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GRAPH RADIOCOLORING CONCEPTS

R. KALFAKAKOU, G. NIKOLAKOPOULOU,
E. SAVVIDOU, M. TSOUROS

*Aristotle University of Thessaloniki, Faculty of Engineering
Thessaloniki, Greece*

Abstract: The evolution of the telecommunication technology, such as mobile telecommunication equipment, radio, television, teleconferencing, etc necessitated the existence of numerously broadcast channels. However the corresponding frequencies may interfere for reason of distance, geographical or atmospherical structure or for a variety of other factors.

In the present paper we give the definition of the graph radio coloring concept which is a graph coloring generalization. We also introduce radio coloring invariants which are of a great practical importance to problems regarding the allocation of diverse frequencies in order to avoid interference occurrences between channel transmissions. A heuristic algorithm that finds approximate values of the radio chromatic number and radio chromatic cost of a given graph is developed and the result of a computational experiment of the proposed algorithm is given.

Keywords: Graph, radio coloring, algorithm, channel assignment.

1. INTRODUCTION

Graph colouring is one of the oldest concepts in the theory of graphs, it has preoccupied a large number of people as a distraction puzzle during the 19^{th} century and later in the framework of scientific research, since this conception exhibits a significant interest from a theoretical and practical point of view. Many applications are modelised and investigated with the use of graph coloring. Because of the technology evolution, new problems arised that can be expressed effectively by handling invariants that are generalizations of graph coloring, see [1].

Here we present the notion of radio coloring which is a generalization of Graph coloring. The term radio coloring is due to F.Harary [2].

For reason of self-reliance and from the fact that graph terminology and notations is not yet unified, we give in the next Section definitions and notations of the terms used in the subsequent pages. Section 3 is devoted to a procedure that we call algorithm A that finds a good approximation of the chromatic number and the chromatic cost respectively for an arbitrary graph.

The concept of radio coloring is presented in the fourth Section and new coloring invariants that derive from the radio coloring notion are stated. A suitable modification of algorithm A results to algorithm Radio that finds an approximation (hopefully satisfactory) of the radio chromatic number and radio chromatic cost. The last Section is devoted to a relevant computational experiment of algorithms Radio and A and to the conclusions.

2. DEFINITIONS AND NOTATIONS

Let $V=\{v_1,v_2,\ldots,v_n\}$ be a nonempty set and $E\subseteq V\times V$ a subset of unordered couples (v_i,v_j), $v_i,v_j\in V$. The ordered pair (V,E) defines a graph $G=(V,E)$, see [3, 4]. The elements of V are usually called vertices, nodes or points and the elements of E links, lines or edges. A graph can easily be drawn in the plane displaying a good image of the connection structure of the elements of V for relatively small graphs.

Two nodes v_i and v_j are adjacent if they define an edge, i.e., $(v_i,v_j)\in E$. The set of adjacent nodes of a node $v_i\in V$ is denoted by $\Gamma(v_i)$, i.e., $\Gamma(v_i)=\{y$ such that $(v_i,y)\in E\ \}$ while the nonadjacent nodes of v_i are symbolized by $\overline{\Gamma}(v_i)=V-\Gamma(v_i)$. The degree $\deg(v_i)$ of a node $v_i\in V$ expresses the number of adjacent nodes to v_i, obviously $\deg(v_i)=\left|\,\Gamma(v_i)\,\right|$ and the density d of G is defined here to be the value of the ratio $d=2e/(n(n-1))$, where e is the number of edges in G.

The set of all adjacent nodes of all nodes in a set $S\subset V$ in the set $\Gamma(S)=\bigcup_{v_i\in S}\Gamma(v_i)$.

A path is a sequence $p=\{v_{i_1},v_{i_2},\ldots,v_{i_k}\}$ of consecutive edges $(v_{i_{j-1}},v_{i_j})$ such that $v_{i_j}\in\Gamma(v_{i_{j-1}})$ for $j=2,\ 3,\ \ldots,\ k$.

The distance $d(x,y)$ between two nodes $x,y\in V$ is the minimum number of edges that joins x and y with a path.

