

ON SOME ASPECTS OF THE MATRIX DATA PERTURBATION IN LINEAR PROGRAM*

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Abstract: Linear program under changes in the system matrix coefficients has proved to be more complex than changes of the coefficients in objective functions and right hand sides. The most of the previous studies deals with problems where only one coefficient, a row (column), or few rows (columns) are linear functions of a parameter. This work considers a more general case, where all the coefficients are polynomial (in the particular case linear) functions of the parameter $t \in T \subseteq R$. For such problems, assuming that some non-singularity conditions hold and an optimal base matrix is known for some particular value \bar{t} of the parameter, corresponding explicit optimal basic solution in the neighbourhood of \bar{t} is determined by solving an augmented LP problem with real system matrix coefficients. Parametric LP can be utilized for example to model the production problem where, technology, resources, costs and similar categories vary with time.

Keywords: Linear parametric programming, parameter-dependent constraint matrix.

1. INTRODUCTION

For the parametric linear programming problems with arbitrary matrix parameterization results of rather theoretical character exist (Finkelstein 1965 [12], Dantzig 1967 [7], Klatte 1979 [26], Schubert and Zimmermann 1985 [37], Pateva 1991 [32]). Parametric linear programs, where the matrix coefficients are polynomial functions of a scalar parameter were investigated in Jodin and Goldstein 1965 [23], Dragan 1966 [9], Weickmeir 1978 [39], Kon-Popovska 1992 [27], while linear parameterization was investigated in Satty 1959 [35], Valiaho 1979 [38], Freund 1985 [14]. More work, both theoretical and practical, has been done on the easier problems

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in which only some particular coefficients of the constraints matrix (one coefficient only, or one or more selected rows/columns) are linear functions of a scalar parameter, see e.g. Courtillot 1961 [6], Maurion 1965 [30], Kaska 1967 [24], Gal 1968 [15], 1973 [16], Dinkelbach 1969 [8], Kim 1971 [25], Finkelstein and Gumenok 1975 [13], Dřek 1979 [10].

In the most of the cited papers, an optimal base approach that starts with an existing and known regular base of the problem, optimal for the parameter $t = \bar{t}$ was utilized. Simple formulas expressing inversion of the parameterized matrix (see section 3), simplex procedure over the field of rational functions Satty 1959 [35], Weickenmeier 1978 [39], decomposition procedure over the field of polynomials (theory of λ -matrices) Kon-Popovska 1992 [27] or Taylor series Freund 1985 [14] near some fixed value $t = \bar{t}$ were developed or utilized.

More recent approaches, beside theoretical investigations concerning sensitivity, stability and parametric analysis of general optimisation problems, where linear problems can be viewed as a special case (see e.g. Bank 1982 [3], Fiacco 1983 [11], Levitin 1994 [29]), include tolerance approach (see e.g. Wendel 1984 [40], Ravi and Wendell 1989 [34]) and interior point approach (see e.g. Adler and Monteiro 1992 [2], Jansen et al. 1992 [22], Berkelaar et al 1997 [5], Greenberg 1997 [21]). For further overall references in the field see Dinkelbach [8], Gal 1984[17], 1995 [18], Bank 1982 [3], Gal and Greenberg 1997 [19]).

This paper deals with a class of scalar parametric linear programs with polynomial (in particular linear) change of the system matrix coefficients. Assuming that some non-singularity conditions hold and that for some particular value \bar{t} of the parameter $t \in T \subseteq R$ a regular (optimal) basic matrix exists and is known, corresponding explicit (optimal) solution in the neighborhood of the \bar{t} is determined by solving an augmented LP problem with real system matrix coefficients.

2. PROBLEM DEFINITION

Consider a linear program in standard form:

$$P: \max \{z = \mathbf{c}\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}, \quad (2.1)$$

where $\mathbf{A} = (\mathbf{a}^1, \dots, \mathbf{a}^n)$, $\mathbf{a}^j = (a_{1j}, \dots, a_{mj})^T \in R^m$, $j = 1, \dots, n$, $m \leq n$ is a given full rank matrix of real coefficients, $\mathbf{c} = (c_1, \dots, c_n) \in R^n$, $\mathbf{b} = (b_1, \dots, b_m)^T \in R^m$ are vectors of objective function and right-hand side coefficients respectively and $\mathbf{x} = (x_1, \dots, x_n)^T \in R^n$ is a solution vector. Let coefficients of the matrix \mathbf{A} and the right-hand side \mathbf{b} be polynomials of parameter $t \in T \subseteq R$:

