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FURTHER NOTES ON CONVERGENCE OF THE WEISZFELD ALGORITHM

Jack BRIMBERG

Department of Business Administration
Royal Military College of Canada
Kingston, Ontario, Canada
and
GERAD, 3000, chemin de la Côte-Sainte-Catherine
Montréal, Québec
Canada H3T 2A7

Abstract: The Fermat-Weber problem is one of the most widely studied problems in classical location theory. In his previous work, Brimberg (1995) attempts to resolve a conjecture posed by Chandrasekaran and Tamir (1989) on a convergence property of the Weiszfeld algorithm, a well-known iterative procedure used to solve this problem. More recently, Cánovas, Marín and Cañavate (2002) provide counterexamples that appear to reopen the question. However, they do not attempt to reconcile their counterexamples with the previous work. We now show that in the light of these counterexamples, the proof is readily modified and the conjecture of Chandrasekaran and Tamir reclosed.

Keywords: Fermat-Weber problem, minisum location, Weiszfeld algorithm.

1. INTRODUCTION

The Fermat-Weber problem, also referred to as the continuous single facility location problem, requires finding a point in space that minimizes the sum of weighted Euclidean distances to m given (or fixed) points. This problem is a cornerstone of classical location theory, and forms the basis of many other more advanced models. For an entertaining account of its long history, the reader is referred to Wesolowsky [8]; also see Love, Morris and Wesolowsky [7].

One aspect of the Fermat-Weber problem that has puzzled researchers for some time relates to the convergence of the Weiszfeld algorithm. It is well known that the sequence of points, $\{x^q; q = 0, 1, \ldots\}$, generated by the algorithm converges to the optimal solution provided that no iterate coincides with one of the fixed points. In such an eventuality, the iteration functions are undefined, and the algorithm will terminate prematurely. The question then relates to the nature of the set of "bad" starting points $\{x^0\}$ that will result in the early termination of the algorithm. Indeed this question is of theoretical interest only, since in practice an iterate is rarely observed to land exactly on a fixed point (that is, within the numerical precision of the computations).

In the seminal convergence proof by Kuhn [6], it is concluded that whenever the fixed points are noncollinear, $\{x^0\}$ will be a denumerable set. This is based on the premise that the system $T(x) = a_i$ has a finite number of roots, where $T(x)$ is the vector of iteration functions and a_i denotes any one of the fixed points. However, Chandrasekaran and Tamir [5] demonstrate with counterexamples that this premise is incorrect; that is, the set of "bad" starting points may not be denumerable for the noncollinear case as originally believed. In each of the counterexamples, the fixed points are contained in an affine subspace of $\mathbb{R}^n$. These authors then conjecture that a sufficient condition for $\{x^0\}$ to be denumerable is that the convex hull of the fixed points be of full dimension $'n'$.

Brimberg [1] attempts to resolve the open question of Chandrasekaran and Tamir by an analysis of the Jacobian matrix of the iteration functions. The analysis concludes that having a convex hull of full dimension is both a necessary and sufficient condition for $\{x^0\}$ to be denumerable. Now, most recently, this result is being refuted by Cánovas et al. [4]. These authors provide counterexamples, but do not attempt to examine or rectify the work in [1].

The purpose of this note is to reconsider the analysis in [1] in light of the new counterexamples [4]. We show that some modifications are required to the original work, but the main conclusion remains intact. Fortunately, the question posed by Chandrasekaran and Tamir may be reclosed.

2. ANALYSIS

The Fermat-Weber location problem is defined as follows:

$$\min W(x) = \sum_{i=1}^{m} w_i d(x, a_i)$$

where

$a_i = (a_{i1}, ..., a_{in})^T$ is the known position of the i^{th} fixed point, $i = 1,...,m$;

$x = (x_1, ..., x_n)^T$ is the unknown position of the new facility;

$w_i > 0$ is a weighting constant for fixed point (customer) i, $i = 1,...,m$; and

$d(x, y) = \| x - y \|$ is the Euclidean distance between any two points $x, y \in \mathbb{R}^n$.