A graph $G'=(V',E')$ is a sub graph of $G=(V,E)$ if $V'\subset V$ and E' comprises only the edge of E that are produced by nodes in V', i.e., $E'=\{\ (x,y)\in E$ and $x,y\in V'\ \}$. We also denote the sub graph G' of G by $G(V')$. The adjacent and nonadjacent nodes of $v\in V'$ in sub graph $G(V')$ are expressed by $\Gamma_{V'}(v)$ and $\overline{\Gamma}_{V'}(v)$ respectively and the corresponding degree of $v\in V'$ by $\deg_{V'}(v)$.

A subset $S\subset V$ is called *independent set* in $G=(V,E)$ if for every pair $\{v_i,v_j\}\subset S$, v_i and v_j are not adjacent i.e., $v_i\notin\Gamma(v_j)\Leftrightarrow v_j\notin\Gamma(v_i)$. An independent set is *maximal* if it is not contained in any other independent set.

An assignment of colors to the nodes of a graph G so that adjacent nodes have different colors is a coloring of G. A coloring of G with n colors is a $n-$coloring and G is $n-$colorable. The minimum number of colors needed to perform a coloring of G

is called the chromatic number $\gamma(G)$ of G. Obviously if $n=|V|$ and $\gamma(G)\le q\le n$ then G is $q-$colorable. In a coloring, the nodes that are assigned the same color form a coloring class. Clearly any coloring class $C \subseteq V$ is an independent set.

We define a reciprocal correspondence between the set of colors and the set of positive integers $I^+=\{1,2,3,...\}$. In the continuation the colors will be referred by numbers in I^+. A graph coloring is a positive function c from the nodes of V to I^+, i.e., $c:V\to I^+$, where $c(y)$ express the color assigned to node $y\in V$.

3. GRAPH COLORING APPROXIMATION PROCEDURES

Many heuristic methods have been developed for finding an approximation of the chromatic number of a given arbitrary graph, see [4].

We develop here a new approximate chromatic number coloring procedure that we call algorithm A, a suitable modification of which enable us to detect good approximations of the radio chromatic number and the radio chromatic cost respectively.

Algorithm A belongs to the class of sequential vertex selection.

The sequential vertex selection methods for finding an approximate chromatic number of a given graph $G=(V,E)$, with $n=|V|$, apply in general the following steps, where

m: Number of coloured vertices.
x: Natural number expressing a color.

Step 1: The vertices of G are ordered according to some criterion.
Set $m\leftarrow 0$ and $x\leftarrow 1$.
Step 2: Select an uncolored vertex with the use of some measure and colour it with color x. Set $m\leftarrow m+1$.
Step 3: The uncoloured vertices which are not adjacent to a vertex already coloured with color x are assigned color x.
To every color assignment Set $m\leftarrow m+1$.
Step 4: If $m=n$ then yield x and stop
Else set $x\leftarrow x+1$ and proceed to Step 2.

In general the augmentation of the density of a graph raises its chromatic number while a node removal reduces the value of its chromatic number. Algorithm A takes advantage of these evident observations.

Let $G=(V,E)$, we denote by F and $\overline{F}=V-F$ the subsets that contain the coloured and uncoloured vertices of V respectively. For every $v\in\overline{F}$ we form set $Q\ (v,\overline{F})=\{y_1,y_2,...,y_k\}$, $k=|\overline{F}|-\deg_{\overline{F}}(v)$ of nonadjacent elements of v in graph $G(\overline{F})$, namely $Q(v,\overline{F})=V-(\Gamma(v)\cup F)=\ \overline{F}-(\Gamma_{\overline{F}}(v))$.

We form a maximal independent set $H(v)$ which contains $v \in \overline{F}$ as follows.

The elements $v_i \in Q\ (v,\overline{F})$ are selected in increasing order of their indices and added to $H(v)$ whenever $\Gamma_{\overline{F}}(v_i) \cap H(v) = \varnothing$. The augmentation of $H(v)$ ends when all elements of $Q(v,\overline{F})$ are examined. Sub graph $G(\overline{F} - H(v))$ is sparse enough since set $H(v)$ was constructed in a greedy manner so to contain nodes with greater degree in $G(\overline{F})$.

