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IMPACT OF TRIGONOMETRIC SIMILARITY MEASURES FOR PYTHAGOREAN FUZZY SETS AND THEIR APPLICATIONS

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Abstract: In fuzzy set theory, the similarity measure is a significant device that measures the degree of correlation between two objects. An extension to intuitionistic fuzzy sets (IFS), Pythagorean fuzzy sets (PFS) have been widely employed in numerous disciplines. It is critical to investigate the similarity measure of PFS. The study proposes the trigonometric function to suggest new similarity measures of PFS to handle the uncertainty that the existing similarity measures are unable to differentiate. Firstly, axiomatic descriptions of similarity measures for the proposed measures are proved. Then, an example is used to validate the proposed measures. Application to pattern recognition and medical diagnosis is also discussed in real-life scenarios. The validity of the suggested similarity measures is proved by comparing the results to the effectiveness of current equivalent similarity measures. Finally, a comparative study of these real-life examples reveals that the novel similarity measures are more flexible and dependable than the current similarity measures in dealing with various real application difficulties.

Keywords: Intuitionistic fuzzy sets (IFS), Pythagorean fuzzy sets (PFS), similarity measures, pattern recognition, medical diagnosis.

MSC: 03E72, 94D05, 03B52.

1. INTRODUCTION

Decision-making is the process of choosing a course of action or selecting a solution from among various alternatives. It is a fundamental cognitive process that individuals, groups, or organizations engage in to address problems, achieve goals, or respond to opportunities. Decision-making involves assessing information, evaluating options, considering potential outcomes, and making a choice whereas multi-criteria decision-making (MCDM) is a specialized approach to decision-making that considers multiple criteria or factors simultaneously. In many real-life scenarios, decisions involve multiple and often conflicting criteria that need to be considered. MCDM methods provide a structured framework for systematically analysing and evaluating alternatives based on multiple criteria, helping decision-makers make informed and balanced choices. TOPSIS (Technology for Order of Preference by Similarity to Ideal Solution) is a valuable method for MCDM issues in the real world proposed by Hwang & Yoon [1]. TOPSIS provides a practical and effective framework for multi-criteria decision-making, enabling decision-makers to systematically evaluate and rank alternatives in complex decision environments [2,3,4].

Zadeh [5] defined fuzzy sets for non-statistical modelling and incomplete information data [6]. Since its inception, fuzzy sets become an interesting area for researchers for decision-making purposes in various domains of the universe. The generalized version of the fuzzy set known as IFS given by Atanassov [7] by including membership grade (MG), non-membership grade (NMG) and hesitation grade (HG) or intuitionistic index. In the past many decades, researchers have proposed entropies based on IFS, which have applications in the fields of decision-making, optimization, pattern recognition, etc. Many new measures of IF entropy that establish a mathematical relationship between the entropy for FS and IFS have been proposed by many researchers [8, 9, 10, 11, 12]. These entropies satisfy all the axioms for the IF entropy and can be applied in many areas of research.

Yager [13] proposed an extension of IFS called the Pythagorean fuzzy set. The new concept derived from IFS is a quadratic form of the IFS, which means that the new modality of fuzzy set has a larger range of the change of variables and therefore has more potential in indicating the probability of various objects. The new form of a fuzzy set is also extended into different forms, such as interval-valued Pythagorean fuzzy set [14], decision-making [15, 16, 17], and some other applications [18]. PFS features have been developed by many scientists in decision-making situations with numerous attributes. the extensions of TOPSIS methods, optimization techniques to MCDM problems with Pythagorean and hesitant fuzzy sets were proposed by Zhang & Xu [19]. Many researchers [20, 21, 22, 23, 24, 25, 26, 27, 28] simplified the concept of PFS and established various Pythagorean fuzzy operators in solving MCDM problems. Zhang [29] and Zhang et al. [30] considered a novel approach based on similarity measure for Pythagorean fuzzy MCDM. Researchers [31, 32, 33, 34, 35, 36] generalized Pythagorean fuzzy aggregation operators and proposed norms for information measures with applications in MCDM problems.

The similarity measure is a valuable means to establish the degree of similarity between the two sets. These measures have been used by researchers across various domains. Mohd & Abdullah [37] introduced the cosine similarity measure and Euclidean distance measure for PFSs. Ejegwa [38] surveyed three grades of PFSs and proposed that these new similarity measures for PFSs with more consistent and efficient results. Cotangent similarity measure by proposed by Immaculate et al. [39] for rough IFS by considering a medical diagnosis problem to verify the proposed measure. Some new similarity measures for PFS have been proposed by Ejegwa [34] and applied to decision-making problems. New cosine similarity measures for imprecise sets and IFS have been proposed by Shi & Ye [41] and Ye [42, 43]. Cotangent similarity measures are defined by Maoying [44] and Rajarajeswari & Uma [45]. Many researchers [46, 47] developed various similarity measures between FSs and IFSs, which have applications to tackle the problems of pattern recognition and medical diagnosis. Mao & Zhang [48] proposed similarity measure for group decision-making problems and is a geometric distance measure between of IFSs. Hung [49] developed similarity measure based on the likelihood of IFSs for bacteria classification problems. Ejegwa & Agbetayo [50] introduced a novel similarity-distance technique with a better performance rating. A comparative analysis was presented to showcase the advantages of the novel similarity-distance over similar existing approaches. Some attributes of the similarity-distance technique were presented. Ejegwa & Onyeke [51] developed a novel distance measure between PFSs and its weighted version to enhance reliability in terms of applications. To show the suitability of the measures, they characterized the distance measure and its weighted version with some results. In addition, certain decision-making problems involving cases of pattern recognition and disease diagnosis were discussed based on the measures. Some novel distance measures for PFSs by incorporating the conventional parameters that describe PFS were proposed by Ejegwa & Awolola [52]. A numerical example to illustrate the validity and applicability of the distance measures for PFS was also discussed. Ejegwa [53] formulated Modified Zhang & Xu’s distance measure for Pythagorean fuzzy sets and discussed its application to pattern recognition problems. Measures of similarity between PFS are an important tool for MADM Problems, medical diagnosis, decision-making, pattern recognition, machine learning, image processing, and in other real-world problems. Recently, some researchers have been engaged in the development of similarity measure of PFS and its applications in MCDM [54, 55, 56], clustering [57], medical diagnosis [58], admission process [59], pattern recognition [60], transportation problem [61], waste-to-energy technology selection [62].

