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A PRODUCTION INVENTORY MODEL FOR AMELIORATING AND DETERIORATING ITEMS WITH TWO LEVELS OF PRODUCTION UNDER TIME DEPENDENT QUADRATIC DEMAND AND PRESERVATION TECHNOLOGY INVESTMENT

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Abstract: A production inventory model for ameliorating and deteriorating items is established with time varying quadratic demand under the influence of preservation technology. Two different production rates are considered such that the initial production rate switches to another rate of production after some interval of time. Preservation technology is used to lower the deterioration effect. The main objective is to obtain the optimal production time, cycle length and preservation technology cost to minimize the total cost per unit time of the system. Numerical examples under different situations are presented to illustrate the model. The effect of the parameters on optimal solution is investigated through sensitivity analysis while various observation obtained are discussed accordingly.

Keywords: Production inventory model, amelioration, deterioration, preservation technology investment, time dependent quadratic demand, two level production rates.

MSC: 90B05.

1. INTRODUCTION

It can be observed that there exist a certain section of items whose value, utility and stock increase rapidly over time. These type of items are known as ameliorating items. These type of items includes various kinds of high breed fishes in the pond or young and fast growing livestock such as chicken, duck, goat etc. Due to the effect of amelioration, a significant increase in the stock, utility and value of these items can be observed over time. However, at the same time, the deterioration effect can also be observed in the inventory of these items. Due to the effect of deterioration, a certain decline in the stock and value of these items can be observed over time. Hwang [1] first analyzed an EOQ model for ameliorating items considering Weibull distribution amelioration rate. Mondal et al. [2] with Weibull distribution amelioration rate and selling price dependent demand established their inventory model for ameliorating items. Moon et al. [3] in their developed model for ameliorating and deteriorating items considered constant amelioration and deterioration rate with time varying demand. Law and Wee [4] considered time discounting and multiple deliveries in an integrated production inventory model with Weibull distribution deterioration and amelioration rate under constant demand. Wee et al. [5] investigated an inventory model for ameliorating and deteriorating items under the effects of finite planning horizon and time value of money. Dem and Singh [6] and Goyal et al. [7] with time varying production and demand rate developed production inventory model for ameliorating items. Mahata and De [8] discussed partial trade credit policy in their EOQ model for ameliorating items with price dependent demand. Vandana and Srivastava [9] in their study assumed trapezoidal demand under the effects of inflation and time discounting for ameliorating and deteriorating items. Mondal et al. [10] presented their models for ameliorating and deteriorating items under crisp and interval environments with selling price and advertisement frequency dependent demand. Nodoust et al. [11] investigated a inventory model for ameliorating and deteriorating items with imperfect production process under the effect of inflation and time dependent demand rate. Rai [12] discussed trade credit policy for ameliorating and deteriorating items between supplier, manufacturer and retailer with Weibull distribution amelioration and deterioration rate. Vandana and Sana [13] established their two-echelon inventory model considering single vendor and multiple buyers for ameliorating and deteriorating items. Hatibaruah and Saha [14] considered the effect of preservation technology in an inventory model for ameliorating and deteriorating items under stock and price dependent demand. Khedlekar and Singh [15] with price and time varying demand investigated an integrated inventory model for ameliorating items considering infinite time horizon. Padhy et al. [16] discussed an EOQ model for ameliorating and deteriorating items in crisp and fuzzy environment with exponentially increasing demand.

Another, natural phenomenon which cannot be ignored in the development of inventory models is deterioration. Deterioration may be defined as the decay, spoilage or expiration of products over time. Due to the effect of deterioration, the quality or utility of any product declines with respect to time. Higher rate of deterioration leads to severe economic loss to any manufacturing company. So, nowadays, many companies adopt different preservation methods to minimize the effect of deterioration. For example, by using refrigeration technique, the quality and freshness of many food products, fruits and flowers can be preserved for a longer period of time. In the inventory literature, many models are developed by considering different rates of preservation technology. Hsieh and Dye [17] considered the effect of preservation technology investment in a production inventory model with time dependent demand and constant deterioration rate. Mishra [18] established an inventory model considering trapezoidal type demand and preservation technology investment. Singh et al. [19] investigated an EOQ model for deteriorating items with stock dependent demand under preservation technology effect and trade credit policy. Mishra et al. [20] in their study for deteriorating items considered preservation technology investment under stock and price dependent demand. Pal et al. [21] considered preservation technology in an inventory model for non-instantaneous deteriorating items with constant demand and random deterioration start time. Bardhan et al. [22] derived an inventory model for non-instantaneous deteriorating items under stock dependent demand with preservation technology investment. Iqbal and Sarkar [23] established an integrated production system for short lifetime deteriorating items with time varying demand and deterioration rate under preservation technology investment. Das et al. [24] assumed two different preservation and backlogging rates in an inventory model for non-instantaneous deteriorating items under selling price dependent demand. Khanna et al. [25] developed an inventory model for deteriorating items considering demand to be dependent on stock under the effect of preservation technology. Priyamvada et al. [26] studied an imperfect production inventory model for deteriorating items under constant deterioration and production rate with preservation technology investment. Rahman et al. [27] in their study considered hybrid price and stock dependent demand with preservation technology investment under advance payment and discount facility. Rahaman et al. [28] in their EPQ model considered the application of preservation technology with unit selling price and stock dependent demand while production rate is a linear function of stock. Roy et al. [29] considered an inventory model under advance payment policy and constant demand with preservation technology investment. Shah et al. [30] developed an integrated production inventory model for deteriorating items considering environmental pollution with demand dependent on stock using preservation technology investment.

Generally, it is observed that several factors influence the demand of a product. The demand of a product is mainly dependent on factors such as selling price, time, inventory level, frequency of advertisement, etc. A vast number of inventory models are available in the existing literature where the demand rate is dependent on these factors. Roy and Chaudhuri [31] established two production inventory models with demand dependent on stock. Sarkar and Moon [32] presented their EPQ model under the effect of inflation with stochastic demand. Pal et al. [33] studied an EPQ model with two parameter Weibull distribution deterioration rate and ramp type demand under the effect of inflation. Sivashankari and Panayappan

[34] in their study considered a production inventory model for deteriorating items with two different production rates under constant demand. Chakrabarty et al. [35] investigated a production inventory model for defective items considering demand rate as quadratic and exponential decreasing function of selling price under inflation and time value of money. Viji and Karthikeyan [36] developed an EPQ model for deteriorating items considering three different production levels with constant demand and production rate. Ruidas et al. [37] studied single period production inventory model under time, promotional effort and selling price dependent demand with price revision. Shen et al. [38] developed their production inventory model for deteriorating items under the effect of preservation technology investment and carbon tax policy with constant production and demand rate. Dari and Sani [39] presented an EPQ model for non-instantaneous deteriorating items considering quadratic demand and time dependent holding cost. Dolai and Mondal [40] developed an EPQ model for imperfect quality items considering preservation technology investment with stochastic demand. Sepehri et al. [41] discussed a sustainable production inventory model for imperfect quality items under quality improvement and preservation technology investment. Shaikh et al. [42] derived an EPQ model for deteriorating items considering partial trade credit policy under the effect of inflation and reliability with price dependent demand. Mishra et al. [43] studied a four level production manufacturing system for deteriorating items assuming rebate value and selling price dependent demand. Singh et al. [44] analyzed an inventory model with three levels of production considering quadratic demand rate and constant production rate.

Till now, researchers developed a vast number of production inventory models for deteriorating items. But, very few models are studied considering the effect of amelioration. Also, the application of preservation technology is very rarely considered in the development of production inventory models for ameliorating items. The main contribution of our study is to develop a production inventory model for ameliorating and deteriorating items with two levels of production under quadratic demand and preservation technology investment. It is assumed that production started at one rate switched to another rate after some interval of time. The rate of amelioration and deterioration is assumed to follow two parameter Weibull distribution. Preservation technology is considered by the manufacturing company to reduce the deterioration rate involved in the items. Two different preservation technology rates are considered in this study. The rate of production is considered to be greater than the demand rate. The primary goal of this study is to obtain the optimal values of preservation technology investment, production time and cycle length such that the total cost of the system is minimized.

This paper is organized as follows: We have presented the notations and assumptions of this study in section 2. We have formulated the mathematical model in section 3. In section 4, some theoretical results are proved regarding convexity of the total cost function. An algorithm is developed to obtain the optimal solutions in section 5. Some numerical examples are solved considering different situations in section 6. In section 7, we have performed sensitivity analysis while various results and managerial insights obtained are discussed in section 8. In section 9, we have discussed the conclusions, limitations and future research directions of this study.

2. NOTATIONS AND ASSUMPTIONS

2.1. Notations

The following notations are used in this study:

C_1 Ordering cost / order.
C_2 Production cost / unit.
C_3 Deterioration cost / unit.
C_4 Amelioration cost / unit.
h Holding cost / unit / unit time.
ξ Preservation technology cost / unit time.
T_i Unit time in period, i=1 and 2.
T Cycle length.
S_1 On hand inventory level at time T_1.
S_2 On hand inventory level at time T_2.
$I_1(t)$ Inventory level at time t, $0 \le t \le T_1$.
$I_2(t)$ Inventory level at time t, $T_1 \le t \le T_2$.
$I_3(t)$ Inventory level at time t, $T_2 \le t \le T$.

2.2. Assumptions

1. The inventory system considers a single item with zero lead time.
2. The demand rate $D(t)$ is considered to be a quadratic function of time, $D(t) = u + vt + wt^2$ where, $u > 0$, $v \neq 0$ and $w \neq 0$.
3. The production rate $P(t)$ is considered to be greater than the demand rate $D(t)$. Thus, $P(t) = \lambda D(t)$ where $\lambda > 1$.
4. The rate of amelioration $A(t)$ is described by two parameter Weibull distribution, $A(t) = \alpha\beta t^{\beta-1}$ where α $(0 < \alpha << 1)$ is scale parameter and β $(\beta > 0)$ is shape parameter.
5. The rate of deterioration, $\theta(t)$ is described by two parameter Weibull distribution, $\theta(t) = xyt^{y-1}$ where, x $(0 < x << 1)$ is scale parameter and y $(y > 0)$ is shape parameter.
6. The effect of deterioration is controlled by investing in preservation technology. The preservation technology function $m(\xi)$ is considered as $m(\xi) = 1 - e^{-\gamma\xi}$, $\gamma > 0$ and $m(\xi) = \frac{\gamma\xi}{1+\gamma\xi}$, $\gamma > 0$ (Das *et* al. [24]).
7. Shortages are not allowed to occur.
8. Since the values of α $(0 < \alpha << 1)$ and x $(0 < x << 1)$ are very small, so second and higher powers of α and x are ignored from the model.
9. In the present model, it is considered that the production rate is lower at the initial stage. However, the production rate increases gradually over the period. The present model represents the production of finished products manufactured from ameliorating items such as livestock, which have a very

low unit production cost. But if there is any damage or spoilage to such an item, the loss may be considered as the item's current market value. This results in a higher deterioration cost per unit of the item. So, in the present model, the cost of deterioration per unit is higher than the unit production cost.