Recall that the iteration function in the Weiszfeld procedure for the t^{th} coordinate is given by:

$$f_t(x) = \sum_{i=1}^{m} \alpha_i(x) a_{it}, \quad t = 1,...,n, \tag{1}$$

where

$$\alpha_i(x) = \frac{w_i / d(x,a_i)}{\sum_{i=1}^{m} w_i / d(x,a_i)}, \quad i = 1,...,m. \tag{2}$$

Letting $f(x) = (f_1(x),...,f_n(x))^T$, we define the following mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$T(x) = \begin{cases} f(x), & \text{if } x \notin \{a_1,...,a_m\} \\ a_i, & \text{if } x = a_i \text{ for any } i = 1,...,m \end{cases} \tag{3}$$

Weiszfeld's algorithm is then given by the simple one-point iterative scheme:

$$x^{q+1} = T(x^q), \quad q = 0,1,2,... . \tag{4}$$

It is well known that the mapping T is continuous everywhere, and infinitely differentiable everywhere except at the fixed points a_i. Furthermore, if an iterate coincides with a fixed point ($x^q = a_i,$ for some i and q), the vector $f(x)$ is undefined due to division by zero in the components, and we see that the algorithm terminates at that fixed point $(x^{q+r} = T(a_i) = a_i, \ \forall r \geq 1)$. Otherwise, the algorithm is guaranteed to converge to the optimal solution. (See the global convergence proof of Kuhn [6] and a generalization to l_p norms by Brimberg and Love [3]).

Let us now further examine the set of (bad) starting points that result in termination of the algorithm at some a_i after a finite number of iterations. From (2) it follows that for any $x \in \mathbb{R}^n \setminus \{a_1,...,a_m\}, \ 0 < \alpha_i(x) < 1, \ i = 1,...,m,$ and $\sum_{i=1}^{m} \alpha_i(x) = 1$. Hence $T(x)$ must be a point in the interior of the convex hull of the set of fixed points (denoted by $ch\{a_1,...,a_m\}$). This leads immediately to the following result.

Property 1. Suppose each a_i is an extreme point of $ch\{a_1,...,a_m\}$. Then

$$\{x \mid T(x) = a_i, \ x \in \mathbb{R}^n \setminus \{a_1,...,a_m\}, \ i \in \{1,...,m\}\}$$

is the null set.

It follows in this case that the set of "bad" starting points, defined as

$$S = \{x^0 \mid x^0 \notin \{a_1,...,a_m\}, \ T(x^q) = a_i \text{ for some } i \text{ and finite } q\} ,$$

is also empty. Since, for example, m can be less than $n+1$, this clearly demonstrates that having $ch\{a_1,...,a_m\}$ of full dimension is not a necessary condition for S to be denumerable as originally claimed in [1]. Cánovas et al. [4] illustrate this property in their counterexample 1.

What if one or more of the fixed points are in the interior of the convex hull? Cánovas et al. [4] show using their counterexample 2 that S may still be empty when $ch\{a_1,...,a_m\}$ is not of full dimension. It will be helpful to examine counterexample 2 in further detail. There are four fixed points in the horizontal plane: $a_1=(1,0,0)$, $a_2=(0,1,0)$, $a_3=(-1,-1,0)$, $a_4=(0,0,0)$. As before let $w_1=w_2=w_4=1$, but generalize the problem by allowing $w_3=k$, where $k>1$. Then setting $T(x)=a_4$ is equivalent to the system:

$$\frac{d(x,a_3)}{d(x,a_1)}=\frac{w_3}{w_1}=k \tag{5}$$

$$\frac{d(x,a_3)}{d(x,a_2)}=\frac{w_3}{w_2}=k \tag{6}$$

It is readily shown that the points satisfying (5) form a sphere centered at $\left(\frac{k^2+1}{k^2-1},\frac{1}{k^2-1},0\right)$ with radius $\left(\frac{k\sqrt{5}}{k^2-1}\right)$; while the points satisfying (6) form a sphere of the same radius centered at $\left(\frac{1}{k^2-1},\frac{k^2+1}{k^2-1},0\right)$. Thus, if $1<k<\sqrt{10}$, the two spheres intersect along a circle; if $k=\sqrt{10}$, the intersection of the two spheres degenerates to a single point, $P=\left(\frac{2}{3},\frac{2}{3},0\right)$; finally, if $k>\sqrt{10}$, the intersection is the null set.