Research Gap and Motivation

The study of similarity measures for Pythagorean fuzzy sets represents a relatively new and evolving area within the broader field of fuzzy set theory. Despite the growing interest in PFS, notable research gaps create opportunities for further investigation. Firstly, the current literature lacks a thorough examination of the performance of similarity measures in specific applications. Different domains may require tailored similarity measures, and understanding their effectiveness in diverse contexts is essential. Secondly, many studies focus on the theoretical aspects of PFS, but research is needed to integrate these sets with real-world problems. Bridging this gap involves identifying practical scenarios where similarity measures for Pythagorean fuzzy sets can offer tangible benefits. Thirdly, existing measures are less informative, they have certain drawbacks regarding accuracy and consistency with the PFS notion that need to be addressed to produce more accurate results.

The motivation for delving into the study of similarity measures for Pythagorean fuzzy sets and their applications is driven by several compelling factors:

- Improving similarity measures by considering membership, non-membership, and hesitancy degrees can enhance decision-making processes by providing more accurate and nuanced comparisons.
- The significance stems from the need to address practical challenges in decision support systems, optimization, and pattern recognition where uncertainty is inherent. Robust similarity measures for Pythagorean fuzzy sets can contribute to overcoming these challenges and improving the reliability of various applications.

The motivation lies in addressing the identified gaps and, in doing so, advancing the understanding of Pythagorean fuzzy sets and their effective utilization in decision-making processes across various domains.

This article is organized by introducing a few fundamental concepts of PFSs in Section 2. Some similarity measures for PFSs have been projected with its verification through numerical examples in section 3. The utility of the suggested similarity measures in pattern recognition and medical diagnosis problems has been examined in section 4. A comparative study of the proposed similarity measures with the similarity measures proposed by Wei & Wei [63] has been established in section 5. Lastly, section 6 concludes with directions for impending studies.

2. PRELIMINARIES

In this section, basic theories related to FSs, IFSs, and PFSs used in the outcome have been given:

Definition 1 (Zadeh [5]). Assume a fuzzy set $\boldsymbol{\mathcal{F}}$ in $\hat{\mathrm{Y}}=\{\mathbb{y}_{\mathbf{1}},\mathbb{y}_{\mathbf{2}},\dots,\mathbb{y}_{\boldsymbol{n}}\}$ where $\hat{\mathrm{Y}}$ is non-empty defined by MG as

$$\boldsymbol{\mathcal{F}}=\left\{\langle\mathbb{y},\boldsymbol{\mu}_{\boldsymbol{\mathcal{F}}}(\mathbb{y}_{\boldsymbol{i}})\rangle\middle|\mathbb{y}\in\hat{\mathrm{Y}}\right\} \tag{1}$$

where $\boldsymbol{\mu}_{\boldsymbol{\mathcal{F}}}:\hat{\mathrm{Y}}\to[\mathbf{0},\mathbf{1}]$ is a measure of MG of an object $\mathbb{y}\in\hat{\mathrm{Y}}$ in $\boldsymbol{\mathcal{F}}$.

Definition 2 (Atanassov [7]). An IFS $\boldsymbol{\mathcal{F}}$ in $\hat{\mathrm{Y}}$ is defined as

$$\boldsymbol{\mathcal{F}}=\left\{\langle\mathbb{y},\boldsymbol{\mu}_{\boldsymbol{\mathcal{F}}}(\mathbb{y}_{\boldsymbol{i}}),\boldsymbol{\nu}_{\boldsymbol{\mathcal{F}}}(\mathbb{y}_{\boldsymbol{i}})\rangle\middle|\mathbb{y}\in\hat{\mathrm{Y}}\right\} \tag{2}$$

where $\boldsymbol{\mu_{\mathcal{F}}}, \boldsymbol{\nu_{\mathcal{F}}}: \hat{Y} \to [\mathbf{0}, \mathbf{1}]$

Also, $\boldsymbol{\mu_{\mathcal{F}}} + \boldsymbol{\nu_{\mathcal{F}}} \in [\mathbf{0}, \mathbf{1}]$, $\forall\ \mathbb{y} \in \hat{Y}$ and $\boldsymbol{\mu_{\mathcal{F}}}(\mathbb{y}_i), \boldsymbol{\nu_{\mathcal{F}}}(\mathbb{y}_i)$ represents the MG and NMG respectively of an object $\mathbb{y} \in \hat{Y}$ in $\boldsymbol{\mathcal{F}}$.

F, or every IFS $\boldsymbol{\mathcal{F}}$ in $\hat{Y}$, we have

$$\boldsymbol{\pi_{\mathcal{F}}}(\mathbb{y}_i) = \mathbf{1} - \boldsymbol{\mu_{\mathcal{F}}}(\mathbb{y}_i) - \boldsymbol{\nu_{\mathcal{F}}}(\mathbb{y}_i),\ \forall\ \mathbb{y} \in \hat{Y} \tag{3}$$

is the HG.