The selected node to be first coloured with a new color is a node w that leads to assign color $c(w)$ to a greater number of nodes, so to reduce the possible number of nodes in the remaining graph $G(\overline{F} - H(v))$ of uncoloured vertices. Thus

$$\left|H(w)\right| = \max_{v \in \overline{F}} \{H(v)\} = q$$

In the case where we have alternative solutions $\left|H(w_1)\right|, \left|H(w_2)\right|, \left|H(w_3)\right|, \ldots,$ $\left|H(w_z)\right|$, we choose to color first with the new color the node L that correspond to the largest degree in sub graph $G(\overline{F})$, namely

$$H(L) = \max\{\deg_{\overline{F}}(w_i), \text{ for which } \left|H(w_i)\right| = q,\ i = 1,2,\ldots z\ \}.$$

The above reasoning is incorporated and applied in algorithm A that follows.

Algorithm A

STEP 1: (Insert Data)

Read n and $\Gamma(x)\quad \forall x \in V = \{x_1, x_2, \ldots, x_n\}$

Set $F \leftarrow \varnothing$, $\overline{F} \leftarrow V$, $r \leftarrow 1$, $p \leftarrow n$.

STEP 2: (Create $Q(v_i)$, $v_i \in \overline{F}$)

For every $v_i \in \overline{F}$, Set $Q(v_i,\overline{F}) \leftarrow \overline{F} - \Gamma_{\overline{F}}(v_i)$

The $k = \left|\overline{F}\right| - \deg_{\overline{F}}(v_i)$ nonadjacent nodes of v_i in $G(\overline{F})$ form set

$Q(v_i,\overline{F}) = \{v_1, v_2, \ldots, v_k\}$ such that $\deg_{\overline{F}}(v_m) \ge \deg_{\overline{F}}(v_j)$

$\forall\ m, j \in \{1,2,\ldots,k\}$ for which $m < j$.

STEP 3.0: (Construction of sets $H\left|(v_i)\right|$, $\forall v_i \in \overline{F}$)

Set $i \leftarrow 0$, $alternative \leftarrow \varnothing$, $MX \leftarrow 0$.

STEP 3.1: (Find $H(v_i)$)

Set $i \leftarrow i+1$, If $i > p$ then proceed to Step 4

Else Set $H(v_i) \leftarrow v_i$, $NX \leftarrow 1$.

STEP 3.2:

If $Q(v_i,\overline{F}) = \varnothing$ proceed to Step 3.3. Else

Set $z \leftarrow \min\ \{j$ such that $v_j \in Q(v_i,\overline{F})$ and $\Gamma_{\overline{F}}(v_j) \cap H(v_i) = \varnothing\}$

$H(v_i) \leftarrow H(v_i) + \{v_z\}$, $Q(v_i,\overline{F}) \leftarrow Q(v_i,\overline{F}) - (\{v_z\} \cup \Gamma(v_z))$

$NX \leftarrow NX + 1$ and repeat Step 3.2.

STEP 3.3: (Check for max $\{|H(v)|\}$ and alternatives)

If $NX = MX$ then

Set alternative $\leftarrow$ alternative $\cup \{v_i\}$ and proceed to Step 3.1

Else If $NX > MX$

Set $MX \leftarrow NX$, alternative $\leftarrow \{v_i\}$ and proceed to Step 3.1

STEP 4: (Selection among alternatives $H(v)$)

Choose node $L \in \bar{F}$ such that $\deg_{\bar{F}}(L) \geq \deg(v),\ v \in \bar{F}$.

STEP 5: (Node coloring - Termination test)

Set $C_r \leftarrow H(L)$, $\bar{F} \leftarrow \bar{F} - H(L)$.

If $\bar{F} = \varnothing$, printout the chromatic classes $C_i,\ i = 1,2,\ldots,r$ and Stop.

Else Set $r \leftarrow r+1$, $p \leftarrow |\bar{F}|$ and proceed to Step 2.

4. RADIOCOLORING

The notion of radio coloring is a generalization of the graph coloring concept and it is a convenient invariant in order to model ordinary frequency assignment problems, see [5, 6] and for recent corresponding work see [7, 8]. We correspond a unique positive integer to every color, therefore in the sequel the colors are expressed by natural numbers.

A graph G is radio colored if the colors $c(v_i)$ assigned to every node $v_i \in V$ verify the following two conditions.