Since P is outside $ch\{a_1,...,a_m\}$, the equation $T(x)=P$ has no solution. It follows that S is nondenumerable if $1<k<\sqrt{10}$, S contains the single point P if $k=\sqrt{10}$, and $S=\varnothing$ if $k>\sqrt{10}$. This example illustrates that when the fixed points are contained in an affine subspace of $\mathbb{R}^n$ ($ch\{a_1,...,a_m\}$ is not of full dimension), the set of "bad" starting points may also be denumerable and nonempty.

How can these different cases be explained and can they be reconciled with the analysis in [1]? To this end let us re-examine $f'(x)$, the Jacobian matrix of $f(x)$. It is shown in [1] that

$$\nabla f_t(x)=\frac{1}{s(x)}\sum_{i=1}^{m}\frac{w_i(f_t(x)-a_{it})}{(d(x,a_i))^3}(x-a_i),\quad t=1,...,n, \tag{7}$$

where $s(x)=\sum_{i=1}^{m} w_i/d(x,a_i)$ and ∇ denotes the gradient operator. Furthermore, if $\{a_1,...,a_m\}$ is contained in an affine subspace of $\mathbb{R}^n$, then $f'(x)$ is singular $\forall x\in\mathbb{R}^n\setminus\{a_1,...,a_m\}$ (see lemma 1 in [1]). If, on the other hand, $ch\{a_1,...,a_m\}$ has full dimension n, then $f'(x)$ is invertible everywhere except at a subset of points of measure zero in $\mathbb{R}^n$ (lemma 2 in [1]).

2.1. Fixed Points in Affine Subspace

Let the set of fixed points be contained in an affine subspace of $\mathbb{R}^n$, and furthermore, assume that $ch\{a_1,...,a_m\}$ has dimension $(n-1)$. (The following discussion is readily extended if $ch\{a_1,...,a_m\}$ is contained in the intersection of two or more hyperplanes in $\mathbb{R}^n$.) Referring to [1], it follows that the rank of $f'(x)$ must be less than or equal to the dimension of $ch\{a_1,...,a_m\}$:

$$rank\,[f'(x)] \le n-1, \quad \forall x \in \mathbb{R}^n \setminus \{a_1,...,a_m\}.$$

The exceptional case observed in the numerical example with $k=\sqrt{10}$ occurs as a result of the level surfaces, $f_1 = 0$ and $f_2 = 0$, just touching each other. In effect, the intersection of the set of level surfaces degenerates to a single point ($x = P$ in the example). This degeneracy (not foreseen in [1]) is only possible when $rank[f'(x)] < n-1$ (in the example, $rank[f'(P)] = 1 < 2$), as proven in the next result.

Property 2. Let $ch\{a_1,...,a_m\}$ have dimension $(n-1)$, and suppose an $x^0 \in \mathbb{R}^n \setminus \{a_1,...,a_m\}$ may be found such that $T(x^0) = a_i$ for some i. If $rank[f'(x^0)] = n-1$, a continuous trajectory of length >0 passing through x^0 exists such that $T(x) = a_i$ for all points on the trajectory.

Proof: Suppose that $a_{ir} = k$, a constant, $\forall i = 1,...,m,$ and some $r \in \{1,...,n\}$ (as in the counterexamples in [5] and [4]). From (1) and (2) it follows that $f_r(x) = k$, and $\nabla f_r(x) = 0,\ \forall x \in \mathbb{R}^n \setminus \{a_1,...,a_m\}$. In effect the r^{th} coordinate drops out.