Definition 3 (Yager [13]). Consider a finite set $\hat{Y} = \{\mathbb{y}_{\mathbf{1}}, \mathbb{y}_{\mathbf{2}}, \ldots, \mathbb{y}_{\boldsymbol{n}}\}$, we define a PFS $\boldsymbol{\mathcal{F}}$ as $\boldsymbol{\mathcal{F}} = \left\{\langle \mathbb{y}, \boldsymbol{\mu_{\mathcal{F}}}(\mathbb{y}_i), \boldsymbol{\nu_{\mathcal{F}}}(\mathbb{y}_i)\rangle \middle| \mathbb{y} \in \hat{Y}\right\}$ $\boldsymbol{\mu_{\mathcal{F}}}(\mathbb{y}_i), \boldsymbol{\nu_{\mathcal{F}}}(\mathbb{y}_i)$ represents the MG and NMG respectively of an object $\mathbb{y} \in \hat{Y}$ in $\boldsymbol{\mathcal{F}}$.

Also $\mathbf{0} \leq (\boldsymbol{\mu_{\mathcal{F}}})^{\mathbf{2}} + (\boldsymbol{\nu_{\mathcal{F}}})^{\mathbf{2}} \leq \mathbf{1}$ and $\boldsymbol{\pi_{\mathcal{F}}}(\mathbb{y}_i) = \sqrt{\mathbf{1} - \boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{F}}}(\mathbb{y}_i) - \boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{F}}}(\mathbb{y}_i)}$; $\boldsymbol{\pi_{\mathcal{F}}}: \hat{Y} \to [\mathbf{0}, \mathbf{1}]$

$$\text{such that } (\boldsymbol{\mu_{\mathcal{F}}}(\mathbb{y}_i))^{\mathbf{2}} + (\boldsymbol{\nu_{\mathcal{F}}}(\mathbb{y}_i))^{\mathbf{2}} + (\boldsymbol{\pi_{\mathcal{F}}}(\mathbb{y}_i))^{\mathbf{2}} = \mathbf{1}. \tag{4}$$

3. SIMILARITY MEASURES

Primarily, we remind the obvious preposition of similarity for PFS.
Proposition 1 (Ejegwa, [38]). $\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}$, and $\boldsymbol{\mathcal{M}}$ be three PFS in $\hat{Y}$ where $\hat{Y}$ is a non-empty set, then the similarity measure between $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{L}}$ must satisfies the following properties

($\boldsymbol{\mathcal{S}}$1) $\mathbf{0} \leq \boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) \leq \mathbf{1}$

($\boldsymbol{\mathcal{S}}$2) $\boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) = \mathbf{1} \Leftrightarrow \boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{L}}$.

($\boldsymbol{\mathcal{S}}$3) $\boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) = \boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}})$

($\boldsymbol{\mathcal{S}}$4) *Inequality:* If $\boldsymbol{\mathcal{M}}$ is a PFS in $\hat{Y}$ and $\boldsymbol{\mathcal{K}} \subseteq \boldsymbol{\mathcal{L}} \subseteq \boldsymbol{\mathcal{M}}$, then $\boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}) \leq \boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}})$ and $\boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}) \leq \boldsymbol{\mathcal{Sim}}(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{M}})$.

Wei & Wei [63] proposed cosine similarity measures for two PFSs A and B (5-8) as follows:

$$\boldsymbol{S}^{\mathbf{1}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) = \frac{\mathbf{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{cos}\left[\frac{\boldsymbol{\pi}}{\mathbf{2}}\left(\left|\boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right| \vee \left|\boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right|\right)\right] \tag{5}$$

$$\boldsymbol{S}^{\mathbf{2}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) = \frac{\mathbf{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{cos}\left[\frac{\boldsymbol{\pi}}{\mathbf{4}}\left(\left|\boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right| + \left|\boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right|\right)\right] \tag{6}$$

$$\boldsymbol{S}^{\mathbf{3}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) = \frac{\mathbf{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{cos}\left[\frac{\boldsymbol{\pi}}{\mathbf{2}}\left(\left|\boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right| \vee \left|\boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right| \vee \left|\boldsymbol{\pi}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\pi}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right|\right)\right] \tag{7}$$

$$\boldsymbol{S}^{\mathbf{4}}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}}) = \frac{\mathbf{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{cos}\left[\frac{\boldsymbol{\pi}}{\mathbf{4}}\left(\left|\boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\mu}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right| + \left|\boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\upsilon}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right| + \left|\boldsymbol{\pi}^{\mathbf{2}}_{\boldsymbol{\mathcal{K}}}(\mathbb{y}_i) - \boldsymbol{\pi}^{\mathbf{2}}_{\boldsymbol{\mathcal{L}}}(\mathbb{y}_i)\right|\right)\right] \tag{8}$$

Based on the above measures, trigonometric similarity measures between $\mathcal{K}$ and $\mathcal{L}$ is defined as

Consider a finite set $\hat{Y} = \{y_1, y_2, \dots, y_n\}$, we define PFS $\mathcal{K}$ and $\mathcal{L}$ as