3. MATHEMATICAL MODEL FORMULATION

Let us assume that, at time $t = 0$ the production cycle starts with initial stock level as zero. In the interval $[0, T_1]$, the production rate is $P(t)$ and the demand rate is $D(t)$ such that the inventory accumulates at the rate of $P(t) - D(t)$. As a result, the stock level at time $t = T_1$ becomes S_1. During the interval $[T_1, T_2]$, the demand rate shifts to "a" times of $D(t)$, i.e. $aD(t)$ while production rate shifts to "a" times of $P(t)$, i.e. $aP(t)$ where a $(a > 1)$ is a constant such that the inventory accumulates at the rate of $a(P(t) - D(t))$. Consequently, stock reaches to maximum level S_2 at time $t = T_2$. The production stops at time $t = T_2$, after which during the interval $[T_2, T]$, the stock decreases due to the effect of demand rate $D(t)$ and deterioration. The stock level shifts to zero at time $t = T$. The graphical representation of the production inventory system is displayed in the Figure 1.

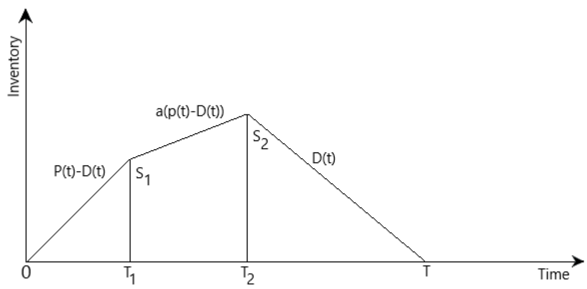

Figure 1: Graphical representation of the inventory system

The production inventory system can be represented with the help of following differential equations

$$\frac{d}{dt}I_1(t)+\{1-m(\xi)\}xyt^{y-1}I_1(t)-\alpha\beta t^{\beta-1}I_1(t) = (\lambda-1)(u+vt+wt^2), \quad 0 \le t \le T_1 \quad (1)$$

where, $I_1(0) = 0$ and $I_1(T_1) = S_1$.

$$\frac{d}{dt}I_2(t)+\{1-m(\xi)\}xyt^{y-1}I_2(t)-\alpha\beta t^{\beta-1}I_2(t) = a(\lambda-1)(u+vt+wt^2), \quad T_1 \le t \le T_2 \quad (2)$$

where, $I_1(T_1) = I_2(T_1)$

and,

$$\frac{d}{dt}I_3(t)+\{1-m(\xi)\}xyt^{y-1}I_3(t)-\alpha\beta t^{\beta-1}I_3(t) = -(u+vt+wt^2), \quad T_2 \le t \le T \quad (3)$$

where, $I_3(T_2) = S_2$ and $I_3(T) = 0$.

For small values of x, we have $\exp(x) \approx 1 + x$. This result is used to obtain the solutions of the differential equations.
The solutions of the differential equations (1), (2) and (3) are given by,

$$I_1(t) = (\lambda - 1)\Big[ut + \frac{v}{2}t^2 + \frac{w}{3}t^3 - \{1 - m(\xi)\}xy\Big\{\frac{u}{y+1}t^{y+1} + \frac{v}{2(y+2)}t^{y+2} + \frac{w}{3(y+3)}t^{y+3}\Big\} + \alpha\beta\Big\{\frac{u}{\beta+1}t^{\beta+1} + \frac{v}{2(\beta+2)}t^{\beta+2} + \frac{w}{3(\beta+3)}t^{\beta+3}\Big\}\Big], \quad 0 \le t \le T_1 \tag{4}$$

$$\begin{aligned} I_2(t) = (\lambda - 1)\Big[&u(at + T_1 - aT_1) + \frac{v}{2}(at^2 + T_1^2 - aT_1^2) + \frac{w}{3}(at^3 + T_1^3 - aT_1^3) \\ &+ \{1 - m(\xi)\}x\Big\{\frac{u}{y+1}\{(1-a)T_1^{y+1} + (y+1)(a-1)T_1t^y - yat^{y+1}\} \\ &+ \frac{v}{2(y+2)}\{2(1-a)T_1^{y+2} + (a-1)(y+2)T_1^2t^y - yat^{y+2}\} \\ &+ \frac{w}{3(y+3)}\{3(1-a)T_1^{y+3} - yat^{y+3} + (a-1)(y+3)T_1^3t^y\}\Big\} \\ &- \alpha\Big\{\frac{u}{\beta+1}\{(1-a)T_1^{\beta+1} + (a-1)(\beta+1)T_1t^\beta - \beta at^{\beta+1}\} \\ &+ \frac{v}{2(\beta+2)}\{2(1-a)T_1^{\beta+2} + (a-1)(\beta+2)T_1^2t^\beta - \beta at^{\beta+2}\} \\ &+ \frac{w}{3(\beta+3)}\{3(1-a)T_1^{\beta+3} + (a-1)(\beta+3)T_1^3t^\beta - \beta at^{\beta+3}\}\Big\}\Big] \end{aligned} \tag{5}$$

$$T_1 \le t \le T_2$$

and,

$$\begin{aligned} I_3(t) = u\Big\{&(T - t) + \frac{\{1 - m(\xi)\}x}{y+1}\Big\{T^{y+1} - (y+1)Tt^y + yt^{y+1}\Big\} \\ &- \frac{\alpha}{\beta+1}\Big\{T^{\beta+1} - (\beta+1)Tt^\beta + \beta t^{\beta+1}\Big\}\Big\} + v\Big\{\frac{1}{2}(T^2 - t^2) \\ &+ \frac{\{1 - m(\xi)\}x}{2(y+2)}\Big\{2T^{y+2} - (y+2)T^2t^y + yt^{y+2}\Big\} \\ &- \frac{\alpha}{2(\beta+2)}\Big\{2T^{\beta+2} - (\beta+2)T^2t^\beta + \beta t^{\beta+2}\Big\}\Big\} + \\ w\Big\{&\frac{1}{3}(T^3 - t^3) + \frac{\{1 - m(\xi)\}x}{3(y+3)}\Big\{3T^{y+3} - (y+3)T^3t^y + yt^{y+3}\Big\} \\ &- \frac{\alpha}{3(\beta+3)}\Big\{3T^{\beta+3} - (\beta+3)T^3t^\beta + \beta t^{\beta+3}\Big\}\Big\}, \qquad T_2 \le t \le T \end{aligned} \tag{6}$$

Inventory level,

$S_1 = I_1(T_1)$

$$= (\lambda-1)\Big[uT_1 + \frac{v}{2}T_1^2 + \frac{w}{3}T_1^3 - \{1-m(\xi)\}xy\Big\{\frac{u}{y+1}T_1^{y+1} + \frac{v}{2(y+2)}T_1^{y+2}$$
$$+ \frac{w}{3(y+3)}T_1^{y+3}\Big\} + \alpha\beta\Big\{\frac{u}{\beta+1}T_1^{\beta+1} + \frac{v}{2(\beta+2)}T_1^{\beta+2} + \frac{w}{3(\beta+3)}T_1^{\beta+3}\Big\}\Big] \quad (7)$$

Inventory level,

$$S_2 = I_3(T_2)$$
$$= u\Big\{(T-T_2) + \frac{\{1-m(\xi)\}x}{y+1}\Big\{T^{y+1} - (y+1)TT_2^y + yT_2^{y+1}\Big\}$$
$$- \frac{\alpha}{\beta+1}\Big\{T^{\beta+1} - (\beta+1)TT_2^\beta + \beta T_2^{\beta+1}\Big\}\Big\} + v\Big\{\frac{1}{2}(T^2 - T_2^2)$$
$$+ \frac{\{1-m(\xi)\}x}{2(y+2)}\Big\{2T^{y+2} - (y+2)T^2T_2^y + yT_2^{y+2}\Big\}$$
$$- \frac{\alpha}{2(\beta+2)}\Big\{2T^{\beta+2} - (\beta+2)T^2T_2^\beta + \beta T_2^{\beta+2}\Big\}\Big\} + w\Big\{\frac{1}{3}(T^3 - T_2^3)$$
$$+ \frac{\{1-m(\xi)\}x}{3(y+3)}\Big\{3T^{y+3} - (y+3)T^3T_2^y + yT_2^{y+3}\Big\}$$
$$- \frac{\alpha}{3(\beta+3)}\Big\{3T^{\beta+3} - (\beta+3)T^3T_2^\beta + \beta T_2^{\beta+3}\Big\}\Big\} \quad (8)$$

The different costs involved in the inventory model are :

1. Ordering cost, $OC = C_1$
2. Production cost,

$$PC = C_2\Big[\int_0^{T_1} P(t)dt + \int_{T_1}^{T_2} aP(t)dt\Big]$$
$$= C_2\lambda\Big[(1-a)uT_1 + (1-a)\frac{v}{2}T_1^2 + (1-a)\frac{w}{3}T_1^3$$
$$+ a(uT_2 + \frac{v}{2}T_2^2 + \frac{w}{3}T_2^3)\Big]$$