Except for this case it is readily shown by extension of the proof in [1] that $\nabla f_t(x^0) \ne 0,\ \forall t$ (otherwise $rank[f'(x^0)] < n-1$). Thus, x^0 is not a critical point of any of the iteration functions, and $f_t(x) = f_t(x^0)$ corresponds to a level surface through x^0 for each $t = 1,...,n$. Furthermore, since $rank[f'(x^0)] = n-1$, the intersection of any $(n-1)$ of these hypersurfaces must yield a continuous trajectory C of nonzero length passing through $x = x^0$. Arbitrarily choose the first $(n-1)$ hypersurfaces:

$$f_t(x) = f_t(x^0) = a_{it}, \quad t = 1,...,n-1.$$

Recall that $f(x) \in ch\{a_1,...,a_m\}$, so that $\sum_{t=1}^{n} c_t f_t(x) = c_0,\ \forall x \in \mathbb{R}^n \setminus \{a_1,...,a_m\}$, where c_0 and the c_t are constants not all equal to zero. Therefore, assuming without loss of generality that $c_n \neq 0$, (otherwise, reselect the $(n-1)$ hypersurfaces), $f_n(x) = \frac{1}{c_n}\left[c_0 - \sum_{t=1}^{n-1} c_t a_{it}\right] = a_{in} = f_n(x^0),\ \forall x \in C$. We conclude that

$$T(x) = f(x) = a_i, \quad \forall x \in C .$$ ♦

Alternatively, consider a unit vector $v(x)$ tangent to C at any point x on C. Since C is an intersection of level surfaces, it follows that $v(x)$ is normal to $\nabla f_t(x)$, i.e., $v(x) \cdot \nabla f_t(x) = 0,\ \forall x \in C,\ t = 1,...,n-1$. The singularity of $f'(x)$ implies that $\nabla f_n(x)$ is a linear combination of $\nabla f_t(x),\ t = 1,...,n-1$ (see also [1]); so that $v(x)$ is also normal to $\nabla f_n(x),\ \forall x \in C$. Thus, $f_n(x) = f_n(x^0),\ \forall x \in C$, and we arrive at the same conclusion. We may also conclude that if one point x^0 is found obeying Property 2, the set of "bad" starting points must now be nondenumerable.

Consider the location problem in the plane $(n = 2)$, where all the fixed points are contained on a straight line: $H = \{x \mid c_1 x_1 + c_2 x_2 = c_0\}$. Also assume without loss in generality that $c_1, c_2 \neq 0$. It is clear that for any $x \notin H$, the vectors, $(x - a_i), i = 1,...,m,$ are contained in a cone at x. Also, $f(x) \in ch\{a_1,...,a_m\}$ must be a point on H somewhere between the two extreme points of the convex hull. It follows that (see (7)) $\nabla f_t(x),\ t = 1,2,$ is a weighted sum of the vectors $(x - a_i),\ i = 1,...,m$, such that negative weights are attributed to the fixed points on H on one side of $f(x)$, and positive weights to those on the other side. Thus, $\nabla f_t(x)$ is a sum of vectors with tails at x, all pointing outwards on the same side of the line through $f(x)$ and x. We conclude that $\nabla f_t(x) \neq 0,\ \forall t,$ and also, $rank[f'(x)] = 1$. Thus, if $T(x^0) = a_i$ for some i and $x^0 \notin H,$ then by Property 2, a continuous trajectory passes through x^0 such that $T(x) = a_i$ for all points on the trajectory.

This result appears to generalize to higher dimensional space $(\mathbb{R}^n)$. Let H denote the affine subspace containing $\{a_1,...,a_m\}$. Assume without loss of generality that $c_t \neq 0,\ t = 1,...,n,$ and $ch\{a_1,...,a_m\}$ has dimension $(n-1)$. Using a similar reasoning, it follows that for any $x \notin H,\ \nabla f_t(x) \neq 0, \forall t$. The rotation of the gradient vector for different coordinates (see (7)) also implies that $rank[f'(x)] = n-1$. Thus, if $T(x^0) = a_i$ for some i and $x^0 \notin H,$ the set of "bad" starting points must be nondenumerable. In other words, it is sufficient to find one "bad" starting point outside the affine subspace containing the fixed points for the set of "bad" starting points to be nondenumerable.