$\mathcal{K} = \{\langle y, \mu_{\mathcal{K}}(y_i), \nu_{\mathcal{K}}(y_i)\rangle | y \in \hat{Y}\}$; $\mathcal{L} = \{\langle y, \mu_{\mathcal{L}}(y_i), \nu_{\mathcal{L}}(y_i)\rangle | y \in \hat{Y}\}$, then

$$S_1(\mathcal{K},\mathcal{L}) = 1 - \frac{1}{n}\sum_{i=1}^{n} \sin\left[\frac{\pi}{2}\left(\left|\mu_{\mathcal{K}}^2(y_i) - \mu_{\mathcal{L}}^2(y_i)\right| \Lambda \left|\upsilon_{\mathcal{K}}^2(y_i) - \upsilon_{\mathcal{L}}^2(y_i)\right| \Lambda \left|\pi_{\mathcal{K}}^2(y_i) - \pi_{\mathcal{L}}^2(y_i)\right|\right)\right] \quad (9)$$

$$S_2(\mathcal{K},\mathcal{L}) = \frac{1}{n}\sum_{i=1}^{n} \tan\left[\frac{\pi}{4} - \frac{\pi}{4}\left(\left|\mu_{\mathcal{K}}^2(y_i) - \mu_{\mathcal{L}}^2(y_i)\right| \Lambda \left|\upsilon_{\mathcal{K}}^2(y_i) - \upsilon_{\mathcal{L}}^2(y_i)\right| \Lambda \left|\pi_{\mathcal{K}}^2(y_i) - \pi_{\mathcal{L}}^2(y_i)\right|\right)\right] \quad (10)$$

$$S_3(\mathcal{K},\mathcal{L}) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi}{2}\left(\left|\mu_{\mathcal{K}}^4(y_i) - \mu_{\mathcal{L}}^4(y_i)\right| \vee \left|\upsilon_{\mathcal{K}}^4(y_i) - \upsilon_{\mathcal{L}}^4(y_i)\right| \vee \left|\pi_{\mathcal{K}}^4(y_i) - \pi_{\mathcal{L}}^4(y_i)\right|\right)\right] \quad (11)$$

where the symbol $\vee$ ***and*** Λ signifies maximum and minimum operations.

Also, $\pi_{\mathcal{K}}(y_i) = \sqrt{1 - \mu_{\mathcal{K}}^2(y_i) - \upsilon_{\mathcal{K}}^2(y_i)}$; $\pi_{\mathcal{L}}(y_i) = \sqrt{1 - \mu_{\mathcal{L}}^2(y_i) - \upsilon_{\mathcal{L}}^2(y_i)}$

Theorem 1. The Pythagorean fuzzy similarity measures $S_1(\mathcal{K},\mathcal{L})$, $S_2(\mathcal{K},\mathcal{L})$ and $S_3(\mathcal{K},\mathcal{L})$ defined in Eq. (9) to Eq. (11) are valid measures of Pythagorean fuzzy similarity.

Proof. All the essential conditions ($\mathcal{S}$1- $\mathcal{S}$4) given in proposition needs to be fulfilled by the proposed similarity measures as follows:

($\mathcal{S}$1) $0 \leq S_1(\mathcal{K},\mathcal{L}), S_2(\mathcal{K},\mathcal{L})$ ***and*** $S_3(\mathcal{K},\mathcal{L}) \leq 1$

lie. For $S_1(\mathcal{K},\mathcal{L})$: As the sine function lies in [0,1], $S_1(\mathcal{K},\mathcal{L})$ will always lies in [0,1].

Thus, $0 \leq S_1(\mathcal{K},\mathcal{L}) \leq 1$. Similarly, measures: $S_2(\mathcal{K},\mathcal{L})$ and $S_3(\mathcal{K},\mathcal{L})$ can be proved.

($\mathcal{S}$2) $S_1(\mathcal{K},\mathcal{L}), S_2(\mathcal{K},\mathcal{L}), S_3(\mathcal{K},\mathcal{L}) = 1 \Leftrightarrow \mathcal{K} = \mathcal{L}$.

Proof. For $S_1(\mathcal{K},\mathcal{L})$: Let $\mathcal{K} = \{\langle y, \mu_{\mathcal{K}}(y_i), \nu_{\mathcal{K}}(y_i)\rangle | y \in \hat{Y}\}$; $\mathcal{L} = \{\langle y, \mu_{\mathcal{L}}(y_i), \nu_{\mathcal{L}}(y_i)\rangle | y \in \hat{Y}\}$ be two PFS in $\hat{Y} = \{y_1, y_2, \dots, y_n\}$.

If $\mathcal{K} = \mathcal{L}$, then $\mu_{\mathcal{K}}^2(y_i) = \mu_{\mathcal{L}}^2(y_i)$; $\upsilon_{\mathcal{K}}^2(y_i) = \upsilon_{\mathcal{L}}^2(y_i)$ and $\pi_{\mathcal{K}}^2(y_i) = \pi_{\mathcal{L}}^2(y_i)$.

Thus, $\left|\mu_{\mathcal{K}}^2(y_i) - \mu_{\mathcal{L}}^2(y_i)\right| = 0$; $\left|\upsilon_{\mathcal{K}}^2(y_i) - \upsilon_{\mathcal{L}}^2(y_i)\right| = 0$ and $\left|\pi_{\mathcal{K}}^2(y_i) - \pi_{\mathcal{L}}^2(y_i)\right| = 0$.

Since sin0 =0, therefore, $S_1(\mathcal{K},\mathcal{L}) = 1$.

If $S_1(\mathcal{K},\mathcal{L}) = 1$, this implies that

$\left|\mu_{\mathcal{K}}^2(y_i) - \mu_{\mathcal{L}}^2(y_i)\right| = 0$; $\left|\upsilon_{\mathcal{K}}^2(y_i) - \upsilon_{\mathcal{L}}^2(y_i)\right| = 0$ and $\left|\pi_{\mathcal{K}}^2(y_i) - \pi_{\mathcal{L}}^2(y_i)\right| = 0$.