3. Deterioration cost,

$$DC = C_3\Big[\int_0^{T_1}\{1-m(\xi)\}xyt^{y-1}I_1(t)dt + \int_{T_1}^{T_2}\{1-m(\xi)\}xyt^{y-1}I_2(t)dt$$
$$+ \int_{T_2}^{T}\{1-m(\xi)\}xyt^{y-1}I_3(t)dt\Big]$$
$$= C_3\Big[\{1-m(\xi)\}xy(\lambda-1)\Big\{\frac{u}{y+1}T_1^{y+1} + \frac{v}{2(y+2)}T_1^{y+2} + \frac{w}{3(y+3)}T_1^{y+3}\Big\}$$
$$+ \{1-m(\xi)\}xy(\lambda-1)\Big[\frac{u}{y(y+1)}\{yaT_2^{y+1} + (1-a)(y+1)T_1T_2^y$$
$$+ (a-y-1)T_1^{y+1}\} + \frac{v}{2y(y+2)}\{ayT_2^{y+2} + (1-a)(y+2)T_1^2T_2^y$$
$$+ (2a-y-2)T_1^{y+2}\} + \frac{w}{3y(y+3)}\{yaT_2^{y+3} + (1-a)(y+3)T_1^3T_2^y$$

$$+ (3a - y - 3)T_1^{y+3}\}\Big] + \{1 - m(\xi)\}xy\Big[\frac{u}{y(y+1)}\{T^{y+1} - (y+1)TT_2^y$$
$$+ yT_2^{y+1}\} + \frac{v}{2y(y+2)}\{2T^{y+2} - (y+2)T^2T_2^y + yT_2^{y+2}\}+$$
$$\frac{w}{3y(y+3)}\{3T^{y+3} - (y+3)T^3T_2^y + yT_2^{y+3}\}\Big]\Big]$$

4. Amelioration cost,

$$AC = C_4\Big[\int_0^{T_1} \alpha\beta t^{\beta-1} I_1(t)dt + \int_{T_1}^{T_2} \alpha\beta t^{\beta-1} I_2(t)dt + \int_{T_2}^{T} \alpha\beta t^{\beta-1} I_3(t)dt\Big]$$
$$= C_4\Big[\alpha\beta(\lambda-1)\Big[\frac{u}{\beta+1}T_1^{\beta+1} + \frac{v}{2(\beta+2)}T_1^{\beta+2} + \frac{w}{3(\beta+3)}T_1^{\beta+3}\Big]+$$
$$\alpha\beta(\lambda-1)\Big[\frac{u}{(\beta+1)\beta}\{\beta a T_2^{\beta+1} + (1-a)(\beta+1)T_1T_2^\beta + (a-\beta-1)T_1^{\beta+1}\}$$
$$+ \frac{v}{2(\beta+2)\beta}\{\beta a T_2^{\beta+2} + (1-a)(\beta+2)T_1^2T_2^\beta + (2a-\beta-2)T_1^{\beta+2}\}$$
$$+ \frac{w}{3(\beta+3)\beta}\{\beta a T_2^{\beta+3} + (1-a)(\beta+3)T_1^3T_2^\beta + (3a-\beta-3)T_1^{\beta+3}\}\Big]+$$
$$\alpha\beta\Big[\frac{u}{\beta(\beta+1)}\{T^{\beta+1} - (\beta+1)TT_2^\beta + \beta T_2^{\beta+1}\} + \frac{v}{2\beta(\beta+2)}\{2T^{\beta+2}-$$
$$(\beta+2)T^2T_2^\beta + \beta T_2^{\beta+2}\} + \frac{w}{3\beta(\beta+3)}\{3T^{\beta+3} - (\beta+3)T^3T_2^\beta + \beta T_2^{\beta+3}\}\Big]\Big]$$

5. Holding cost,

$$HC = h\Big[\int_0^{T_1} I_1(t)dt + \int_{T_1}^{T_2} I_2(t)dt + \int_{T_2}^{T} I_3(t)dt\Big]$$
$$= h\Big[(\lambda-1)\Big[\frac{u}{2}T_1^2 + \frac{v}{6}T_1^3 + \frac{w}{12}T_1^4 - \{1-m(\xi)\}xy\Big\{\frac{u}{(y+1)(y+2)}T_1^{y+2}$$
$$+ \frac{v}{2(y+2)(y+3)}T_1^{y+3} + \frac{w}{3(y+3)(y+4)}T_1^{y+4}\Big\} + \alpha\beta\Big\{\frac{u}{(\beta+1)(\beta+2)}T_1^{\beta+2}$$
$$+ \frac{v}{2(\beta+2)(\beta+3)}T_1^{\beta+3} + \frac{w}{3(\beta+3)(\beta+4)}T_1^{\beta+4}\Big\}\Big]+$$
$$(\lambda-1)\Big[\frac{u}{2}(T_2-T_1)\{a(T_2+T_1) + 2T_1 - 2aT_1\} + \frac{v}{6}\{a(T_2^3 - T_1^3)+$$
$$3(T_2-T_1)(T_1^2 - aT_1^2)\} + \frac{w}{12}\{a(T_2^4 - T_1^4) + (T_2-T_1)(4T_1^3 - 4aT_1^3)\}+$$
$$\{1-m(\xi)\}x\Big\{\frac{u}{y+1}\{(1-a)T_1^{y+1}(T_2-T_1) + (a-1)T_1(T_2^{y+1} - T_1^{y+1})-$$
$$\frac{ya}{y+2}(T_2^{y+2} - T_1^{y+2})\} + \frac{v}{2(y+2)}\{2(1-a)T_1^{y+2}(T_2-T_1)+$$
$$(a-1)\left(\frac{y+2}{y+1}\right)T_1^2(T_2^{y+1} - T_1^{y+1}) - \frac{ya}{y+3}(T_2^{y+3} - T_1^{y+3})\}+$$

$$\frac{w}{3(y+3)}\{3(1-a)T_1^{y+3}(T_2-T_1)-\frac{ya}{y+4}(T_2^{y+4}-T_1^{y+4})+$$
$$(a-1)\left(\frac{y+3}{y+1}\right)T_1^3(T_2^{y+1}-T_1^{y+1})\}\Bigg\}-\alpha\Bigg\{\frac{u}{\beta+1}\{(1-a)T_1^{\beta+1}(T_2-T_1)$$
$$+(a-1)T_1(T_2^{\beta+1}-T_1^{\beta+1})-\frac{\beta a}{\beta+2}(T_2^{\beta+2}-T_1^{\beta+2})\}+$$
$$\frac{v}{2(\beta+2)}\{2(1-a)T_1^{\beta+2}(T_2-T_1)+(a-1)\left(\frac{\beta+2}{\beta+1}\right)T_1^2(T_2^{\beta+1}-T_1^{\beta+1})$$
$$-\frac{\beta a}{\beta+3}(T_2^{\beta+3}-T_1^{\beta+3})\}+\frac{w}{3(\beta+3)}\{3(1-a)T_1^{\beta+3}(T_2-T_1)+$$
$$(a-1)\left(\frac{\beta+3}{\beta+1}\right)T_1^3(T_2^{\beta+1}-T_1^{\beta+1})-\frac{\beta a}{\beta+4}(T_2^{\beta+4}-T_1^{\beta+4})\}\Bigg\}\Bigg]+$$
$$u\Bigg\{\frac{1}{2}(T-T_2)^2+\frac{\{1-m(\xi)\}x}{(y+1)(y+2)}\{y(T^{y+2}-T_2^{y+2})-(y+2)T_2T(T^y-T_2^y)\}$$
$$-\frac{\alpha}{(\beta+1)(\beta+2)}\{\beta(T^{\beta+2}-T_2^{\beta+2})-(\beta+2)TT_2(T^\beta-T_2^\beta)\}\Bigg\}+$$
$$v\Bigg\{\frac{1}{6}(2T^3-3T^2T_2+T_2^3)+\frac{\{1-m(\xi)\}x}{2(y+2)}\Bigg\{\frac{2y(y+2)}{(y+1)(y+3)}T^{y+3}-2T^{y+2}T_2$$
$$+\left(\frac{y+2}{y+1}\right)T^2T_2^{y+1}-\frac{y}{y+3}T_2^{y+3}\Bigg\}-\frac{\alpha}{2(\beta+2)}\Bigg\{\frac{2\beta(\beta+2)}{(\beta+1)(\beta+3)}T^{\beta+3}-$$
$$2T^{\beta+2}T_2+\left(\frac{\beta+2}{\beta+1}\right)T^2T_2^{\beta+1}-\frac{\beta}{\beta+3}T_2^{\beta+3}\Bigg\}\Bigg\}+$$
$$w\Bigg\{\frac{1}{12}(3T^4-4T^3T_2+T_2^4)+\frac{\{1-m(\xi)\}x}{3(y+3)}\Bigg\{\frac{3y(y+3)}{(y+1)(y+4)}T^{y+4}-3T^{y+3}T_2$$
$$+\left(\frac{y+3}{y+1}\right)T^3T_2^{y+1}-\frac{y}{y+4}T_2^{y+4}\Bigg\}-\frac{\alpha}{3(\beta+3)}\Bigg\{\frac{3\beta(\beta+3)}{(\beta+1)(\beta+4)}T^{\beta+4}-$$
$$3T^{\beta+3}T_2+\left(\frac{\beta+3}{\beta+1}\right)T^3T_2^{\beta+1}-\frac{\beta}{\beta+4}T_2^{\beta+4}\Bigg\}\Bigg\}\Bigg]$$