$$P(t): \max \{z = \mathbf{c}\mathbf{x} : (\sum_{k=0}^q \mathbf{A}_k t^k) \mathbf{x} = \sum_{k=0}^p \mathbf{b}_k t^k, \mathbf{x} \geq \mathbf{0}, p \geq 0, q \geq 1\} \quad (2.2)$$

where $\mathbf{b}_k = (b_{1k}, \dots, b_{mk})^T \in R^m$, $k = 0, \dots, p$, and $\mathbf{A}_k = (\mathbf{a}^{1k}, \dots, \mathbf{a}^{nk})$, $k = 0, \dots, q$, $\mathbf{a}^{jk} = (a_{1jk}, \dots, a_{mjk})^T \in R^m$, $j = 1, \dots, n$. We are looking for the corresponding solutions: $x = x(t)$, $z = z(t)$ of the new problem, as functions of the parameter t .

Let $\mathbf{B}(t) = (\mathbf{a}^{j_1}(t), \dots, \mathbf{a}^{j_m}(t))$ be regular matrix at the particular $t = \bar{t} \in T$ and in its neighborhood, $(\det \mathbf{B}(\bar{t}) \neq 0)$ i.e. basis for $P(t)$, where $\beta = \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$ and $\alpha = \{1, \dots, n\} \setminus \beta$ are the corresponding sets of basic and non-basic indices. Then, the corresponding explicit primal basic solution at \bar{t} is given by $\mathbf{x}_\beta(t) = \mathbf{B}^{-1}(t)\mathbf{b}(t)$, $\mathbf{x}_\alpha(t) \equiv 0$, the vector $\mathbf{y} = (y_1, \dots, y_m) \in R^m$ by $\mathbf{y}_\beta(t) = \mathbf{c}_\beta \mathbf{B}^{-1}(t)$ and the objective function by $z(t) = \mathbf{c}\mathbf{x}_\beta(t)$. At \bar{t} , basis $\mathbf{B}(t)$ is primal feasible if $\mathbf{x}_\beta(t) \geq \mathbf{0}$, \mathbf{y} is dual feasible solution if $\mathbf{y}_\beta \mathbf{A}(t) - \mathbf{c} \geq \mathbf{0}$, optimal if both the primal and dual solutions are feasible at \bar{t} and non-degenerate optimal if $\mathbf{y}_\beta \mathbf{A}(t) - \mathbf{c} + \mathbf{x}(t) > \mathbf{0}$.

2.1. Preliminary remarks

Our task is to determine the explicit form of the basic solution $\mathbf{x}_\beta(t) = \mathbf{B}^{-1}(t)\mathbf{b}(t)$ in the neighborhood of $t = \bar{t} \in T$, where an optimal base $\mathbf{B}(\bar{t}) = (\mathbf{a}^{j_1}(\bar{t}), \dots, \mathbf{a}^{j_m}(\bar{t}))$ exists and is known, by solving the system $\mathbf{B}(t)\mathbf{x} = \mathbf{b}(t)$. Determination of the dual feasible solution $\mathbf{y}_\beta(t) = \mathbf{c}_\beta \mathbf{B}^{-1}(t)$ i.e. $\mathbf{y}_\beta^T(t) = (\mathbf{B}^{-1}(t))^T \mathbf{c}_\beta^T$, and transformation of the column vectors, $\mathbf{y}^j(t) = \mathbf{B}^{-1}(t)\mathbf{a}^j(t)$, $j \in \alpha$ can be done by using the same method for solving the systems $\mathbf{B}(t)^T \mathbf{y}_\beta^T(t) = \mathbf{c}_\beta^T$ and $\mathbf{B}(t)\mathbf{y}^j = \mathbf{a}^j(t)$, $j \in \alpha$ respectively. Determination of the (optimal) solution as explicit functions of t suffice that if needed, other values i.e. (optimal) objective function $z(t) = \mathbf{c}\mathbf{x}_\beta(t)$, its k -derivative $z^k(t) = \mathbf{c}\mathbf{x}_\beta^k(t)$ and reduced costs $\mathbf{y}_\beta(t)\mathbf{A}(t) - \mathbf{c}$ etc. could be obtained. Critical optimality region of the solution (see e.g. Dinclebah [8], Gal [18], Freund [14]) having explicit expressions for $\mathbf{x}_\beta(t)$, $\mathbf{y}_\beta(t)$, $\mathbf{y}_\beta(t)\mathbf{A}(t) - \mathbf{c}$, $\mathbf{y}^j(t)$, $j \in \alpha$ and $\det \mathbf{B}(t)$, could be found using the methods of polynomial roots determination.