(i) $\left|c(v_i) - c(v_j)\right| \geq 2$ for every $(v_i, v_j) \in E$

(ii) if $d(v_i, v_j) = 2$ then $c(v_i) \neq c(v_j)$

Frequencies are allocated on stations that are represented by the nodes of $G = (V,E)$. An edge $(v_i, v_j) \in E$ means that stations v_i and v_j might interfere for some reason, eg. (distance, geographical or atmospherical structure, etc). To every frequency we associate a color, therefore if two stations might interfere, i.e., $(v_i, v_j) \in E$ then the frequency difference must be greater or equal than a number k . The radio coloring concept assumes that $k = 2$. Also stations that are distance two apart should not be assigned the same color, this means that the intermediate nodes of the couple $\{v_i, v_j\}$ when $d(v_i, v_j) = 2$ must not be influenced by the same frequency, as shown in Figure 1. The squared nodes are intermediate nodes of the couple $\{v_i, v_j\}$ for which $d(v_i, v_j) = 2$.

A radio coloring that uses q colors is a $q-$radio coloring. Let $G = (V,E)$ be a graph with m nodes. We represent a particular radio coloring, say R , with the use of a linear array of m elements, where the i^{th} element of R expresses the color assigned to node v_i , namely, $R(i) = c(v_i)$, see [9].

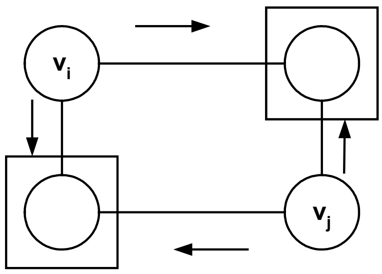

Figure 1: Intermediate nodes

The following invariants can be defined in a graph G :

- The *radio chromatic score* $rs(G,R)$ of a radio coloring R is the number of used colors.
- The number of colors used in a radio coloring with the minimum score is the *radio chromatic number* $rn(G)$ of G .
- The *radio chromatic price* $rp(G,R)$ of a radio coloring R is the value of the largest used color.
- The largest used color in a radio coloring with the minimum price is the *radio chromatic value* $rv(G)$ of G .
- The *radio chromatic weight* $rw(G,R)$ of a radio coloring R is the sum of the used colors.
- The sum of the used colors in a radio coloring with the minimum weight is the *radio chromatic cost* $rc(G)$ of G .
- The *radio chromatic gap* $rg(G,R)$ of a radio coloring R is the difference between the smallest and the largest used colors.
- The difference between the smallest and the largest used colors in a radio coloring with the minimum gap is the *radio chromatic bandwidth* $rb(G)$ of G .