6. Preservation technology cost, $PTC=\xi T$.

   Total cost per unit time,

$$TC(T_2,T,\xi)=\frac{1}{T}\Bigg[OC+PC+DC+AC+HC+PTC\Bigg]$$
$$=\frac{1}{T}\Bigg[C_1+C_2\lambda\Bigg[(1-a)uT_1+(1-a)\frac{v}{2}T_1^2+(1-a)\frac{w}{3}T_1^3$$
$$+a(uT_2+\frac{v}{2}T_2^2+\frac{w}{3}T_2^3)\Bigg]+C_3\Bigg[\{1-m(\xi)\}xy(\lambda-1)\Bigg\{\frac{u}{y+1}T_1^{y+1}+$$
$$\frac{v}{2(y+2)}T_1^{y+2}+\frac{w}{3(y+3)}T_1^{y+3}\Bigg\}+\{1-m(\xi)\}xy(\lambda-1)\Bigg[\frac{u}{y(y+1)}\{yaT_2^{y+1}$$
$$+(1-a)(y+1)T_1T_2^y+(a-y-1)T_1^{y+1}\}+\frac{v}{2y(y+2)}\{ayT_2^{y+2}+$$

$$(1-a)(y+2)T_1^2T_2^y+(2a-y-2)T_1^{y+2}\}+\frac{w}{3y(y+3)}\{yaT_2^{y+3}+$$
$$(1-a)(y+3)T_1^3T_2^y+(3a-y-3)T_1^{y+3}\}\Big]+\{1-m(\xi)\}xy\Big[\frac{u}{y(y+1)}\{T^{y+1}$$
$$-(y+1)TT_2^y+yT_2^{y+1}\}+\frac{v}{2y(y+2)}\{2T^{y+2}-(y+2)T^2T_2^y+yT_2^{y+2}\}+$$
$$\frac{w}{3y(y+3)}\{3T^{y+3}-(y+3)T^3T_2^y+yT_2^{y+3}\}\Big]\Big]+C_4\Big[\alpha\beta(\lambda-1)\Big[\frac{u}{\beta+1}T_1^{\beta+1}$$
$$+\frac{v}{2(\beta+2)}T_1^{\beta+2}+\frac{w}{3(\beta+3)}T_1^{\beta+3}\Big]+\alpha\beta(\lambda-1)\Big[\frac{u}{(\beta+1)\beta}\{\beta aT_2^{\beta+1}+$$
$$(1-a)(\beta+1)T_1T_2^\beta+(a-\beta-1)T_1^{\beta+1}\}+\frac{v}{2(\beta+2)\beta}\{\beta aT_2^{\beta+2}+$$
$$(1-a)(\beta+2)T_1^2T_2^\beta+(2a-\beta-2)T_1^{\beta+2}\}+\frac{w}{3(\beta+3)\beta}\{\beta aT_2^{\beta+3}+$$
$$(1-a)(\beta+3)T_1^3T_2^\beta+(3a-\beta-3)T_1^{\beta+3}\}\Big]+\alpha\beta\Big[\frac{u}{\beta(\beta+1)}\{T^{\beta+1}$$
$$-(\beta+1)TT_2^\beta+\beta T_2^{\beta+1}\}+\frac{v}{2\beta(\beta+2)}\{2T^{\beta+2}-(\beta+2)T^2T_2^\beta+\beta T_2^{\beta+2}\}+$$
$$\frac{w}{3\beta(\beta+3)}\{3T^{\beta+3}-(\beta+3)T^3T_2^\beta+\beta T_2^{\beta+3}\}\Big]\Big]+h\Big[(\lambda-1)\Big[\frac{u}{2}T_1^2+\frac{v}{6}T_1^3$$
$$+\frac{w}{12}T_1^4-\{1-m(\xi)\}xy\Big\{\frac{u}{(y+1)(y+2)}T_1^{y+2}+\frac{v}{2(y+2)(y+3)}T_1^{y+3}$$
$$+\frac{w}{3(y+3)(y+4)}T_1^{y+4}\Big\}+\alpha\beta\Big\{\frac{u}{(\beta+1)(\beta+2)}T_1^{\beta+2}+\frac{v}{2(\beta+2)(\beta+3)}T_1^{\beta+3}$$
$$+\frac{w}{3(\beta+3)(\beta+4)}T_1^{\beta+4}\Big\}\Big]+(\lambda-1)\Big[\frac{u}{2}(T_2-T_1)\{a(T_2+T_1)+2T_1-2aT_1\}$$
$$+\frac{v}{6}\{a(T_2^3-T_1^3)+3(T_2-T_1)(T_1^2-aT_1^2)\}+\frac{w}{12}\{a(T_2^4-T_1^4)+$$
$$(T_2-T_1)(4T_1^3-4aT_1^3)\}+\{1-m(\xi)\}x\Big\{\frac{u}{y+1}\{(1-a)T_1^{y+1}(T_2-T_1)+$$
$$(a-1)T_1(T_2^{y+1}-T_1^{y+1})-\frac{ya}{y+2}(T_2^{y+2}-T_1^{y+2})\}+$$
$$\frac{v}{2(y+2)}\{2(1-a)T_1^{y+2}(T_2-T_1)+(a-1)\left(\frac{y+2}{y+1}\right)T_1^2(T_2^{y+1}-T_1^{y+1})-$$
$$\frac{ya}{y+3}(T_2^{y+3}-T_1^{y+3})\}+\frac{w}{3(y+3)}\{3(1-a)T_1^{y+3}(T_2-T_1)-$$
$$\frac{ya}{y+4}(T_2^{y+4}-T_1^{y+4})+(a-1)\left(\frac{y+3}{y+1}\right)T_1^3(T_2^{y+1}-T_1^{y+1})\}\Big\}-$$
$$\alpha\Big\{\frac{u}{\beta+1}\{(1-a)T_1^{\beta+1}(T_2-T_1)+(a-1)T_1(T_2^{\beta+1}-T_1^{\beta+1})-$$
$$\frac{\beta a}{\beta+2}(T_2^{\beta+2}-T_1^{\beta+2})\}+\frac{v}{2(\beta+2)}\{2(1-a)T_1^{\beta+2}(T_2-T_1)+$$
$$(a-1)\left(\frac{\beta+2}{\beta+1}\right)T_1^2(T_2^{\beta+1}-T_1^{\beta+1})-\frac{\beta a}{\beta+3}(T_2^{\beta+3}-T_1^{\beta+3})\}+$$

$$\frac{w}{3(\beta+3)}\{3(1-a)T_1^{\beta+3}(T_2-T_1)+(a-1)\left(\frac{\beta+3}{\beta+1}\right)T_1^3(T_2^{\beta+1}-T_1^{\beta+1})-$$
$$\frac{\beta a}{\beta+4}(T_2^{\beta+4}-T_1^{\beta+4})\}\Bigg\}\Bigg]+u\Bigg\{\frac{1}{2}(T-T_2)^2+\frac{\{1-m(\xi)\}x}{(y+1)(y+2)}\{y(T^{y+2}-T_2^{y+2})$$
$$-(y+2)T_2T(T^y-T_2^y)\}-\frac{\alpha}{(\beta+1)(\beta+2)}\{\beta(T^{\beta+2}-T_2^{\beta+2})-$$
$$(\beta+2)TT_2(T^\beta-T_2^\beta)\}\Bigg\}+v\Bigg\{\frac{1}{6}(2T^3-3T^2T_2+T_2^3)+$$
$$\frac{\{1-m(\xi)\}x}{2(y+2)}\Bigg\{\frac{2y(y+2)}{(y+1)(y+3)}T^{y+3}-2T^{y+2}T_2+\left(\frac{y+2}{y+1}\right)T^2T_2^{y+1}-$$
$$\frac{y}{y+3}T_2^{y+3}\Bigg\}-\frac{\alpha}{2(\beta+2)}\Bigg\{\frac{2\beta(\beta+2)}{(\beta+1)(\beta+3)}T^{\beta+3}-2T^{\beta+2}T_2+\left(\frac{\beta+2}{\beta+1}\right)T^2T_2^{\beta+1}-$$
$$\frac{\beta}{\beta+3}T_2^{\beta+3}\Bigg\}\Bigg\}+w\Bigg\{\frac{1}{12}(3T^4-4T^3T_2+T_2^4)+\frac{\{1-m(\xi)\}x}{3(y+3)}\Bigg\{\frac{3y(y+3)}{(y+1)(y+4)}T^{y+4}$$
$$-3T^{y+3}T_2+\left(\frac{y+3}{y+1}\right)T^3T_2^{y+1}-\frac{y}{y+4}T_2^{y+4}\Bigg\}-$$
$$\frac{\alpha}{3(\beta+3)}\Bigg\{\frac{3\beta(\beta+3)}{(\beta+1)(\beta+4)}T^{\beta+4}-3T^{\beta+3}T_2+$$
$$\left(\frac{\beta+3}{\beta+1}\right)T^3T_2^{\beta+1}-\frac{\beta}{\beta+4}T_2^{\beta+4}\Bigg\}\Bigg\}\Bigg]+\xi T\Bigg] \tag{9}$$

The optimal solutions of the decision variables can be obtained if the following necessary conditions are satisfied : $\frac{\partial}{\partial T_2}TC(T_2,T,\xi)=0$, $\frac{\partial}{\partial T}TC(T_2,T,\xi)=0$ and $\frac{\partial}{\partial \xi}TC(T_2,T,\xi)=0$.
Hence, the non-linear problem becomes,

$$\begin{aligned}&\text{Minimize } TC(T_2,T,\xi)\\ &\text{subject to, } T_1\le T_2,\ T_2\le T\\ &\text{and } 0\le\xi,\ 0\le T_2,\ 0\le T\end{aligned}$$

4. SOME THEORETICAL RESULTS ON OPTIMALITY

Proposition 1. *For fixed ξ and T_2, the total cost function $TC(T_2,T,\xi)$ given by equation (9) is convex with respect to T if $T^2\frac{\partial^2 g_1}{\partial T^2}-2T\frac{\partial g_1}{\partial T}+2g_1>0$, where the function g_1 is given in equation (A.1) of Appendix A.*

Proof. The total cost function $TC(T_2,T,\xi)$ given by equation (9) can be expressed in the form,

$$TC(T_2,T,\xi)=\frac{g_1(T_2,T,\xi)}{T} \tag{10}$$

where $g_1(T_2,T,\xi)$ is given in equation (A.1) of Appendix A.
Differentiating equation (10) partially with respect to T we have,

$$\frac{\partial TC}{\partial T}=\frac{T\frac{\partial g_1}{\partial T}-g_1}{T^2} \tag{11}$$

Differentiating equation (11) again partially with respect to T we have,

$$\frac{\partial^2 TC}{\partial T^2} = \frac{T^2 \frac{\partial^2 g_1}{\partial T^2} - 2T\frac{\partial g_1}{\partial T} + 2g_1}{T^3} \tag{12}$$

From, equation (12) we see that $\frac{\partial^2 TC}{\partial T^2} > 0$ if

$$T^2 \frac{\partial^2 g_1}{\partial T^2} - 2T\frac{\partial g_1}{\partial T} + 2g_1 > 0$$

□

Proposition 2. *For fixed T_2 and T, the total cost function $TC(T_2, T, \xi)$ given by equation (9) is convex with respect to ξ if $g_3(T_2, T) > 0$, where the function $g_3(T_2, T)$ is given in equation (B.2) of Appendix B.*

Proof. The total cost function $TC(T_2, T, \xi)$ given by equation (9) can be expressed in the form,

$$TC(T_2, T, \xi) = \frac{1}{T}\Big[g_2(T_2, T) + \{1 - m(\xi)\}g_3(T_2, T) + \xi T\Big] \tag{13}$$

where the functions $g_2(T_2, T)$ and $g_3(T_2, T)$ are given in equations (B.1) and (B.2) of Appendix B respectively.