Having these remarks in mind, it suffices to show how to solve $\mathbf{B}(t)\mathbf{x} = \mathbf{b}(t)$.

3. REVIEW OF KNOWN RESULTS

At first we review several solutions for problem where parameterization is simpler. For further references and proofs see e.g. Sherman and Morrison 1950 [36], Kaska 1973 [24], Gal 1968 [15], 1973 [16], Dinkelbach 1969 [8], Finkelstein 1977 [13], Willner 1967 [41], Arana 1977 [1], Freund 1985 [14].

3.1. Linear rank 1 matrix parameterization ($\mathbf{A} - \mathbf{u}\mathbf{v}^T$)

Here \mathbf{A} is a regular $m \times n$ -matrix and $\mathbf{u} = (u_1, \dots, u_m)^T$, $\mathbf{v} = (v_1, \dots, v_n)$ are vectors of real coefficients. If \mathbf{B} is base of matrix \mathbf{A} and \mathbf{v}_β vector of corresponding coefficients, the following is true: for every $t \in T$, where matrix $(\mathbf{B} - \mathbf{u}\mathbf{v}_\beta^T)$ is non-singular, its inverse matrix is given by

$$\begin{aligned}
(\mathbf{B} - \mathbf{u}\mathbf{v}_\beta t)^{-1} &= \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{u}\mathbf{v}_\beta\mathbf{B}^{-1}t}{1 - \mathbf{v}_\beta\mathbf{B}^{-1}\mathbf{u}t} = (\mathbf{I} - \frac{\mathbf{B}^{-1}\mathbf{u}\mathbf{v}_\beta t}{1 - \mathbf{v}_\beta\mathbf{B}^{-1}\mathbf{u}t})\mathbf{B}^{-1} \\
&= (\mathbf{I} - \frac{\mathbf{B}^{-1}\mathbf{u}\mathbf{v}_\beta t}{1 - \text{tr}(\mathbf{B}^{-1}\mathbf{u}\mathbf{v}_\beta)t})\mathbf{B}^{-1}
\end{aligned} \tag{3.1}$$

and corresponding basic solution by $\mathbf{x}_\beta(t) = (\mathbf{I} - \frac{\mathbf{B}^{-1}\mathbf{u}\mathbf{v}_\beta t}{1 - \mathbf{v}_\beta\mathbf{B}^{-1}\mathbf{u}t})^{-1}\mathbf{x}$.

Special cases of linear matrix parameterization can be achieved by various choices of vectors \mathbf{u} and/or \mathbf{v} .

Let b_{ij} be i, j -element, $(\mathbf{B}^{-1})^j$ j -column and $(\mathbf{B}^{-1})_i$ i -row of matrix \mathbf{B}^{-1} .

Than:

a) $\mathbf{u} = (0, \dots, 1_i, \dots, 0)^T$ and $\mathbf{v} = (0, \dots, a'_j, \dots, 0)$ give one element parameterization

$$(\mathbf{B} - \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \dots & a'_{ij} & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} t)\mathbf{x} = \mathbf{b},$$

where $\mathbf{B}(t)^{-1} = \mathbf{B}^{-1} - \frac{(\mathbf{B}^{-1})^i \mathbf{a}'_{ij} (\mathbf{B}^{-1})^j t}{1 - b_{ji} a'_{ij} t}$ (Sherman-Morrison formula)

b) $\mathbf{u} = (0, \dots, 1_i, \dots, 0)^T$ and $\mathbf{v} = (a'_1, \dots, a'_j, \dots, a'_n)$ give one row parameterization

$$(\mathbf{B} - \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ a'_{i1} & \dots & a'_{ij} & \dots & a'_{in} \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} t)\mathbf{x} = \mathbf{b},$$

where $\mathbf{B}(t)^{-1} = \mathbf{B}^{-1} - \frac{(\mathbf{B}^{-1})^i \mathbf{v}_\beta \mathbf{B}^{-1} t}{1 - \mathbf{v}_\beta (\mathbf{B}^{-1})^i t}$.