Since sin0 = 0, $\therefore$ $\mu_{\mathcal{K}}^2(y_i) = \mu_{\mathcal{L}}^2(y_i)$; $\upsilon_{\mathcal{K}}^2(y_i) = \upsilon_{\mathcal{L}}^2(y_i)$ and $\pi_{\mathcal{K}}^2(y_i) = \pi_{\mathcal{L}}^2(y_i)$.

Hence $\mathcal{K} = \mathcal{L}$. Similarly, measures: $S_2(\mathcal{K},\mathcal{L})$ and $S_3(\mathcal{K},\mathcal{L})$ can be proved.

(S3) $S_1(\mathcal{K},\mathcal{L}) = S_1(\mathcal{L},\mathcal{K})$; $S_2(\mathcal{K},\mathcal{L}) = S_2(\mathcal{L},\mathcal{K})$ and $S_3(\mathcal{K},\mathcal{L}) = S_3(\mathcal{L},\mathcal{K})$

Proofs are direct and self-evident.

(S4) If $\mathcal{M}$ is a PFS in Ŷ and $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{M}$, then

$S_i(\mathcal{K},\mathcal{M}) \leq S_i(\mathcal{K},\mathcal{L})$; $S_i(\mathcal{K},\mathcal{M}) \leq S_i(\mathcal{L},\mathcal{M})$, where $i = 1, 2, 3.$

Proof. For $S_1(\mathcal{K},\mathcal{L})$: If $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{M}$, then for $y_i \in$ Ŷ,

We have, $0 \leq \mu_{\mathcal{K}}(y_i) \leq \mu_{\mathcal{L}}(y_i) \leq \mu_{\mathcal{M}}(y_i) \leq 1$; $1 \geq v_{\mathcal{K}}(y_i) \geq v_{\mathcal{L}}(y_i) \geq v_{\mathcal{M}}(y_i) \geq 0$ and $0 \leq \pi_{\mathcal{K}}(y_i) \leq \pi_{\mathcal{L}}(y_i) \leq \pi_{\mathcal{M}}(y_i) \leq 1$

$\Rightarrow 0 \leq \mu^2_{\mathcal{K}}(y_i) \leq \mu^2_{\mathcal{L}}(y_i) \leq \mu^2_{\mathcal{M}}(y_i) \leq 1$; $1 \geq v^2_{\mathcal{K}}(y_i) \geq v^2_{\mathcal{L}}(y_i) \geq v^2_{\mathcal{M}}(y_i) \geq 0$ and

$0 \leq \pi^2_{\mathcal{K}}(y_i) \leq \pi^2_{\mathcal{L}}(y_i) \leq \pi^2_{\mathcal{M}}(y_i) \leq 1$.

Thus,

$$\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{L}}(y_i)\right| \leq \left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right| ; \left|\mu^2_{\mathcal{L}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right| \leq \left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right|,$$

$$\left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{L}}(y_i)\right| \leq \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{M}}(y_i)\right| ; \left|v^2_{\mathcal{L}}(y_i) - v^2_{\mathcal{M}}(y_i)\right| \leq \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{M}}(y_i)\right|,$$

$$\left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{L}}(y_i)\right| \leq \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right| ; \left|\pi^2_{\mathcal{L}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right| \leq \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right|$$

We can deduce the following from the above

$$\left[\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{L}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{L}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{L}}(y_i)\right|\right]$$
$$\leq \left[\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{M}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right|\right]$$

$$\Rightarrow \frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{L}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{L}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{L}}(y_i)\right|\right\}$$
$$\leq \frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{M}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right|\right\}$$

$$\Rightarrow \sin\left[\frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{L}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{L}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{L}}(y_i)\right|\right\}\right]$$
$$\leq \sin\left[\frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{M}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right|\right\}\right]$$

$$\Rightarrow -\frac{1}{n}\sum_{i=1}^{n} \sin\left[\frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{L}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{L}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{L}}(y_i)\right|\right\}\right]$$
$$\geq -\frac{1}{n}\sum_{i=1}^{n} \sin\left[\frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{M}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{M}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{M}}(y_i)\right|\right\}\right]$$

$$\Rightarrow 1 - \frac{1}{n}\sum_{i=1}^{n} \sin\left[\frac{\pi}{2}\left\{\left|\mu^2_{\mathcal{K}}(y_i) - \mu^2_{\mathcal{L}}(y_i)\right| \wedge \left|v^2_{\mathcal{K}}(y_i) - v^2_{\mathcal{L}}(y_i)\right| \wedge \left|\pi^2_{\mathcal{K}}(y_i) - \pi^2_{\mathcal{L}}(y_i)\right|\right\}\right]$$

$\geq \mathbf{1} - \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{sin}\left[\frac{\pi}{2}\left\{ \left|\boldsymbol{\mu}_{\mathcal{K}}^{2}(\mathrm{y}_i) - \boldsymbol{\mu}_{\mathcal{M}}^{2}(\mathrm{y}_i)\right| \wedge \left|\boldsymbol{v}_{\mathcal{K}}^{2}(\mathrm{y}_i) - \boldsymbol{v}_{\mathcal{M}}^{2}(\mathrm{y}_i)\right| \wedge \left|\boldsymbol{\pi}_{\mathcal{K}}^{2}(\mathrm{y}_i) - \boldsymbol{\pi}_{\mathcal{M}}^{2}(\mathrm{y}_i)\right|\right\}\right]$

$\Rightarrow \boldsymbol{S_1}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}) \leq \boldsymbol{S_1}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}})$. Similarly, $\boldsymbol{S_1}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}) \leq \boldsymbol{S_1}(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{M}})$.