Differentiating equation (13) partially with respect to ξ we have,

$$\frac{\partial TC}{\partial \xi} = \frac{1}{T}\Big[-m^{'}(\xi)g_3(T_2, T) + T\Big] \tag{14}$$

Differentiating equation (14) partially again with respect to ξ we have,

$$\frac{\partial^2 TC}{\partial \xi^2} = -\frac{1}{T}m^{''}(\xi)g_3(T_2, T) \tag{15}$$

Since, $m^{''}(\xi) < 0 \implies -m^{''}(\xi) > 0$.
From, equation (15) we see that $\frac{\partial^2 TC}{\partial \xi^2} > 0$ if $g_3(T_2, T) > 0$. □

The convexity of $TC(T_2, T, \xi)$ with respect to T_2 is shown graphically through numerical examples.

5. SOLUTION PROCEDURE

The following algorithm is developed to obtain the optimal values of the decision variables of the given model.

Algorithm

Step 1: Initialize the values of u, v, w, α, β, x, y, λ, γ, C_1, C_2, C_3, C_4, h, a and T_1.

Step 2: Minimize the total cost function $TC(T_2, T, \xi)$ given by equation (9) subject to the constraints $T_1 \leq T_2$, $T_2 \leq T$, $0 \leq T_2$, $0 \leq T$ and $0 \leq \xi$ using software Mathematica.

Step 3: Obtain the optimal values $TC^*(T_2^*, T^*, \xi^*)$, ξ^*, T_2^* and T^*.

Step 4: Determine the optimal value of S_1 (S_1^*) by substituting the optimal value ξ^* in equation (7). Also, determine the optimal value of S_2 (S_2^*) by substituting the optimal value T^*, T_2^* and ξ^* in equation (8).

6. NUMERICAL EXAMPLES

6.1. When $m(\xi) = 1 - e^{-\gamma\xi}$

Example 1. Consider the following values of the parameters, $u = 20$, $v = 10$, $w = 5$, $\alpha = 0.4$, $\beta = 1.2$, $x = 0.25$, $y = 0.35$, $\lambda = 1.3$, $\gamma = 0.8$, $C_1 = \$100$ per order, $C_2 = \$0.3$ per unit, $C_3 = \$3$ per unit, $C_4 = \$0.6$ per unit, $h = \$0.2$ per unit per month, $a = 1.5$ and $T_1 = 1.1$ months.
The optimal solution using the above algorithm is given by $T_2^* = 1.6663$ months, $T^* = 2.8863$ months, $\xi^* = \$1.5719$, $S_1^* = 10.9348$ units, $S_2^* = 54.0154$ units and $TC^*(T_2^*, T^*, \xi^*) = \$58.4082$.

Example 2. In this example all the parameters are considered to be exactly same as in Example 1. However, a restriction is put on the maximum investment in the preservation technology. Let $0 \leq \xi \leq \bar{\xi}$, where $\bar{\xi}$ is the maximum investment in preservation technology. Here, $\bar{\xi} = 1$ i.e. $0 \leq \xi \leq 1$.
In this case, the optimal solution using the above algorithm is given by $T_2^* = 1.6938$ months, $T^* = 2.8707$ months, $\xi^* = \$1$, $S_1^* = 10.8473$ units, $S_2^* = 53.7858$ units and $TC^*(T_2^*, T^*, \xi^*) = \$58.5453$.

6.2. When $m(\xi) = \frac{\gamma\xi}{1+\gamma\xi}$

Example 3. Consider the following values of the parameters, $u = 20$, $v = 10$, $w = 5$, $\alpha = 0.4$, $\beta = 1.2$, $x = 0.25$, $y = 0.35$, $\lambda = 1.3$, $\gamma = 0.8$, $C_1 = \$100$ per order, $C_2 = \$0.3$ per unit, $C_3 = \$3$ per unit, $C_4 = \$0.6$ per unit, $h = \$0.2$ per unit per month, $a = 1.5$ and $T_1 = 1.1$ months.
The optimal solution using the above algorithm is given by $T_2^* = 1.7082$ months, $T^* = 2.8609$ months, $\xi^* = \$1.0144$, $S_1^* = 10.7928$ units, $S_2^* = 53.5921$ units and $TC^*(T_2^*, T^*, \xi^*) = \$58.9862$.

Example 4. In this example all the parameters are considered to be exactly same as in Example 3. However, a restriction is put on the maximum investment in the preservation technology. Let $0 \leq \xi \leq \bar{\xi}$, where $\bar{\xi}$ is the maximum investment in preservation technology. Here, $\bar{\xi} = 1$ i.e. $0 \leq \xi \leq 1$.
In this case, the optimal solution using the above algorithm is given by $T_2^* = 1.7087$ months, $T^* = 2.8606$ months, $\xi^* = \$1$, $S_1^* = 10.7910$ units, $S_2^* = 53.5848$ units and $TC^*(T_2^*, T^*, \xi^*) = \$58.9863$.

It is observed from the results of Example 3 and Example 4 that the restriction put on the maximum investment in preservation technology does not have a significant effect on the optimal values of the total cost and decision variables. That is why the manufacturing company can spend more money on preservation technology to reduce the deterioration rate.
The convexity of the total cost function is shown in the Figures 2, 3, 4 and 5.

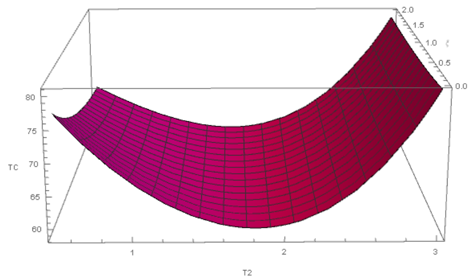

Figure 2: Convexity of $TC(T_2, T, \xi)$ of Example 1 with respect to T_2 and ξ for fixed T.

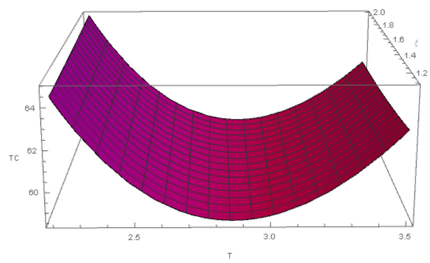

Figure 3: Convexity of $TC(T_2, T, \xi)$ of Example 1 with respect to T and ξ for fixed T_2.

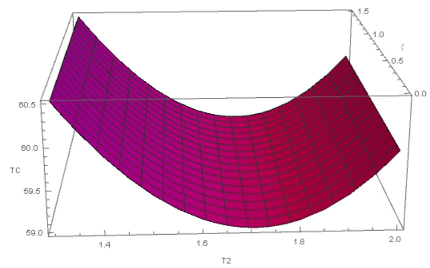

Figure 4: Convexity of $TC(T_2, T, \xi)$ of Example 3 with respect to T_2 and ξ for fixed T.

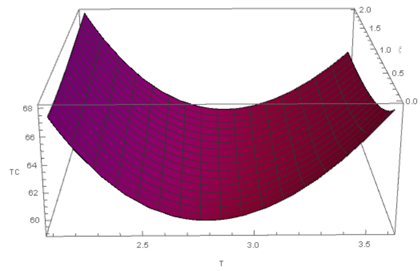

Figure 5: Convexity of $TC(T_2, T, \xi)$ of Example 3 with respect to T and ξ for fixed T_2.

The comparison of optimal solution for different cases are given in Tables 1, 2, 3 and 4.

Table 1: Comparison of optimal values for different cases in Example 1

Case	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
Without deterioration ($x = 0$)	1.6007	2.9119	0	11.0855	54.0277	55.5281
Constant deterioration ($y = 1$)	1.6413	2.8584	3.3386	11.0080	53.4555	60.1844
Without amelioration ($\alpha = 0$)	1.6347	3.4591	2.3103	8.9970	145.8840	49.1429
Constant amelioration ($\beta = 1$)	1.7071	3.0361	1.7159	10.7376	69.1350	56.4441

Table 2: Comparison of optimal values for different cases in Example 2

Case	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
Without deterioration ($x = 0$)	1.6007	2.9119	0	11.0855	54.0277	55.5281
Constant deterioration ($y = 1$)	1.7134	2.6392	1	10.5824	47.5742	63.4685
Without amelioration ($\alpha = 0$)	1.7701	3.4019	1	8.8423	133.9590	49.9418
Constant amelioration ($\beta = 1$)	1.7438	3.0113	1	10.6337	68.0377	56.6645

Table 3: Comparison of optimal values for different cases in Example 3

Case	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
Without deterioration ($x = 0$)	1.6007	2.9119	0	11.0855	54.0277	55.5281
Constant deterioration ($y = 1$)	1.7023	2.7153	2.9415	10.7516	50.0007	63.4088
Without amelioration ($\alpha = 0$)	1.7640	3.4054	1.6431	8.8515	134.6040	50.4698
Constant amelioration ($\beta = 1$)	1.7555	3.0019	1.1319	10.5937	67.5910	57.1425

Table 4: Comparison of optimal values for different cases in Example 4

Case	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
Without deterioration ($x = 0$)	1.6007	2.9119	0	11.0855	54.0277	55.5281
Constant deterioration ($y = 1$)	1.7146	2.5915	1	10.4635	45.9691	64.7542
Without amelioration ($\alpha = 0$)	1.8042	3.3805	1	8.7859	130.1400	50.6285
Constant amelioration ($\beta = 1$)	1.7599	2.9980	1	10.5774	67.4077	57.1496

7. SENSITIVITY ANALYSIS

In this section, sensitivity analysis is performed to investigate the effect of the parameters on optimal solution. Here, it is performed for Example 1 in Tables 5, 6 and 7.