c) $\mathbf{u}^T = (a'_1, \dots, a'_j, \dots, a'_n)$ and $\mathbf{v} = (0, \dots, 1_j, \dots, 0)$ give one column parameterization

$$(\mathbf{B} - \begin{bmatrix} 0 & \dots & a'_{1j} & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \dots & a'_{ij} & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \dots & a'_{mj} & \dots & 0 \end{bmatrix} t)\mathbf{x} = \mathbf{b},$$

where $\mathbf{B}(t)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}u(\mathbf{B}^{-1})_j t}{1 - (\mathbf{B}^{-1})_j \mathbf{u}t}$

(Bodewig transformation of Sherman-Morrisson formula).

3.2. Linear rank r parameterization ($\mathbf{A} - \mathbf{UV}t$)

Here \mathbf{A} is a regular $m \times n$ -matrix and $\mathbf{U} = \begin{bmatrix} u_{11} & \dots & u_{1m} \\ \dots & \dots & \dots \\ u_{r1} & \dots & u_{rm} \end{bmatrix}^T$, $\mathbf{V} = \begin{bmatrix} v_{11} & \dots & v_{1n} \\ \dots & \dots & \dots \\ v_{r1} & \dots & v_{rn} \end{bmatrix}$ are

matrices of rank r of real coefficients. If \mathbf{B} is base matrix of \mathbf{A} and \mathbf{V}_β is the matrix of corresponding columns of matrix \mathbf{V} , following is true: for every $t \in T$, were matrix $(\mathbf{B} - \mathbf{UV}_\beta t)$ is non-singular its inverse matrix is given by

expressions
$$\begin{aligned} (\mathbf{B} - \mathbf{UV}_\beta t)^{-1} &= \mathbf{B}^{-1} + \mathbf{B}^{-1} \mathbf{U} t (\mathbf{I} - \mathbf{V}_\beta \mathbf{B}^{-1} \mathbf{U} t)^{-1} \mathbf{V}_\beta \mathbf{B}^{-1} \\ &= (\mathbf{I} + \mathbf{B}^{-1} \mathbf{U} t (\mathbf{I} - \mathbf{V}_\beta \mathbf{B}^{-1} \mathbf{U} t)^{-1} \mathbf{V}_\beta) \mathbf{B}^{-1}. \end{aligned}$$

Here $(\mathbf{I} - \mathbf{V}_\beta \mathbf{B}^{-1} \mathbf{U} t)^{-1}$ is a matrix of order r , so result can be practically useful for $r \ll m$. For the special case $\mathbf{U} = \mathbf{u}$ and $\mathbf{V} = \mathbf{v}$ the expression reduces to (3.1).

3.3. Linear parameterization ($\mathbf{A}_0 - \mathbf{A}_1 t$)

Here \mathbf{A}_0 and \mathbf{A}_1 are $m \times n$ -matrices and we suppose that for some t matrix $(\mathbf{A}_0 - \mathbf{A}_1 t)$ is regular.

Theorem 1. (According Freund [14, theorem 1, part iii and theorem 2]) Let $(\mathbf{B}_0 - \mathbf{B}_1 \bar{t})$ be an optimal basis and $\bar{\mathbf{x}}_\beta$ optimal basic solution at $t = \bar{t}$ for the problem $\max \mathbf{c}\mathbf{x}$ where $(\mathbf{B}_0 - \mathbf{B}_1 \bar{t})\mathbf{x} = \mathbf{b}$, $\mathbf{x} > 0$. Then, except eventually for finite number of values $t \in K \subseteq T$, $(\mathbf{B}_0 - \mathbf{B}_1 t)$ is an optimal basis and $\mathbf{x}_\beta(t) = (\sum_{i=0}^{\infty} (t - \bar{t})^i (-\mathbf{B}_0^{-1} \mathbf{B}_1)^i) \bar{\mathbf{x}}_\beta$ an optimal basic solution of the problem $\max \mathbf{c}\mathbf{x}$ where $(\mathbf{B}_0 - \mathbf{B}_1 t)\mathbf{x} = \mathbf{b}$, $\mathbf{x} > 0$ for all values of t near \bar{t} .

