \ z_2 \le z \le z_3 \\ \frac{z_4-z}{z_4-z_3} & for\ \ z_3 \le z \le z_4 \\ 0 & elsewhere \end{cases} \tag{20}$$

Now the required polynomial function with support ϵ is given by

$$Q(z) = Dz^3 - Ez^2 + Fz - G \tag{21}$$

where

$$\begin{cases} D = \frac{[z_1^2 - z_4^2 + (z_2+z_3)(z_4-z_1)]\epsilon}{(z_2-z_1)(z_4-z_1)(z_3-z_1)(z_4-z_3)(z_4-z_2)} + \frac{[z_2^2 - z_3^2 + (z_1+z_4)(z_3-z_2)]}{(z_2-z_1)(z_4-z_2)(z_3-z_1)(z_4-z_3)(z_3-z_2)} \\ E = \frac{[z_1^3 - z_4^3 + z_1^2 z_2 - z_1 z_3^2 + z_2^2 z_4 + z_3^2 z_4 + z_2 z_3 (z_4-z_1)]\epsilon}{(z_2-z_1)(z_4-z_1)(z_3-z_1)(z_4-z_3)(z_4-z_2)} + \frac{[z_2^3 - z_3^3 + z_1^2 z_3 + z_3 z_4^2 - z_2 z_4^2 - z_1^2 z_2 + z_1 z_4 (z_3-z_2)]}{(z_2-z_1)(z_4-z_2)(z_3-z_1)(z_4-z_3)(z_3-z_2)} \\ F = \frac{[z_2^2 z_4^2 + z_3^2 z_4^3 - z_1^2 z_3^2 - z_1^2 z_2^2 + z_1^3 z_2 + z_1^3 z_3 - z_2 z_4^3 - z_3 z_4^3 + z_2 z_3 (z_4^2 - z_1^2)]\epsilon}{(z_2-z_1)(z_4-z_1)(z_3-z_1)(z_4-z_3)(z_4-z_2)} \\ \quad + \frac{[z_3^2 z_4^2 + z_1^2 z_3^2 - z_2^2 z_4^2 - z_1^2 z_2^2 + z_1 z_2^3 - z_1 z_3^3 - z_3^3 z_4 + z_4 z_2^3 + z_1 z_4 (z_3^2 - z_2^2)]}{(z_2-z_1)(z_4-z_2)(z_3-z_1)(z_4-z_3)(z_3-z_2)} \\ G = \frac{[z_1 z_2^2 z_3^2 + z_3 z_2^2 z_4^3 + z_2 z_3^2 z_4^2 + z_2 z_1^3 z_3 - z_2 z_1^2 z_3^2 - z_2^2 z_1^3 z_3 - z_1^2 z_3^2 z_4 - z_2 z_3 z_4^3]\epsilon}{(z_2-z_1)(z_4-z_1)(z_3-z_1)(z_4-z_3)(z_4-z_2)} \\ \quad + \frac{[z_1 z_4^2 z_3^2 + z_4 z_1^2 z_3^2 + z_2 z_1^2 z_4^2 + z_1 z_2^3 z_4 - z_3 z_1^2 z_4^2 - z_2^2 z_4^2 z_1 - z_1^2 z_2^2 z_4 - z_1 z_4 z_3^3]}{(z_2-z_1)(z_4-z_2)(z_3-z_1)(z_4-z_3)(z_3-z_2)} \end{cases} \tag{22}$$

Thus, utilizing the solution algorithm (subsection 3.1) the obtained optimal ranking values can be put in Table 3 given below.

Table 3: Optimal Result and its comparative Analysis

Methods	Fuzzy set used	x^*	z^*	α^*	Problem
Yager [2], Wierman [18] etc.	Triangular	0.754298	21.87464	1#	$Min\ z = 14.5x + \frac{8.25}{x}$
	Trapezoidal	0.7874992	25.39685	1#	$Min\ z = 16.125x + \frac{10}{x}$
Bellman and Zadeh [34], Delgado et al. [49], Werners [51] etc.	Triangular	0.4659892	24.15762	1#	$Max\ \alpha$ Subject to $z_1 + (z_2 - z_1)\alpha \leq z \leq z_3 - (z_3 - z_2)\alpha$ with condition (14)
	Trapezoidal	0.7626600	21.92950	1#	$Max\ \alpha$ Subject to $z_1 + (z_2 - z_1)\alpha \leq z \leq z_4 - (z_4 - z_3)\alpha$ with condition (20)
Present Research	Triangular	0.794200	21.34001	0.99822	$Max\ \alpha = P(z)$ where $P(z) = -Az^2 + Bz - C$ and the relations (19)
	Trapezoidal	0.9251922	68.41431	0.97793	$Max\ \alpha = P(z)$ where $P(z) = Dz^3 - Ez^2 + Fz - G$ and the relations (22)