Table 5: Sensitivity analysis of *Example 1*

Parameter	% change	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
u	-20	1.7393	2.9319	1.4409	9.3031	51.7050	56.4013
	-10	1.7023	2.9090	1.5079	10.1186	52.8530	57.4115
	10	1.6313	2.8636	1.6336	11.7515	55.1933	59.3921
	20	1.5971	2.8411	1.6929	12.5687	56.3869	60.3635
v	-20	1.7195	2.9532	1.5178	10.5082	52.0139	57.1087
	-10	1.6924	2.9192	1.5453	10.7215	53.0243	57.7654
	10	1.6412	2.8545	1.5979	11.1479	54.9876	59.0379
	20	1.6169	2.8237	1.6232	11.3610	55.9415	59.6550

Table 6: Sensitivity analysis of *Example 1*

Parameter	% change	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
w	-20	1.7201	2.9858	1.5471	10.7826	52.3279	57.1643
	-10	1.6922	2.9339	1.5596	10.8587	53.1980	57.8016
	10	1.6421	2.8424	1.5842	11.0108	54.7872	58.9877
	20	1.6194	2.8018	1.5961	11.0868	55.5193	59.5428
α	-20	1.6662	2.9661	1.6771	10.5459	63.5100	56.9335
	-10	1.6667	2.9249	1.6225	10.7402	58.4802	57.6869
	10	1.6652	2.8500	1.5251	11.1295	50.0280	59.1006
	20	1.6634	2.8159	1.4816	11.3245	46.4464	59.7667
β	-20	1.7138	3.0652	1.7447	10.6949	72.4153	56.0636
	-10	1.6921	2.9768	1.6582	10.8195	62.8120	57.2189
	10	1.6376	2.7957	1.4874	11.0422	46.1321	59.6182
	20	1.6070	2.7068	1.4058	11.1432	39.1857	60.8365
x	-20	1.6663	2.8863	1.2931	10.9348	54.0154	58.1293
	-10	1.6663	2.8863	1.4403	10.9348	54.0154	58.2765
	10	1.6663	2.8863	1.6911	10.9348	54.0154	58.5274
	20	1.6663	2.8863	1.7999	10.9348	54.0154	58.6361
y	-20	1.6687	2.8893	1.2495	10.9226	54.0755	58.0848
	-10	1.6675	2.8878	1.4183	10.9288	54.0456	58.2541
	10	1.6651	2.8848	1.7134	10.9404	53.9854	58.5502
	20	1.6639	2.8833	1.8448	10.9458	53.9552	58.6820
λ	-20	2.7069	3.4182	0.0765	1.4116	55.1252	51.2992
	-10	2.0334	3.0601	1.1312	6.1603	55.9341	55.8311
	10	1.4169	2.7796	1.8510	15.7164	51.9997	60.1363
	20	1.2318	2.7065	2.0549	20.5017	50.1814	61.3855
γ	-20	1.6802	2.8789	1.5819	10.8929	53.9202	58.7616
	-10	1.6726	2.8830	1.5866	10.9163	53.9772	58.5748
	10	1.6610	2.8889	1.5467	10.9496	54.0417	58.2595
	20	1.6565	2.8910	1.5156	10.9618	54.0599	58.1263
C_1	-20	1.4918	2.6609	1.4979	10.9256	48.7089	51.2019
	-10	1.5827	2.7785	1.5361	10.9304	51.4493	54.8781
	10	1.7439	2.9861	1.6057	10.9388	56.4318	61.8137
	20	1.8163	3.0791	1.6375	10.9425	58.7174	65.1109
C_2	-20	1.8626	2.9728	1.4471	10.9189	54.9908	56.4144
	-10	1.7635	2.9286	1.5066	10.9267	54.5595	57.4561
	10	1.5707	2.8456	1.6425	10.9430	53.3691	59.2747
	20	1.4765	2.8064	1.7177	10.9513	52.6303	60.0589
C_3	-20	1.6675	2.8864	1.2944	10.8973	54.2288	58.1320
	-10	1.6669	2.8863	1.4409	10.9181	54.1105	58.2778
	10	1.6659	2.8862	1.6906	10.9484	53.9376	58.5264
	20	1.6655	2.8862	1.7989	10.9598	53.8725	58.6343

Table 7: Sensitivity analysis of *Example 1*

Parameter	% change	T_2^*	T^*	ξ^*	S_1^*	S_2^*	TC^*
C_4	-20	1.6314	2.9773	1.7362	10.9533	56.4919	56.7294
	-10	1.6514	2.9293	1.6478	10.9436	55.2720	57.5999
	10	1.6774	2.8473	1.5059	10.9266	52.7656	59.1653
	20	1.6856	2.8115	1.4475	10.9190	51.5450	59.8793
h	-20	1.6704	2.9236	1.6135	10.9397	55.0452	57.5851
	-10	1.6684	2.9048	1.5926	10.9372	54.5309	57.9999
	10	1.6641	2.8682	1.5517	10.9323	53.5003	58.8104
	20	1.6619	2.8504	1.5317	10.9298	52.9861	59.2065
a	-20	1.9613	3.0417	1.3964	10.9120	56.2131	56.9421
	-10	1.8050	2.9582	1.4861	10.9240	55.1337	57.7671
	10	1.5418	2.8232	1.6548	10.9444	52.8893	58.8968
	20	1.4288	2.7669	1.7352	10.9532	51.7750	59.2569

8. RESULTS AND MANAGERIAL INSIGHTS

The following results are obtained from sensitivity analysis:

(a) The optimal total cost per unit time ($TC^*(T_2^*, T^*, \xi^*)$) increases with the increase in the value of the parameters u, v, w, α, β, x, y, λ, C_1, C_2, C_3, C_4, h and a while $TC^*(T_2^*, T^*, \xi^*)$ decreases with the decrease in the value of the parameters u, v, w, α, β, x, y, λ, C_1, C_2, C_3, C_4, h and a. $TC^*(T_2^*, T^*, \xi^*)$ decreases with the increase in the value of the parameter γ while $TC^*(T_2^*, T^*, \xi^*)$ increases with the decrease in the value of the parameter γ.

(b) The optimal preservation technology investment (ξ^*) increases with the increase in the value of the parameters u, v, w, x, y, λ, C_1, C_2, C_3 and a while ξ^* decreases with the decrease in the value of the parameters u, v, w, x, y, λ, C_1, C_2, C_3 and a. ξ^* decreases with the increase in the value of the parameters α, β, γ, C_4 and h while ξ^* increases with the decrease in the value of the parameters α, β, γ, C_4 and h.

(c) The optimal production time (T_2^*) increases with the increase in the value of the parameters C_1 and C_4 while T_2^* decreases with the decrease in the value of the parameters α, C_1 and C_4. T_2^* decreases with the increase in the value of the parameters u, v, w, α, β, y, λ, γ, C_2, C_3, h and a while T_2^* increases with the decrease in the value of the parameters u, v, w, β, y, λ, γ, C_2, C_3, h and a.

(d) The optimal cycle length (T^*) increases with the increase in the value of the parameters γ and C_1 while T^* decreases with the decrease in the value of the parameters γ and C_1. T^* decreases with the increase in the value of the parameters u, v, w, α, β, y, λ, C_2, C_3, C_4, h and a while T^* increases with the decrease in the value of the parameters u, v, w, α, β, y, λ, C_2, C_3, C_4, h and a. T^* remain unchanged for any change in the value of the parameter x.

(e) The optimal inventory level (S_1^*) increases with increase in the value of the parameters u, v, w, α, β, y, λ, γ, C_1, C_2, C_3 and a while S_1^* decreases with decrease in the value of the parameters u, v, w, α, β, y, λ, γ, C_1, C_2, C_3 and a. S_1^* decreases with increase in the value of the parameters C_4 and h

while S_1^* increases with decrease in the value of the parameters C_4 and h. S_1^* remain unchanged for any change in the value of the parameter x.

(f) The optimal inventory level (S_2^*) increases with increase in the value of the parameters u, v, w, γ and C_1 while S_2^* decreases with decrease in the value of the parameters u, v, w, γ and C_1. S_2^* decreases with increase in the value of the parameters α, β, y, λ, C_2, C_3, C_4, h and a while S_2^* increases with decrease in the value of the parameters α, β, y, λ, C_2, C_3, C_4, h and a. S_2^* remains unchanged for any change in the value of the parameter x.

The following managerial insights can be obtained from sensitivity analysis :

a. When the holding cost per unit (h) increases then the optimal production time (T_2^*), cycle length (T^*), preservation technology cost (ξ^*), inventory level (S_1^* and S_2^*) decrease while total cost per unit time ($TC^*(T_2^*, T^*, \xi^*)$) increases. This result indicates that with rise in the holding cost, the manufacturing company will shorten the length of production time as well as cycle length, decrease the preservation technology cost and manufacture a smaller quantity to keep the inventory level as low as possible. Hence, the manufacturing company is suggested to produce and store smaller lot size to reduce excessive holding cost.

b. Increment in the deterioration parameter (x) and the deterioration cost per unit (C_3) increases the optimal preservation technology cost (ξ^*) and total cost per unit time ($TC^*(T_2^*, T^*, \xi^*)$) which is an obvious result. Actually any rise in the deterioration parameter increases the deterioration rate, which further increases the total cost. Hence the manufacturing company will increase the investment in preservation technology to reduce the deterioration effect. Thus investing necessary amount on better and useful preservation methods is always important for any manufacturing company to avoid losses due to deterioration effect.

c. When the value of the demand parameters u, v and w increases then the optimal production time (T_2^*) and cycle length (T^*) decreases while preservation technology cost (ξ^*), inventory level (S_1^* and S_2^*) and total cost per unit time ($TC^*(T_2^*, T^*, \xi^*)$) increases. Increase in the parameters u, v and w will lead to increase in the demand rate. Hence, the manufacturing company will increase the production level of the items. Consequently, the company will also increase the investment in preservation technology to lower the deterioration rate. As a result, the total cost increases. With the increase in the parameters u, v and w, the company will decrease the production time and cycle length to store the items for a lesser amount of time.

d. When the production rate parameter (λ) increases then the optimal production time (T_2^*), cycle length (T^*) and inventory level (S_2^*) decreases but preservation technology cost (ξ^*), inventory level (S_1^*) and total cost ($TC^*(T_2^*, T^*, \xi^*)$) increases. Increment in the production rate parameter (λ) increases the production rate. As a result, the production level of the stock increases while production time decreases. It is due to the fact that it require less time to produce more items when the production rate is high. If the items are produced at a high rate it may increase the storage cost and chance of the items to deteriorate. So, the manufacturing company must invest a certain amount in preservation technology to reduce the deterioration effect. Hence the total cost increases.

e. When the production cost per unit (C_2) increases then the optimal production time (T_2^*), cycle length (T^*) and inventory level (S_2^*) decreases while preservation technology cost (ξ^*), inventory level (S_1^*) and total cost per unit time ($TC^*(T_2^*, T^*, \xi^*)$) increases. It means that due to higher production cost, the manufacturing company will reduce the production time to reduce the produced inventory level. Thus, if the company reduces the production cost, then the total cost automatically decreases.

f. If the parameter γ increases then the optimal preservation technology investment (ξ^*) decreases. A higher value of the parameter γ decreases the investment in preservation technology. Thus, when the value of γ is high, the manufacturing company needs to invest less money in preservation technology to reduce deterioration. On the other hand, when the parameter γ decreases, then the values of the optimal preservation technology investment (ξ^*) increase first and then decrease. A smaller value of the parameter γ represents an insignificant improvement in reducing the effect of deterioration with the investment in preservation technology. This results in a decrease in investment in preservation technology. Thus, the manufacturing company reduces the preservation technology cost when γ is significantly small.