2.2. Convex Hull of Full Dimension

Finally let us suppose that $ch\{a_1,...,a_m\}$ has full dimension n. In this case, the Jacobian matrix $f'(x)$ has full rank of n except at a set of points of measure zero in $\mathbb{R}^n$ (lemma 2 in [1]). If $L = \{x \mid rank[f'(x)] < n\}$, and a point $x^0 \notin L$ is found such that $T(x^0) = a_i$ for some i, then by the fundamental inverse function theorem of calculus, it follows that a neighbourhood of x^0 exists such that x^0 is the only point in that neighbourhood mapping onto a_i (also see the discussion in [1]). In fact, the invertibility of $f'(x)$ implies that in the vicinity of x^0 the level surfaces, $f_t(x) = a_{it}$, $t = 1,...,n$, intersect at the unique point x^0.

Cánovas et al. [4] in their counterexample 3 wish to infer that the set of "bad" starting points (S) may still be nondenumerable. However, this counterexample is completely unrelated to the Fermat-Weber location problem. We now show by clarifying the proof in [1] that having $ch\{a_1,...,a_m\}$ of full dimension n is sufficient for S to be denumerable.

Consider the counterexample in $\mathbb{R}^2$. The mapping is given by:

$$G(x) = (G_1(x), G_2(x)) = (x_1^2, x_1 g_2(x_2)) .$$

The Jacobian matrix,

$$G'(x) = \begin{bmatrix} 2x_1 & 0 \\ g_2(x_2) & x_1 g_2'(x_2) \end{bmatrix},$$

is invertible everywhere except on $L = \{x \mid x_1 = 0\}$. Also note that $\nabla G_1(x)$ is the zero vector and $\nabla G_2(x) = (g_2(x_2), 0)$ is normal to $L, \forall x \in L$. The level curves, $G_1(x) = 0$ and $G_2(x) = 0$, coincide with L, and thus, all points in L map onto the origin $(0,0)$. However, this fabricated example has nothing in common with the problem we are looking at.

In the context of the Fermat-Weber problem in $\mathbb{R}^2$, it follows from the inverse function theorem of calculus that the set S will be nondenumerable only if level curves of $f_1(x)$ and $f_2(x)$ coincide with L over a finite length, or equivalently, $f_1(x)$ and $f_2(x)$ have level curves that are identical over a finite length. However, this is impossible given the functional forms of $f_1(x)$ and $f_2(x)$.

The above argument applies to higher dimensional space $(\mathbb{R}^n)$. In effect, the set L of points where $f'(x)$ is singular $(rank[f'(x)] < n)$ corresponds analogously to the case where $ch\{a_1,...,a_m\}$ has dimension $(n-1)$ and $rank[f'(x)] < n-1$: a zero gradient vector $(\nabla f_t(x))$ may exist at x or a level surface of an $f_t(x)$ may be just tangent to another level surface or the intersection of a combination of such level surfaces at x. However, the functional forms of the $f_t(x)$ do not permit the level surfaces to all coincide on a continuous trajectory of nonzero length.

Thus, the sufficient condition in the theorem in [1] is salvaged, and we restate this theorem as follows:

> Given that $ch\{a_1,...,a_m\}$ has full dimension n, the set of starting points that will terminate the Weiszfeld algorithm at some fixed point a_i after a finite number of iterations is denumerable.

Corollary 1 in [1], that the set S is denumerable whenever x^0 is restricted to the smallest affine subspace containing $\{a_1,...,a_m\}$, is also seen to hold.

3. CONCLUSIONS

The counterexamples provided by Cánovas et al. [4] have identified some problems with the proof in Brimberg [1]. However, upon closer examination, these problems are resolved. We see that when the convex hull of the fixed points is contained in an affine subspace of $\mathbb{R}^n$, the set of starting points that terminate the Weiszfeld algorithm prematurely at a fixed point will be nondenumerable under general conditions specified above. When the convex hull has full dimension n, this set is guaranteed to be denumerable. Thus, the open question posed by Chandrasekaran and Tamir [5] is reclosed.

The brief reference in [4] to a related work by Brimberg and Chen [2] is puzzling, since the problem (and results) for general l_p norms is substantially different.

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