Note 4: 1# indicates the case of classical set. The crisp optimal of $Min\ z = 15x + \frac{8}{x}$ is $x_* = 0.7302967$ and $z_* = 21.9089$

Table 3 discusses the optimal solution of a nonlinear programming problem (Example 2) under various methods that shows the novelty of our proposed approach. It is seen that not a single method has been able to solve the problem whose optimum membership grade assumes value other than 1 except the proposed method. This outputted result also reveals that the choice of better optimality depends on the choice of the degree of the polynomial function. Here the polynomial function over triangular fuzzy number gives the better optimum (21.34001) than that of trapezoidal (68.41431) one with respect to the highest aspiration levels 0.9982201 and 0.9779291 respectively.

5. GRAPHICAL ILLUSTRATION

Here we shall draw various graphs for the numerical out puts obtained from various methods due to Triangular fuzzy number using Table 2.

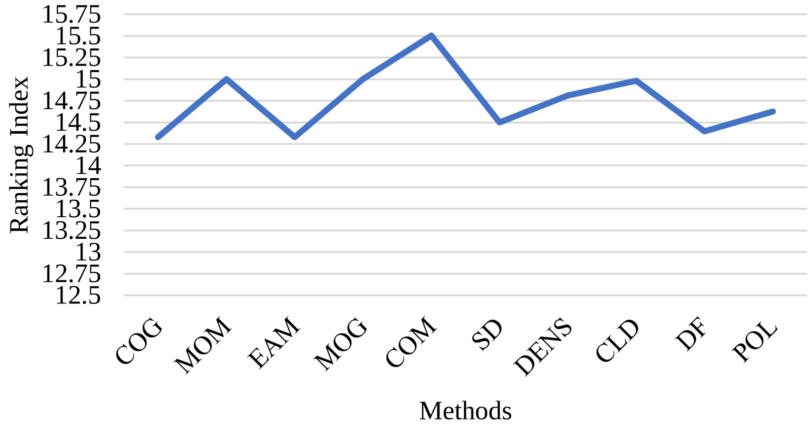

Figure 4: Defuzzified values of Triangular Fuzzy number under various Methods

Figure 4 shows the line curve for the defuzzified values (DV) of a given fuzzy number under various methods. It is seen that, DV gets maximum value (15.5) for the MOG method but it gets minimum value (around 14.6) for the proposed Polynomial approximation method. In cloudy fuzzy defuzzification method and MOM method give the same value (around 15) of the triangular fuzzy number. The other DV or ranking value (RV) lies between the value range [14.25, 15.5] exclusively.

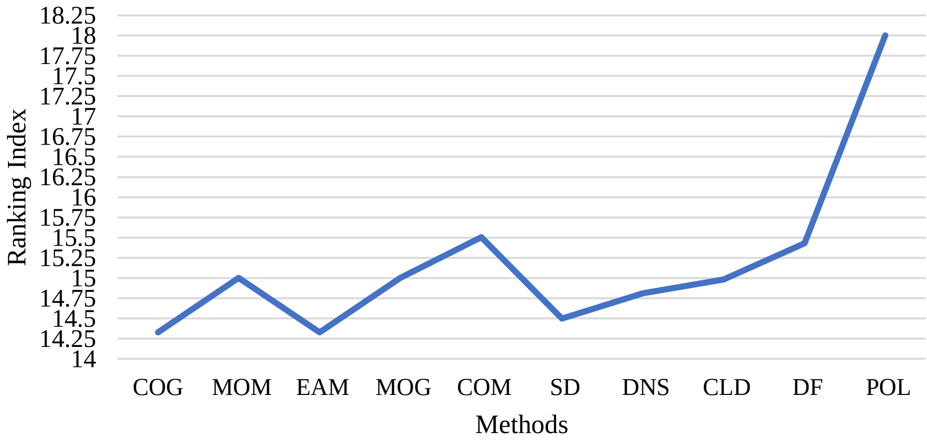

Figure 5:Defuzzified values of Trapezoidal fuzzy number under various Methods

Figure 5 gives the DV under various methods for the Trapezoidal fuzzy number. We see that, the minimum value (14.25) occurs due to the COG / EAM formula and highest value (around 18) occurs from the polynomial approximation formula. The COM/ DF(DOF) method takes DV as 15.5 but for some other formulae the range of DV assumes [14.5, 15.5] alone.