9. CONCLUSIONS

In this article, a production inventory model for ameliorating and deteriorating items is discussed considering quadratic demand under the effect of preservation technology. Two different production rates are considered such that the initial production rate switches to another rate of production after some interval of time. Also, two different preservation technology rates are considered in the developed model. The amelioration and deterioration rates are represented by two parameter Weibull distribution. The optimal production time, cycle length, preservation technology cost are obtained from the model such that the total cost per unit time is minimized.

The model is first mathematically formulated. Then some theoretical results are proved regarding the convexity of total cost function. Several solved numerical examples are provided to illustrate the model while the convexity is shown graphically. Through sensitivity analysis it can be observed that the manufacturing company must decrease the holding cost to lower the total cost. The manufacturing company will further reduce the production time, cycle length and preservation technology investment when the holding cost is high. The manufacturing company will increase the investment in preservation methods when the deterioration rate is high. Such preservation techniques will decrease the effect of deterioration significantly. Any rise in the demand parameters will increase the demand rate significantly. In such case, the company will increase the production level of the items. Increment in the production level will ensure higher investment in preservation technology by the company. If the manufacturing company lowers the ordering cost, production cost, deterioration cost and amelioration cost then the total cost automatically decreases.

In this study, the shortages are not considered. Thus, this model can be extended by allowing shortages. This model can be developed further by considering three parameter Weibull distribution amelioration and deterioration rate, time dependent holding cost, trade credit policy, advanced payment etc.

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Appendix A.

The total cost function $TC(T_2, T, \xi)$ given by equation (9) can be expressed in the form,

$$TC(T_2, T, \xi) = \frac{g_1(T_2, T, \xi)}{T}$$

where,

$$\begin{aligned}
g_1(T_2, T, \xi) = {} & C_1 + C_2\lambda\Big[(1-a)uT_1 + (1-a)\frac{v}{2}T_1^2 + (1-a)\frac{w}{3}T_1^3 + a(uT_2 + \frac{v}{2}T_2^2 \\
& + \frac{w}{3}T_2^3)\Big] + C_3\Bigg[\{1-m(\xi)\}xy(\lambda-1)\Big\{\frac{u}{y+1}T_1^{y+1} + \frac{v}{2(y+2)}T_1^{y+2} \\
& + \frac{w}{3(y+3)}T_1^{y+3}\Big\} + \{1-m(\xi)\}xy(\lambda-1)\Big[\frac{u}{y(y+1)}\{yaT_2^{y+1} \\
& + (1-a)(y+1)T_1T_2^y + (a-y-1)T_1^{y+1}\} + \frac{v}{2y(y+2)}\{ayT_2^{y+2} + \\
& (1-a)(y+2)T_1^2T_2^y + (2a-y-2)T_1^{y+2}\} + \frac{w}{3y(y+3)}\{yaT_2^{y+3} + \\
& (1-a)(y+3)T_1^3T_2^y + (3a-y-3)T_1^{y+3}\}\Big] + \{1-m(\xi)\}xy\Big[\frac{u}{y(y+1)}\{T^{y+1} \\
& - (y+1)TT_2^y + yT_2^{y+1}\} + \frac{v}{2y(y+2)}\{2T^{y+2} - (y+2)T^2T_2^y + yT_2^{y+2}\} + \\
& \frac{w}{3y(y+3)}\{3T^{y+3} - (y+3)T^3T_2^y + yT_2^{y+3}\}\Big]\Bigg] + C_4\Bigg[\alpha\beta(\lambda-1)\Big[\frac{u}{\beta+1}T_1^{\beta+1} \\
& + \frac{v}{2(\beta+2)}T_1^{\beta+2} + \frac{w}{3(\beta+3)}T_1^{\beta+3}\Big] + \alpha\beta(\lambda-1)\Big[\frac{u}{(\beta+1)\beta}\{\beta aT_2^{\beta+1} + \\
& (1-a)(\beta+1)T_1T_2^\beta + (a-\beta-1)T_1^{\beta+1}\} + \frac{v}{2(\beta+2)\beta}\{\beta aT_2^{\beta+2} + \\
& (1-a)(\beta+2)T_1^2T_2^\beta + (2a-\beta-2)T_1^{\beta+2}\} + \frac{w}{3(\beta+3)\beta}\{\beta aT_2^{\beta+3} + \\
& (1-a)(\beta+3)T_1^3T_2^\beta + (3a-\beta-3)T_1^{\beta+3}\}\Big] + \alpha\beta\Big[\frac{u}{\beta(\beta+1)}\{T^{\beta+1} \\
& - (\beta+1)TT_2^\beta + \beta T_2^{\beta+1}\} + \frac{v}{2\beta(\beta+2)}\{2T^{\beta+2} - (\beta+2)T^2T_2^\beta + \beta T_2^{\beta+2}\} + \\
& \frac{w}{3\beta(\beta+3)}\{3T^{\beta+3} - (\beta+3)T^3T_2^\beta + \beta T_2^{\beta+3}\}\Big]\Bigg] + h\Bigg[(\lambda-1)\Big[\frac{u}{2}T_1^2 + \frac{v}{6}T_1^3 \\
& + \frac{w}{12}T_1^4 - \{1-m(\xi)\}xy\Big\{\frac{u}{(y+1)(y+2)}T_1^{y+2} + \frac{v}{2(y+2)(y+3)}T_1^{y+3} \\
& + \frac{w}{3(y+3)(y+4)}T_1^{y+4}\Big\} + \alpha\beta\Big\{\frac{u}{(\beta+1)(\beta+2)}T_1^{\beta+2} + \frac{v}{2(\beta+2)(\beta+3)}T_1^{\beta+3} \\
& + \frac{w}{3(\beta+3)(\beta+4)}T_1^{\beta+4}\Big\}\Big] + (\lambda-1)\Big[\frac{u}{2}(T_2-T_1)\{a(T_2+T_1) + 2T_1 - 2aT_1\} \\
& + \frac{v}{6}\{a(T_2^3 - T_1^3) + 3(T_2-T_1)(T_1^2 - aT_1^2)\} + \frac{w}{12}\{a(T_2^4 - T_1^4) +
\end{aligned}$$

$$(T_2 - T_1)(4T_1^3 - 4aT_1^3)\} + \{1 - m(\xi)\}x\left\{\frac{u}{y+1}\{(1-a)T_1^{y+1}(T_2 - T_1)+\right.$$
$$(a-1)T_1(T_2^{y+1} - T_1^{y+1}) - \frac{ya}{y+2}(T_2^{y+2} - T_1^{y+2})\}+$$
$$\frac{v}{2(y+2)}\{2(1-a)T_1^{y+2}(T_2 - T_1) + (a-1)\left(\frac{y+2}{y+1}\right)T_1^2(T_2^{y+1} - T_1^{y+1})-$$
$$\frac{ya}{y+3}(T_2^{y+3} - T_1^{y+3})\} + \frac{w}{3(y+3)}\{3(1-a)T_1^{y+3}(T_2 - T_1) - \frac{ya}{y+4}(T_2^{y+4} - T_1^{y+4})$$
$$\left. + (a-1)\left(\frac{y+3}{y+1}\right)T_1^3(T_2^{y+1} - T_1^{y+1})\}\right\} - \alpha\left\{\frac{u}{\beta+1}\{(1-a)T_1^{\beta+1}(T_2 - T_1)\right.$$
$$+ (a-1)T_1(T_2^{\beta+1} - T_1^{\beta+1}) - \frac{\beta a}{\beta+2}(T_2^{\beta+2} - T_1^{\beta+2})\}+$$
$$\frac{v}{2(\beta+2)}\{2(1-a)T_1^{\beta+2}(T_2 - T_1) + (a-1)\left(\frac{\beta+2}{\beta+1}\right)T_1^2(T_2^{\beta+1} - T_1^{\beta+1})$$
$$- \frac{\beta a}{\beta+3}(T_2^{\beta+3} - T_1^{\beta+3})\} + \frac{w}{3(\beta+3)}\{3(1-a)T_1^{\beta+3}(T_2 - T_1)+$$
$$\left.(a-1)\left(\frac{\beta+3}{\beta+1}\right)T_1^3(T_2^{\beta+1} - T_1^{\beta+1}) - \frac{\beta a}{\beta+4}(T_2^{\beta+4} - T_1^{\beta+4})\}\right\}\Bigg]+$$
$$u\left\{\frac{1}{2}(T - T_2)^2 + \frac{\{1-m(\xi)\}x}{(y+1)(y+2)}\{y(T^{y+2} - T_2^{y+2}) - (y+2)T_2T(T^y - T_2^y)\}\right.$$
$$\left. - \frac{\alpha}{(\beta+1)(\beta+2)}\{\beta(T^{\beta+2} - T_2^{\beta+2}) - (\beta+2)TT_2(T^\beta - T_2^\beta)\}\right\}+$$
$$v\left\{\frac{1}{6}(2T^3 - 3T^2T_2 + T_2^3) + \frac{\{1-m(\xi)\}x}{2(y+2)}\left\{\frac{2y(y+2)}{(y+1)(y+3)}T^{y+3} - 2T^{y+2}T_2\right.\right.$$
$$\left. + \left(\frac{y+2}{y+1}\right)T^2T_2^{y+1} - \frac{y}{y+3}T_2^{y+3}\right\} - \frac{\alpha}{2(\beta+2)}\left\{\frac{2\beta(\beta+2)}{(\beta+1)(\beta+3)}T^{\beta+3}-\right.$$
$$\left.\left. 2T^{\beta+2}T_2 + \left(\frac{\beta+2}{\beta+1}\right)T^2T_2^{\beta+1} - \frac{\beta}{\beta+3}T_2^{\beta+3}\right\}\right\} + w\left\{\frac{1}{12}(3T^4-\right.$$
$$4T^3T_2 + T_2^4) + \frac{\{1-m(\xi)\}x}{3(y+3)}\left\{\frac{3y(y+3)}{(y+1)(y+4)}T^{y+4} - 3T^{y+3}T_2+\right.$$
$$\left.\left(\frac{y+3}{y+1}\right)T^3T_2^{y+1} - \frac{y}{y+4}T_2^{y+4}\right\} - \frac{\alpha}{3(\beta+3)}\left\{\frac{3\beta(\beta+3)}{(\beta+1)(\beta+4)}T^{\beta+4}\right.$$
$$\left.\left. - 3T^{\beta+3}T_2 + \left(\frac{\beta+3}{\beta+1}\right)T^3T_2^{\beta+1} - \frac{\beta}{\beta+4}T_2^{\beta+4}\right\}\right\}\Bigg] + \xi T \tag{A.1}$$

Appendix B.