Similarly, measures: $\boldsymbol{S_2}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}})$ and $\boldsymbol{S_3}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{L}})$ can be proved.

From the above, we conclude that proposed similarity measures fulfill all the axiomatic properties.

3.1 Numerica Authentication of the proposed Similarity Measures

Example 1. Suppose $\mathcal{K}, \mathcal{L}, \mathcal{M} \in$ PFS(X) for $\hat{\mathrm{Y}} = \{\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3\}$.

Let $\mathcal{K} = \{\langle x_1, 0.60, 0.20\rangle, \langle x_2, 0.40, 0.60\rangle, \langle x_3, 0.50, 0.30\rangle\}$,

$\mathcal{L} = \{\langle x_1, 0.80, 0.10\rangle, \langle x_2, 0.70, 0.30\rangle, \langle x_3, 0.60, 0.10\rangle\}$ and

$\mathcal{M} = \{\langle x_1, 0.90, 0.20\rangle, \langle x_2, 0.80, 0.20\rangle, \langle x_3, 0.70, 0.30\rangle\}$

Computing similarity measures for the above PFS, we find the following numerical values

Table 1: Example for validation of suggested measures

Measure 1	Values	Measure 2	Values	Measure 3	Values
$S_1(A, B)$	0.968631	$S_2(A, B)$	0.969977	$S_3(A, B)$	0.958569
$S_1(A, C)$	0.859623	$S_2(A, C)$	0.870417	$S_3(A, C)$	0.819468
$S_1(B, C)$	0.931992	$S_2(B, C)$	0.934245	$S_3(B, C)$	0.942274

4. PRACTICAL APPLICATIONS OF PFS

To demonstrate the legitimacy of the proposed similarity measures, discussed in section 3, the applications for Pattern recognition and Medical Diagnosis have been presented in this section.

4.1 Pattern recognition

Suppose there exists four patterns represented as a set of customers $A = \{A_1, A_2, A_3, A_4\}$, in the feature space consists of set of attributes/ criteria C = {Performance(C_1), Price(C_2), Safety(C_3), Features(C_4), Discounts(C_5)} with the sample to be recognized as the brands of cars as B = $\{B_1, B_2, B_4, B_4\}$. The objective for the problems is to classify the class in which the unknown pattern B is labelled by using the proposed similarity measures. Relations of attributes have been given in the Tables 2 & Table 3. To make decisions, the steps given in the form of flow chart (Figure 1) have been used to complete the task.

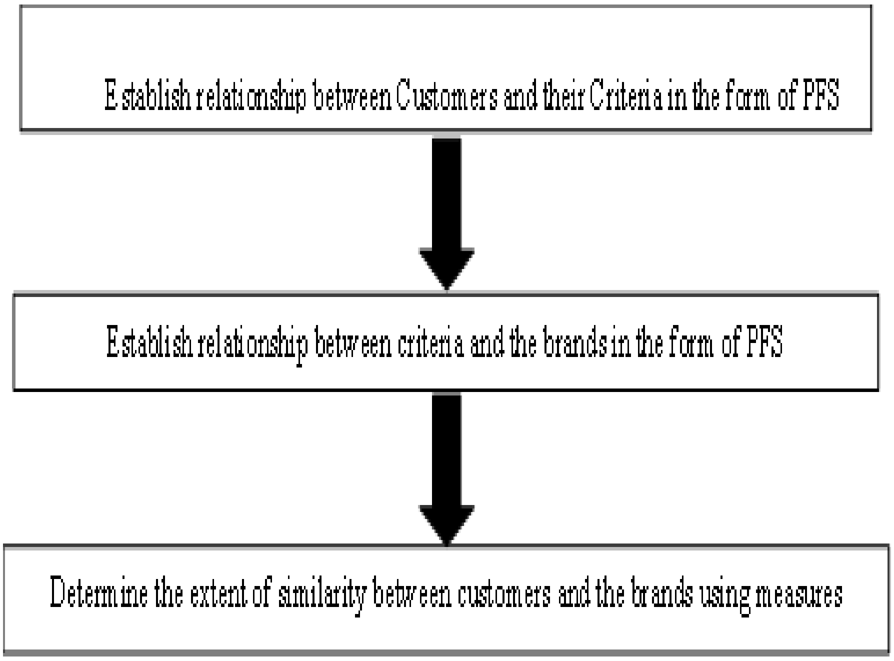

Figure 1: Flow Chart of the Methodology

The relationship between the customer and their criteria is defined as $\boldsymbol{\theta}$: $\mathbf{A} \rightarrow \mathbf{C}$ and is presented in Table 2**.**

Table 2: The relationship between Customers and their Criteria

$\boldsymbol{\theta}$	Performance ($\mathbf{C_1}$)	Price ($\mathbf{C_2}$)	Safety ($\mathbf{C_3}$)	Features ($\mathbf{C_4}$)	Discount ($\mathbf{C_5}$)
$\boldsymbol{A_1}$	<0.8, 0.1>	<0.6, 0.1>	<0.2, 0.8>	<0.6, 0.1>	<0.1, 0.6>
$\boldsymbol{A_2}$	<0.0, 0.8>	<0.4, 0.4>	<0.6, 0.1>	<0.1, 0.7>	<0.1, 0.8>
$\boldsymbol{A_3}$	<0.6, 0.1>	<0.5, 0.4>	<0.3, 0.4>	<0.7, 0.2>	<0.3, 0.4>
$\boldsymbol{A_4}$	<0.7, 0.2>	<0.4, 0.3>	<0.6, 0.2>	<0.8, 0.1>	<0.2, 0.5>

The relationship between the criteria and the likely brand is defined as $\boldsymbol{\varphi}$: $\mathbf{C} \rightarrow \mathbf{B}$ and is given in Table 3**.**