4. LINEAR PARAMETERIZATION

First we consider linear parameterization of the matrix \mathbf{A} , with constant right-hand side \mathbf{b} i.e. case $q = 1$ and $p = 0$ of the problem (2.2). The following definition and lemma will be used.

Definition 1. (According to Wilkinson [42]): The rank $p \leq m$ of the m -vector \mathbf{b} with respect to the $m \times m$ -matrix \mathbf{A} is the minimum value p for which vectors $\mathbf{b}, \mathbf{A}\mathbf{b}, \dots, \mathbf{A}^{p-1}\mathbf{b}, \mathbf{A}^p\mathbf{b}$ are linearly dependent, or equivalently, the minimal polynomial of the m -vector \mathbf{b} with respect to the $m \times m$ -matrix \mathbf{A} is the polynomial with rank p for which the corresponding relation holds $(\mathbf{A}^p + c_{p-1}\mathbf{A}^{p-1} + \dots + c_1\mathbf{A} + c_0\mathbf{I})\mathbf{b} = 0$.

Lemma 1. (According Wilkinson [42]) If \mathbf{b} is m -vector of rank p with respect to the $m \times m$ -matrix \mathbf{A} , then each vector $\mathbf{A}^p \mathbf{b}, \mathbf{A}^{p+1} \mathbf{b}, \dots$ is linear combination of the vectors $\mathbf{b}, \mathbf{A} \mathbf{b}, \dots, \mathbf{A}^{p-1} \mathbf{b}$.

The main result for linear parameterization is as follows:

Theorem 2. In linearly parameterized system of equations

$$(\mathbf{B}_0 + \mathbf{B}_1 t) \mathbf{x}(t) = \mathbf{b}, \tag{4.1}$$

let be \mathbf{B}_0 and \mathbf{B}_1 $m \times m$ matrices, \mathbf{B}_0 is non-singular and \mathbf{b} is m -vector of rank m with respect to the matrix $(\mathbf{B}_1 \mathbf{B}_0^{-1})$. Then, for all $t \in T, t \neq 0$ for which $\det(\mathbf{B}_0 + \mathbf{B}_1 t) \neq 0$, the solution vector $\mathbf{x}(t)$ is determined by

$$\mathbf{x}(t) = \frac{\mathbf{x}_0 + \mathbf{x}_1 t + \dots + \mathbf{x}_{m-1} t^{m-1}}{1 + q_1 t + \dots + q_m t^m}, \tag{4.2}$$

where x_0, x_1, \dots, x_{m-1} are m -column vectors and q_1, \dots, q_m are scalars obtained by solving the following real linear system given in matrix form

$$\begin{bmatrix} 0 & 1 & & m-2 & m-1 & m & m-1 & 1 \\ \mathbf{B}_0 & & & & & & & \\ \mathbf{B}_1 & \mathbf{B}_0 & & & & & & \\ & \mathbf{B}_1 & & & & & & \\ & & \dots & & & & & \\ & & & \dots & & & & \\ & & & & \dots & & & \\ & & & & & \dots & & \\ & & & & & & \mathbf{B}_1 & \mathbf{B}_0 & -\mathbf{b} \\ & & & & & & \mathbf{B}_1 & -\mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \dots \\ \mathbf{x}_{m-1} \\ q_m \\ \dots \\ q_1 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \\ \dots \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.3}$$

Proof: Solution $\mathbf{x}(t)$ of the system $(\mathbf{B}_0 + \mathbf{B}_1 t) \mathbf{x}(t) = \mathbf{b}$ is in general m -vector, each of whose coefficients is rational functions of the form $p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials of the greatest degree $m - 1$ and m respectively. $\mathbf{x}(t) = (\mathbf{B}_0 + \mathbf{B}_1 t)^{-1} \mathbf{b}$, i.e.

$$\mathbf{x}(t) = \frac{\text{adj}(\mathbf{B}_0 + \mathbf{B}_1 t)}{\det(\mathbf{B}_0 + \mathbf{B}_1 t)} \mathbf{b}. \tag{4.4}$$