The total cost function $TC(T_2, T, \xi)$ given by equation (9) can be expressed in the form,

$$TC(T_2, T, \xi) = \frac{1}{T}\left[g_2(T_2, T) + \{1 - m(\xi)\}g_3(T_2, T) + \xi T\right]$$

where,

$$
\begin{aligned}
g_2(T_2,T) = {} & C_1 + C_2\lambda\Big[(1-a)uT_1 + (1-a)\frac{v}{2}T_1^2 + (1-a)\frac{w}{3}T_1^3 \\
& + a(uT_2 + \frac{v}{2}T_2^2 + \frac{w}{3}T_2^3)\Big] + C_4\Big[\alpha\beta(\lambda-1)\Big[\frac{u}{\beta+1}T_1^{\beta+1} \\
& + \frac{v}{2(\beta+2)}T_1^{\beta+2} + \frac{w}{3(\beta+3)}T_1^{\beta+3}\Big] + \alpha\beta(\lambda-1)\Big[\frac{u}{(\beta+1)\beta}\{\beta aT_2^{\beta+1} + \\
& (1-a)(\beta+1)T_1T_2^{\beta} + (a-\beta-1)T_1^{\beta+1}\} + \frac{v}{2(\beta+2)\beta}\{\beta aT_2^{\beta+2} + \\
& (1-a)(\beta+2)T_1^2T_2^{\beta} + (2a-\beta-2)T_1^{\beta+2}\} + \frac{w}{3(\beta+3)\beta}\{\beta aT_2^{\beta+3} + \\
& (1-a)(\beta+3)T_1^3T_2^{\beta} + (3a-\beta-3)T_1^{\beta+3}\}\Big] + \alpha\beta\Big[\frac{u}{\beta(\beta+1)}\{T^{\beta+1} \\
& - (\beta+1)TT_2^{\beta} + \beta T_2^{\beta+1}\} + \frac{v}{2\beta(\beta+2)}\{2T^{\beta+2} - (\beta+2)T^2T_2^{\beta} + \\
& \beta T_2^{\beta+2}\} + \frac{w}{3\beta(\beta+3)}\{3T^{\beta+3} - (\beta+3)T^3T_2^{\beta} + \beta T_2^{\beta+3}\}\Big]\Big] \\
& + h\Big[(\lambda-1)\Big[\frac{u}{2}T_1^2 + \frac{v}{6}T_1^3 + \frac{w}{12}T_1^4 + \alpha\beta\Big\{\frac{u}{(\beta+1)(\beta+2)}T_1^{\beta+2} \\
& + \frac{v}{2(\beta+2)(\beta+3)}T_1^{\beta+3} + \frac{w}{3(\beta+3)(\beta+4)}T_1^{\beta+4}\Big\}\Big] + \\
& (\lambda-1)\Big[\frac{u}{2}(T_2-T_1)\{a(T_2+T_1) + 2T_1 - 2aT_1\} \\
& + \frac{v}{6}\{a(T_2^3-T_1^3) + 3(T_2-T_1)(T_1^2-aT_1^2)\} + \frac{w}{12}\{a(T_2^4-T_1^4) + \\
& (T_2-T_1)(4T_1^3-4aT_1^3)\} - \alpha\Big\{\frac{u}{\beta+1}\{(1-a)T_1^{\beta+1}(T_2-T_1) \\
& + (a-1)T_1(T_2^{\beta+1}-T_1^{\beta+1}) - \frac{\beta a}{\beta+2}(T_2^{\beta+2}-T_1^{\beta+2})\} + \\
& \frac{v}{2(\beta+2)}\{2(1-a)T_1^{\beta+2}(T_2-T_1) + (a-1)\left(\frac{\beta+2}{\beta+1}\right)T_1^2(T_2^{\beta+1}-T_1^{\beta+1}) \\
& - \frac{\beta a}{\beta+3}(T_2^{\beta+3}-T_1^{\beta+3})\} + \frac{w}{3(\beta+3)}\{3(1-a)T_1^{\beta+3}(T_2-T_1) + \\
& (a-1)\left(\frac{\beta+3}{\beta+1}\right)T_1^3(T_2^{\beta+1}-T_1^{\beta+1}) - \frac{\beta a}{\beta+4}(T_2^{\beta+4}-T_1^{\beta+4})\}\Big\}\Big] \\
& + u\Big\{\frac{1}{2}(T-T_2)^2 - \frac{\alpha}{(\beta+1)(\beta+2)}\{\beta(T^{\beta+2}-T_2^{\beta+2}) - \\
& (\beta+2)TT_2(T^{\beta}-T_2^{\beta})\}\Big\} + v\Big\{\frac{1}{6}(2T^3-3T^2T_2+T_2^3) - \\
& \frac{\alpha}{2(\beta+2)}\Big\{\frac{2\beta(\beta+2)}{(\beta+1)(\beta+3)}T^{\beta+3} - 2T^{\beta+2}T_2 + \left(\frac{\beta+2}{\beta+1}\right)T^2T_2^{\beta+1}
\end{aligned}
$$

$$- \frac{\beta}{\beta+3} T_2^{\beta+3} \Big\} \Big\} + w \Big\{ \frac{1}{12}(3T^4 - 4T^3 T_2 + T_2^4) -$$
$$\frac{\alpha}{3(\beta+3)} \Big\{ \frac{3\beta(\beta+3)}{(\beta+1)(\beta+4)} T^{\beta+4} - 3T^{\beta+3}T_2 + \left(\frac{\beta+3}{\beta+1}\right) T^3 T_2^{\beta+1}$$
$$- \frac{\beta}{\beta+4} T_2^{\beta+4} \Big\} \Big\} \Big] \tag{B.1}$$

$$g_3(T_2, T) = C_3 \Big[xy(\lambda-1) \Big\{ \frac{u}{y+1} T_1^{y+1} + \frac{v}{2(y+2)} T_1^{y+2} + \frac{w}{3(y+3)} T_1^{y+3} \Big\}$$
$$+ xy(\lambda-1) \Big[\frac{u}{y(y+1)} \{ yaT_2^{y+1} + (1-a)(y+1)T_1 T_2^y + (a-y-1)T_1^{y+1} \}$$
$$+ \frac{v}{2y(y+2)} \{ ayT_2^{y+2} + (1-a)(y+2)T_1^2 T_2^y + (2a-y-2)T_1^{y+2} \} +$$
$$\frac{w}{3y(y+3)} \{ yaT_2^{y+3} + (1-a)(y+3)T_1^3 T_2^y + (3a-y-3)T_1^{y+3} \} \Big] +$$
$$xy \Big[\frac{u}{y(y+1)} \{ T^{y+1} - (y+1)TT_2^y + yT_2^{y+1} \} + \frac{v}{2y(y+2)} \{ 2T^{y+2}$$
$$- (y+2)T^2 T_2^y + yT_2^{y+2} \} + \frac{w}{3y(y+3)} \{ 3T^{y+3} - (y+3)T^3 T_2^y + yT_2^{y+3} \} \Big] \Big]$$
$$+ h \Big[- (\lambda-1)xy \Big\{ \frac{u}{(y+1)(y+2)} T_1^{y+2} + \frac{v}{2(y+2)(y+3)} T_1^{y+3}$$
$$+ \frac{w}{3(y+3)(y+4)} T_1^{y+4} \Big\} + (\lambda-1)x \Big\{ \frac{u}{y+1} \{ (1-a)T_1^{y+1}(T_2 - T_1) +$$
$$(a-1)T_1(T_2^{y+1} - T_1^{y+1}) - \frac{ya}{y+2}(T_2^{y+2} - T_1^{y+2}) \} +$$
$$\frac{v}{2(y+2)} \{ 2(1-a)T_1^{y+2}(T_2 - T_1) + (a-1)\left(\frac{y+2}{y+1}\right) T_1^2 (T_2^{y+1} - T_1^{y+1}) -$$
$$\frac{ya}{y+3}(T_2^{y+3} - T_1^{y+3}) \} + \frac{w}{3(y+3)} \{ 3(1-a)T_1^{y+3}(T_2 - T_1) -$$
$$\frac{ya}{y+4}(T_2^{y+4} - T_1^{y+4}) + (a-1)\left(\frac{y+3}{y+1}\right) T_1^3 (T_2^{y+1} - T_1^{y+1}) \} \Big\} +$$
$$\frac{ux}{(y+1)(y+2)} \{ y(T^{y+2} - T_2^{y+2}) - (y+2)T_2 T(T^y - T_2^y) \}$$
$$+ \frac{vx}{2(y+2)} \Big\{ \frac{2y(y+2)}{(y+1)(y+3)} T^{y+3} - 2T^{y+2}T_2 + \left(\frac{y+2}{y+1}\right) T^2 T_2^{y+1} -$$
$$\frac{y}{y+3} T_2^{y+3} \Big\} + \frac{wx}{3(y+3)} \Big\{ \frac{3y(y+3)}{(y+1)(y+4)} T^{y+4} - 3T^{y+3}T_2 +$$
$$\left(\frac{y+3}{y+1}\right) T^3 T_2^{y+1} - \frac{y}{y+4} T_2^{y+4} \Big\} \Big] \tag{B.2}$$