Table 3: The relationship between Criteria and the brands

φ	B_1	B_2	B_3	B_4
Performance ($\boldsymbol{C_1}$)	<0.4, 0.0>	<0.7, 0.0>	<0.3, 0.3>	<0.1, 0.8>
Price ($\boldsymbol{C_2}$)	<0.3, 0.5>	<0.2, 0.6>	<0.6, 0.1>	<0.2, 0.4>
Safety ($\boldsymbol{C_3}$)	<0.1, 0.7>	<0.0, 0.9>	<0.2, 0.7>	<0.8, 0.0>
Features ($\boldsymbol{C_4}$)	<0.4, 0.3>	<0.1, 0.8>	<0.2, 0.6>	<0.2, 0.7>
Discounts ($\boldsymbol{C_5}$)	<0.1, 0.7>	<0.8, 0.1>	<0.1, 0.9>	<0.4, 0.5>

Using the proposed similarity measures, the degree of similarity between the customers and the brands are given in Table 4 to Table 6**.**

Table 4: Degree of Similarity between the Customers and the Brands for $S_1(A, B)$

$\boldsymbol{S_1(A,B)}$	$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\boldsymbol{B_4}$
$\boldsymbol{A_1}$	0.95294931	0.856733339	**0.909086562**	**0.899720204**
$\boldsymbol{A_2}$	0.804567426	0.821078922	0.881371491	0.781733468
$\boldsymbol{A_3}$	**0.965512063**	0.896968667	0.881059324	0.832726098
$\boldsymbol{A_4}$	0.878456742	**0.950169464**	0.843159302	0.896797551

Table 5: Degree of Similarity between the Customers and the Brands for $S_2(A, B)$

$\boldsymbol{S_2(A,B)}$	$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\boldsymbol{B_4}$
$\boldsymbol{A_1}$	0.954787566	0.869442272	**0.913543302**	**0.90487781**
$\boldsymbol{A_2}$	0.829333557	0.843941936	0.890438486	0.811325179
$\boldsymbol{A_3}$	**0.96711423**	0.904355671	0.888163636	0.848453655
$\boldsymbol{A_4}$	0.888428191	**0.954550112**	0.861194703	0.903599378

Table 6: Degree of Similarity between the Customers and the Brands for $S_3(A, B)$

$\boldsymbol{S_3(A,B)}$	$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\boldsymbol{B_4}$
$\boldsymbol{A_1}$	**0.899102564**	0.878528476	**0.899718227**	0.865328793
$\boldsymbol{A_2}$	0.894085515	0.813221516	0.816826663	0.813224419
$\boldsymbol{A_3}$	0.854156457	0.897082442	0.848146406	0.810665711
$\boldsymbol{A_4}$	0.890483527	**0.944014083**	0.8697812	**0.89413737**

Results and Discussion:

From the results received from $S_1(A, B)$ and $S_2(A, B)$ presented in Tables 4 & Ta the degree of similarity for A_1 is with brand B_3 and brand B_4 whereas, A_3 is with brand B_1 and A_4 is with brand B_2, the largest. However, from the results by using $S_3(A, B)$ represented in the Table 6, it is observed that degree of similarity of A_1 is with brand B_1 and brand B_3 whereas A_4 prefers brand B_2 and brand B_4, the largest one. These results are desired from the proposed similarity measures. The proposed entropy measures can be used for the problems of pattern recognition for the labelling of most likely classes in the feature space.

4.2 Medical Diagnosis Problem

Medical decision-making problems cannot afford a little uncertainty or vagueness in any serious situation. To demonstrate this problem, let $P = \{P_1, P_2, P_3, P_4, P_5\}$ be the set of patients under the diagnosis of set of diseases as = $\{Viral\ fever, Malaria, Typhoid, Stomach\ problem\ and\ Chest\ problem\}$ with symptoms set as $S = \{Temperature, Headache, Stomach\ pain, Cough\ and\ Chest\ pain\}$. The objective is to classify the disease for which patient P is suffering by using the proposed similarity measures. Relations of attributes have been given in Table 7 & Table 8. To make decisions, the flow chart (Figure 1) is used to complete the task as:

The relationship between the symptoms and their diseases is defined as σ: $S \rightarrow D$ and is given in Table 7. The relationship between the patients and the likely symptom is defined as $\boldsymbol{\omega}$: $\mathbf{P} \rightarrow \mathbf{S}$ and is given in Table 8**.** Using the proposed similarity measures, the degree of similarity between the patients and the diseases are given in Table 9 to Table11**.**

Table 7: The relationship between Symptoms and their Disease

σ	*Viral Fever*	*Malaria*	*Typhoid*	*Stomach Pain*	*Chest Pain*
Temperature	<0.8, 0.1>	<0.6, 0.1>	<0.2, 0.8>	<0.6, 0.1>	<0.1, 0.6>
Headache	<0.9, 0.1>	<0.7, 0.2>	<0.2, 0.8>	<0.7, 0.2>	<0.2, 0.7>
Stomach Pain	<0.0, 0.7>	<0.4, 0.5>	<0.6, 0.2>	<0.2, 0.7>	<0.1, 0.2>
Cough	<0.7, 0.1>	<0.7, 0.1>	<0.0, 0.5>	<0.1, 0.7>	<0.0, 0.6>
Chest pain	<0.5, 0.1>	<0.4, 0.3>	<0.4, 0.5>	<0.8, 0.2>	<0.3, 0.4>