By dividing numerator and denominator with value $q_0 = \det \mathbf{B}_0$ (assuming \mathbf{B}_0 is non-singular), we get (4.2). By corresponding substitution we can rewrite relation (4.1) in the form:

$$(\mathbf{B}_0 + \mathbf{B}_1 t) \frac{\mathbf{x}_0 + \mathbf{x}_1 t + \dots + \mathbf{x}_{m-1} t^{m-1}}{1 + q_1 t + \dots + q_m t^m} = \mathbf{b}.$$

For every $t \in T$, for which denominator $1 + q_1t + \dots + q_mt^m \neq 0$, this relation can be expressed in the form $(\mathbf{B}_0 + \mathbf{B}_1t)(\mathbf{x}_0 + \mathbf{x}_1t + \dots + \mathbf{x}_{m-1}t^{m-1}) = \mathbf{b}(1 + q_1t + \dots + q_mt^m)$, or as $m + 1$ linear systems

$$\begin{aligned}
 \mathbf{B}_0\mathbf{x}_0 & & & = \mathbf{b} \\
 (\mathbf{B}_1\mathbf{x}_0 + \mathbf{B}_0\mathbf{x}_1)t & & - \mathbf{b}q_1t & = \mathbf{0} \\
 (\mathbf{B}_1\mathbf{x}_1 + \mathbf{B}_0\mathbf{x}_2)t^2 & & - \mathbf{b}q_2t^2 & = \mathbf{0} \\
 \cdot & & \cdot & \\
 (\mathbf{B}_1\mathbf{x}_{m-2} + \mathbf{B}_0\mathbf{x}_{m-1})t^{m-1} & & - \mathbf{b}q_{m-1}t^{m-1} & \\
 \mathbf{B}_1\mathbf{x}_{m-1}t^m & & - \mathbf{b}q_mt^m & = \mathbf{0}
 \end{aligned} \tag{4.5}$$

After dividing by $t, t \neq 0$ we get the system compactly written in matrix form (4.3) considered in the theorem. System (4.1) is equivalent to (4.2), (4.3) if the matrix of the last one is non-singular. This holds by the statements of the theorem and can be seen easily by the following assumptions: Columns of the first m blocks $0, 1, \dots, m - 1$ of the system are linearly independent by the assumption that \mathbf{B}_0 is non-singular. So, by solving first m systems regarding $\mathbf{x}_j, j = 0, 1, \dots, m - 1$ we get:

$$\begin{aligned}
 \mathbf{x}_0 &= \mathbf{B}_0^{-1}\mathbf{b} \\
 \mathbf{x}_1 &= q_1\mathbf{B}_0^{-1}\mathbf{b} - \mathbf{B}_0^{-1}(\mathbf{B}_1\mathbf{B}_0^{-1})\mathbf{b} \\
 \mathbf{x}_2 &= q_2\mathbf{B}_0^{-1}\mathbf{b} - q_1\mathbf{B}_0^{-1}(\mathbf{B}_1\mathbf{B}_0^{-1})\mathbf{b} + \mathbf{B}_0^{-1}(\mathbf{B}_1\mathbf{B}_0^{-1})^2\mathbf{b} \\
 \cdot & \\
 \mathbf{x}_{m-1} &= q_{m-1}\mathbf{B}_0^{-1}\mathbf{b} - q_{m-2}\mathbf{B}_0^{-1}(\mathbf{B}_1\mathbf{B}_0^{-1})\mathbf{b} + \dots + (-1)^{m-1}\mathbf{B}_0^{-1}(\mathbf{B}_1\mathbf{B}_0^{-1})^{m-1}\mathbf{b}
 \end{aligned} \tag{4.6}$$

The matrix in (4.3) is non-singular if the columns of the last $(m + 1)$ -block of the system are linearly independent on the previous ones. By replacing \mathbf{x}_{m-1} in the last equation $\mathbf{B}_1\mathbf{x}_{m-1}t^m - \mathbf{b}q_mt^m = 0$ of (4.5) and after rearrangement we get:

$$q_m\mathbf{b} - q_{m-1}(\mathbf{B}_1\mathbf{B}_0^{-1})\mathbf{b} + \dots + (-1)^{m-1}q_1(\mathbf{B}_1\mathbf{B}_0^{-1})^{m-1}\mathbf{b} = -(-1)^m(\mathbf{B}_1\mathbf{B}_0^{-1})^m\mathbf{b}. \tag{4.7}$$

The matrix on the left-hand side is non-singular due to proposition that \mathbf{b} is of rank m with respect to the matrix $(\mathbf{B}_1\mathbf{B}_0^{-1})$. \blacklozenge

Remark 1. The dimension of the new system is $m((m - 1)q + 1) + mq = mmq + m$, but it is almost in lower block triangular form, with at most $2m$ non-zero elements. More over, decomposition in fact needs to be done (due to special structure and replications) only on matrix \mathbf{B}_0 and m spikes in last m columns of (4.3).