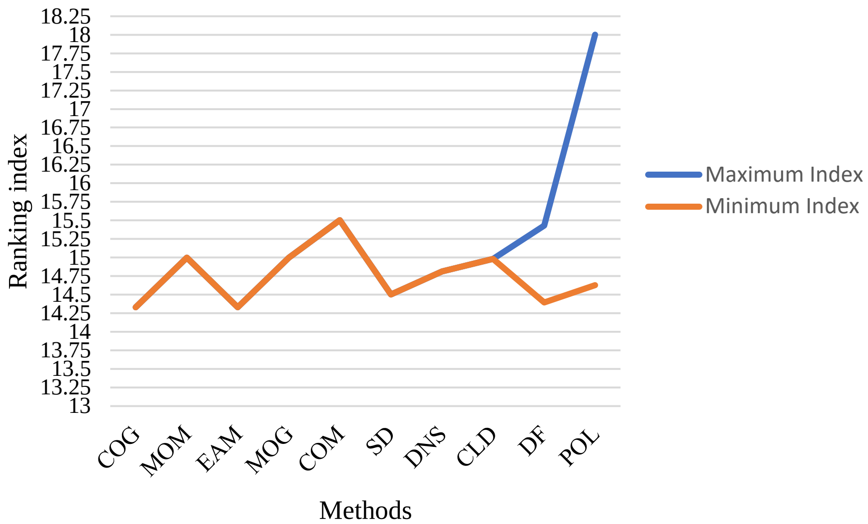

Figure 6: Max-min Defuzzification values under various Methods

Figure 6 shows the various defuzzification values of the Trapezoidal Fuzzy number. We see that the max-min values (ranging from 14.25 to 15.5) are same for the methods like COG, MOM, EAM, MOG, COM SD (Signed distance.). But the methods based on the learning theory gives two different set of defuzzified values of the proposed fuzzy set. The minimum values for the methods under learning theory vary from 14.25 to 15.5 but for maximum it varies from 14.5 to 18. Moreover, the polynomial approximation approach gives the finer optimum 14.625 than any other methods.

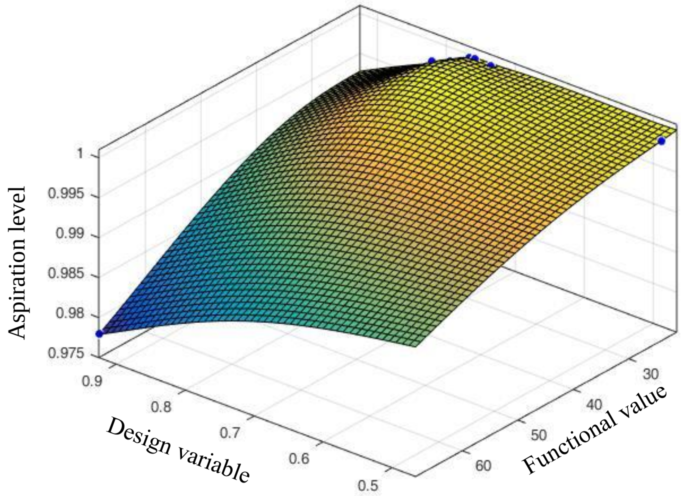

Figure. 7: Aspiration levels with design variable and functional

Figure 7 explores the actual nature of aspiration level under decision making with polynomial fuzzy set (Example 2) and some of the existing methods. It is seen that; the aspiration level assumes 1 with respect to the design variable range 0.6~0.8 and the function value range 35~60 respectively and they came from the existing methods. However, the minimum aspiration level attains above 0.975 and that of the maximum near 0.998 with respect to the function value near 70 and 20 respectively.

6. CONCLUSION

The basic drawbacks of the existing defuzzification methods have their limited scope of application and most of the cases the actual notion of fuzzy flexibility is violated. In such cases, after getting the defuzzified value if we wish to study the aspiration level which has been achieved, we could see that most of them have come from weaker fuzzy (membership grade <0.5) instead of getting strong fuzzy (membership grade > 0.5). Indeed, the traditional fuzzy system allows the membership grade to achieve 0 and 1 values but in fact they fall into classical set. Decision making under such considerations is nothing but the violation of the notion of fuzzy sets nothing else. To overcome this paradox, first of all, we have constructed polynomial approximation function using the proposed polygonal fuzzy numbers. In the literature several defuzzification methods are available but they did not guarantee the aspiration level would be a maximum one. But in this study because of existence of optimality, the actual notion of fuzziness is also conserved and the defuzzification values so obtained are novel and new. The limitation of this study is that it has not been verified with the constrained optimization problem. Beyond this the fundamental contribution of this article is described as follows:

i) This new defuzzification approach can be applied for both the linear and non-linear optimization problems.
ii) The defuzzification algorithm with varying degrees of the polynomial might have its global use in any kind of decision-making problem (with gain or loss).
iii) By this approach the membership grade maxima are always restored.
iv) The actual fuzzy uncertainty/ fuzzy deviation in true sense is conserved.
v) Polynomial functions are continuous hence it is easily differentiable under classical sense to get a finer optimum.
vi) The notion of neutrosophic set can also be explained by taking the function optimum within [-1, 1].
vii) By this approach the notion of non-standard fuzzy set, over set, under set and off sets are explained properly under a unified umbrella/ function.
viii) Using variation function the degree of fuzziness can be calculated to measure the degree of uncertainties of a fuzzy set/ fact of a decision-making problem.
ix) Using vector valued functions, the notion of Type-2 fuzzy numbers with its new defuzzification methods can also be developed.