Table 8: The relationship between Patient and the Diseases

ω	*Temperature*	*Headache*	*Stomach Pain*	*Cough*	*Chest pain*
Alex	*<0.4, 0.0>*	*<0.3, 0.5>*	*<0.1, 0.7>*	*<0.4, 0.3>*	*<0.1, 0.7>*
Chris	*<0.7, 0.0>*	*<0.2, 0.6>*	*<0.0, 0.9>*	*<0.7, 0.0>*	*<0.1, 0.8>*
James	*<0.3, 0.3>*	*<0.6, 0.1>*	*<0.2, 0.7>*	*<0.2, 0.6>*	*<0.1, 0.9>*
Mike	*<0.1, 0.7>*	*<0.2, 0.4>*	*<0.8, 0.0>*	*<0.2,0.7>*	*<0.2, 0.7>*
Shawn	*<0.1, 0.8>*	*<0.0, 0.8>*	*<0.2, 0.8>*	*<0.2,0.8>*	*<0.8, 0.1>*

Table 9: Degree of Similarity between the Diseases and the Patients for $S_1(A, B)$

	Alex	*Chris*	*James*	*Mike*	*Shawn*
Viral Fever	0.8245	0.8187	0.8700	0.7857	0.8758
Malaria	0.8381	0.8598	0.8500	0.8043	0.7621
Typhoid	**0.8873**	0.7137	0.7909	0.9279	0.8286
Stomach Pain	0.8355	**0.9311**	**0.9466**	0.8408	0.8710
Chest Pain	0.8781	0.8937	0.8907	**0.9591**	**0.9187**

Table 10: Degree of Similarity between the Diseases and the Patients for $S_2(A, B)$

	Alex	*Chris*	*James*	*Mike*	*Shawn*
Viral Fever	0.8450	0.8448	0.8830	0.8067	0.8881
Malaria	0.8521	0.8725	0.8610	0.8227	0.7869
Typhoid	**0.8946**	0.7512	0.8135	0.9318	0.8506
Stomach Pain	0.8524	**0.9351**	**0.9486**	0.8540	0.8858
Chest Pain	0.8869	0.9014	0.8985	**0.9605**	**0.9240**

Table 11: Degree of Similarity between the Diseases and the Patients $S_3(A, B)$

	Alex	*Chris*	*James*	*Mike*	*Shawn*
Viral Fever	0.7817	0.8120	0.8081	0.7835	0.7729
Malaria	**0.9088**	0.8547	0.9064	**0.8667**	0.8054
Typhoid	0.8654	0.7931	0.8321	0.8709	0.8747
Stomach Pain	0.9000	**0.8863**	**0.9309**	0.8645	**0.9061**
Chest Pain	0.8401	0.7629	0.8169	0.7859	0.7669

5. COMPARATIVE ANALYSIS

To determine the supremacy of the projected similarity measures, a comparison between the proposed similarity measures and the existing similarity measures proposed by Wei & Wei [63] has been conducted based on the numerical data suggested in pattern recognition and medical diagnosis problems.

Table 12: Comparative study for pattern recognition problem

Comparison	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$	$\boldsymbol{A_3}$	$\boldsymbol{A_4}$
$S^1(A, B)$	B_1	B_4	B_1	B_1
$S^2(A, B)$	B_1	B_4	B_1	B_1
$S^3(A, B)$	B_1	B_4	B_1	B_1
$S^4(A, B)$	B_1	B_4	B_1	B_1
Proposed $S_1(A, B)$	B_3	B_4	B_1	B_1
Proposed $S_2(A, B)$	B_3	B_4	B_1	B_1
Proposed $S_3(A, B)$	B_1	B_4	B_1	B_4

Table 13: Comparative study for medical diagnosis problem

	Alex	*Chris*	*James*	*Mika*	*Shawn*
$S^1(A,B)$	Malaria	Malaria	Stomach pain	Typhoid	Typhoid
$S^2(A,B)$	Chest pain	Malaria	Chest pain	Typhoid	Chest pain
$S^3(A,B)$	Malaria	Malaria	Stomach pain	Typhoid	Typhoid
$S^4(A,B)$	Malaria	Malaria	Stomach pain	Typhoid	Typhoid
Proposed $S_1(A,B)$	Typhoid	Stomach pain	Stomach pain	Chest pain	Chest pain
Proposed $S_2(A,B)$	Typhoid	Stomach pain	Stomach pain	Chest pain	Chest pain
Proposed $S_3(A,B)$	Malaria	Stomach pain	Stomach pain	Malaria	Stomach pain

The comparative evaluation of proposed similarity measures stated in sections (4.1) and (4.2) respectively are presented in Table 12 & Table 13. From the numerical results presented in the tables, it has been noticed that the results obtained by using the proposed similarity measures are analogous to the existing results.

6. CONCLUSIONS

In this article, we have put forward a new way to construct PFS similarity measurements based on sine, cosine, and tangent functions. These trigonometric similarity measures are widely used as effective tools that allow for greater flexibility when handling real-world decision-making situations. The main contribution of the study is as follows:

- Three new similarity measures of the PFS are proposed.
- The desirable combinations and their features are studied in detail.
- To show the efficiency of the proposed similarity measures, we give some numerical examples which show the new similarity measures can effectively overcome the limitations of the existing similarity measures.
- The application to pattern recognition and medical diagnosis has been determined.
- A comparative analysis of the suggested similarity measures with the existing ones is provided to determine the efficacy of the proposed measures.

Similarity measures are recommended to address vulnerabilities in the data and have applications in a variety of disciplines, especially in risk analysis problems, investment problems, selection problems, etc. We will also investigate the utility of established similarity measures in other areas including multi-criteria decision-making, clustering, medical image registration, etc. The PFS has been extended, such as the Pythagorean fuzzy linguistic set.

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