Further we can relax somehow the conditions of the theorem.

Remark 2. Non-singularity of the matrix \mathbf{B}_0 is not necessary condition that statement of the theorem holds. It suffices that for any $t = \bar{t} \in T$ matrix $\mathbf{B}'_0 = \mathbf{B}_0 + \mathbf{B}_1\bar{t}$ is non-singular. By using transformation $\tau = t - \bar{t}$, we get $(\mathbf{B}_0 + \mathbf{B}_1\bar{t} + \mathbf{B}_1\tau)\mathbf{x} = \mathbf{b}$ i.e.

$p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials of degree at most mq and $(m-1)q+p$ respectively i.e.

$$\begin{aligned}\mathbf{x}(t) &= (\mathbf{B}_0 + \mathbf{B}_1 t + \dots + \mathbf{B}_q t^q)^{-1} (\mathbf{b}_0 + \mathbf{b}_1 t + \dots + \mathbf{b}_p t^p), \\ \mathbf{x}(t) &= \frac{\text{adj}(\mathbf{B}_0 + \mathbf{B}_1 t + \dots + \mathbf{B}_q t^q)}{\det(\mathbf{B}_0 + \mathbf{B}_1 t + \dots + \mathbf{B}_q t^q)} (\mathbf{b}_0 + \mathbf{b}_1 t + \dots + \mathbf{b}_p t^p).\end{aligned}\quad (5.4)$$

Dividing the numerator and denominator by $q_0 = \det \mathbf{B}_0$ (assuming \mathbf{B}_0 is non-singular) we get (5.2) that gives

$$(\mathbf{B}_0 + \mathbf{B}_1 t + \dots + \mathbf{B}_q t^q) \frac{\mathbf{x}_0 + \mathbf{x}_1 t + \dots + \mathbf{x}_{mq-q+p} t^{mq-q+p}}{1 + q_1 t + \dots + q_{mq} t^{mq}} = (\mathbf{b}_0 + \mathbf{b}_1 t + \dots + \mathbf{b}_p t^p).$$

For every $t \in T$ for which denominator $1 + q_1 t + \dots + q_{mq} t^{mq} \neq 0$, this relation can be expressed in the form: $(\mathbf{B}_0 + \mathbf{B}_1 t + \dots + \mathbf{B}_q t^q)(\mathbf{x}_0 + \mathbf{x}_1 t + \dots + \mathbf{x}_{mq-q+p} t^{mq-q+p}) = \mathbf{b}(1 + q_1 t + \dots + q_{mq} t^{mq})$, and assuming $p \leq q$, as $mq+p+1$ linear systems similar to (4.5).

After dividing by t , $t \neq 0$ we get system compactly written in matrix form (5.3) given in the theorem. (Similarly it is valid for $p \geq q$). System (5.1) is equivalent to (5.2) (5.3) if the matrix of the last one is non-singular and this holds by statements of the theorem. This can be seen easily by following: columns of the first $(m-1)q+p$ blocks of the system are linearly independent by assumption \mathbf{B}_0 being non-singular, while by conditions in the theorem last mq columns a linearly independent regarding columns of the first $(m-1)q+p$ blocks. (The same holds for the assumption $p \geq q$). \blacklozenge

6. CONCLUSIONS

We proposed an equivalent real system for solving the problem $(\mathbf{B}_0 + \dots + \mathbf{B}_q t^q) \mathbf{x}(t) = \mathbf{b}_0 + \dots + \mathbf{b}_p t^p$ which gives practical way to find explicit solution. Similar transformation can be extended for the case of other functions parameterization. Also, instead of optimal base approach result can be used in interior point approach where matrix $\mathbf{B}(t)$ means active sub-matrix of matrix $\mathbf{A}(t)$ corresponding to optimal partition (see e.g. [20]).

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