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APPENDIX

A.1 Case of Triangular Fuzzy Set

Here the polynomial function $\mu(x) = Ax^2 + Bx + C$, so, for optimality we always have $\mu'(x) = 0 \implies x = -\frac{B}{2A}$ and $\mu''(x) = 2A < 0 \Rightarrow \mu(x)$ has a maximum value at

$$x = \frac{-B}{2A} = \frac{\left[\frac{(a_2+a_3)\epsilon}{(a_2-a_1)(a_3-a_1)} - \frac{(a_1+a_3)}{(a_2-a_1)(a_3-a_2)} + \frac{(a_1+a_2)\epsilon}{(a_3-a_1)(a_3-a_2)}\right]}{2\left[\frac{\epsilon}{(a_2-a_1)(a_3-a_1)} - \frac{1}{(a_2-a_1)(a_3-a_2)} + \frac{\epsilon}{(a_3-a_1)(a_3-a_2)}\right]}$$

$$= \left\{\frac{(a_2+a_3)(a_3-a_2)\epsilon + (a_1+a_3)(a_3-a_1) + (a_1+a_2)(a_2-a_1)\epsilon}{2\{(a_3-a_2)\epsilon + (a_2-a_1)\epsilon + (a_3-a_1)\}}\right\}$$

$$= \left\{\frac{(a_3^2-a_2^2)\epsilon + (a_3^2-a_1^2) + (a_2^2-a_1^2)\epsilon}{2\{(a_3-a_2)\epsilon + (a_2-a_1)\epsilon + (a_3-a_1)\}}\right\} = \frac{(1+\epsilon)a_3^2 - (1+\epsilon)a_1^2}{2(1+\epsilon)(a_3-a_1)} = \frac{(1+\epsilon)(a_3^2-a_1^2)}{2(1+\epsilon)(a_3-a_1)} \frac{(a_3+a_1)(a_3-a_1)}{2(a_3-a_1)} = \frac{a_3+a_1}{2}$$

Eq.(A.1)

For convergency checking we see, $\lim_{a_3 \to a_2 a_1 \to a_2,} \frac{a_1+a_3}{2} = a_2$

A.2 Case of Trapezoidal fuzzy Set

Here the polynomial function be $\mu'(x) = Ax^3 + 2Bx^2 + Cx + D$, so for optimality we always have

$$\mu'(x) = 3Ax^2 + 2Bx + C \qquad \text{Eq.(A.2)}$$

And

$$\mu''(x) = 6Ax + 2B \qquad \text{Eq.(A.3)}$$

Solving Eq.(A.2) we get for single positive root

$$\bar{x} = -\frac{B}{3A} = \frac{\frac{(a_2^2+a_3^2-a_1^2-a_4^2-a_1a_4+a_2a_3)}{(a_2-a_1)(a_3-a_1)(a_4-a_2)(a_4-a_3)}}{\frac{(a_2+a_3-a_1-a_4)}{(a_2-a_1)(a_3-a_1)(a_4-a_2)(a_4-a_3)}} = \frac{(a_2^2+a_3^2-a_1^2-a_4^2-a_1a_4+a_2a_3)}{3(a_2+a_3-a_1-a_4)} \qquad \text{Eq.(A.4)}$$

To check the crisp convergence, we write

$$\operatorname*{Lim}_{a_4\to a_3} \bar{x} = \lim_{a_4\to a_3} \frac{(a_2^2+a_3^2-a_1^2-a_4^2-a_1a_4+a_2a_3)}{3(a_2+a_3-a_1-a_4)} = \frac{(a_2^2-a_1^2-a_1a_3+a_2a_3)}{3(a_2-a_1)} = \frac{(a_2-a_1)(a_1+a_2+a_3)}{3(a_2-a_1)}$$

$$= \frac{a_1+a_2+a_3}{3} \text{ . Again } \lim_{a_3\to a_2 a_1\to a_2,} \frac{a_1+a_2+a_3}{3} = \frac{a_1+a_1+a_1}{3} = a_2$$

Moreover, it is obvious that, by Eq. (A.3), $\mu''(\bar{x}) < 0 \implies \mu(x)$ has a maximum at $\overline{x}$ giving $R(\widetilde{A_2})$.