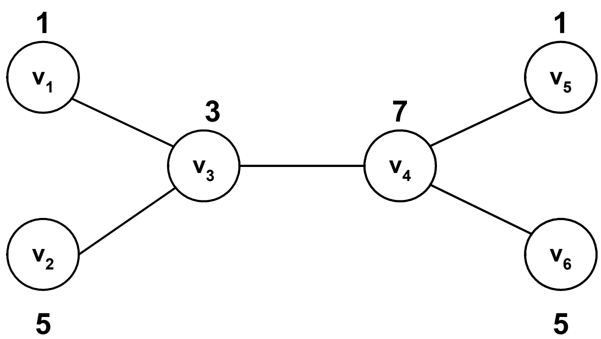

Figure 2: a 4 - radio coloring, $R=[1\ \ 5\ \ 3\ \ 7\ \ 1\ \ 5]$

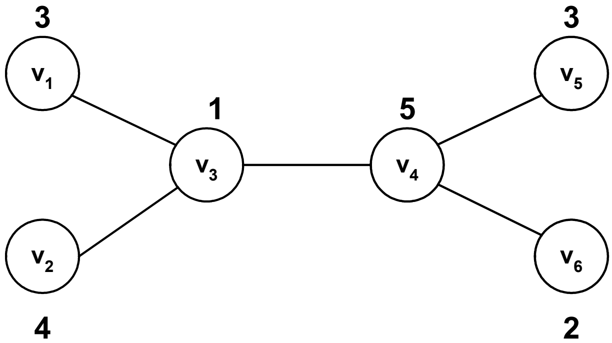

Figure 3: a 5 - radio coloring, $R = [3\ \ 4\ \ 1\ \ 5\ \ 3\ \ 2]$

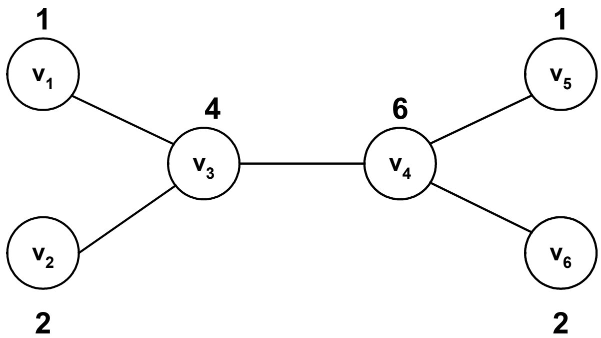

Figure 4: 4 - radio coloring, $R = [1\ \ 2\ \ 4\ \ 6\ \ 1\ \ 2]$

Figures 2, 3 and 4 show three radio colorings of a six node graph G. The values of the invariant for the radio coloring of Figure 2 are

$$\mathbf{rs}(\boldsymbol{G},\boldsymbol{R}) = \mathbf{rn}(\boldsymbol{G}) = \mathbf{4},\ \ \mathrm{rp}(G,R) = 7,\ \ \mathrm{rw}(G,R) = 22,\ \ \mathrm{rg}(G,R) = 6.$$

For the radio coloring of Figure 3 we have

$$\mathrm{rs}(G,R) = 5,\ \ \mathbf{rp}(\boldsymbol{G},\boldsymbol{R}) = \mathbf{rv}(\boldsymbol{G}) = \mathbf{5},\ \ \mathrm{rw}(G,R) = 18,\ \ \mathbf{rg}(\boldsymbol{G},\boldsymbol{R}) = \mathbf{rb}(\boldsymbol{G}) = \mathbf{4}.$$

and for the radio coloring of Figure 4

$$\mathbf{rs}(\boldsymbol{G},\boldsymbol{R}) = \mathbf{rn}(\boldsymbol{G}) = \mathbf{4},\ \ \mathrm{rp}(G,R) = 6,\ \ \mathbf{rw}(\boldsymbol{G},\boldsymbol{R}) = \mathbf{rc}(\boldsymbol{G}) = \mathbf{16},\ \ \mathrm{rg}(G,R) = 5.$$

The relations reflected with bold character indicate the minimum values of the corresponding invariant.

Algorithm Radio finds approximate values of $\mathrm{rn}(G)$ and $\mathrm{rc}(G)$ and uses similar reasoning to algorithm A. The second restriction where

$c(v_i) \neq c(v_j)$ if $d(v_i, v_j) = 2$

is taken into account during the construction of sets $H(v)$. Therefore Step 3.2 and Step 5 of A are replaced in Radio by the following steps while the other steps remain unchanged.

STEP 3.2:

If $Q(v_i, \bar{F}) = \varnothing$ proceed to Step 3.3.

Else

Set $z \leftarrow \min\{j$ such that $v_j \in Q(v_i, \bar{F})$ and $\Gamma_{\bar{F}}(v_j) \cap H(v_i) = \varnothing$ and $d(v_j, v_k) > 2,\ v_k \in H(v)\}$

$H(v_i) \leftarrow H(v_i) + \{v_z\},\ Q(v_i, \bar{F}) \leftarrow Q(v_i, \bar{F}) - (\{v_z\} \cup \Gamma(v_z))$

$NX \leftarrow NX + 1$ and repeat Step 3.2.

STEP 5: (Node coloring - Termination test)

Set $C_r \leftarrow H(L)$.

If $\bar{F} = \varnothing$, printout the chromatic classes $C_i,\ i = 1, 2, \ldots, r$ and Stop.

Else Set $r \leftarrow r + 1,\ p \leftarrow |\bar{F}|$ and proceed to Step 2.

5. COMPUTATIONAL EXPERIMENTS AND CONCLUSIONS

Table 1 gives the result of the application of algorithm Radio on graphs with various densities d and number of vertices n. Each item of the table represent the mean value of the approximate radio chromatic number of 10 running of algorithm Radio on random graphs with the same n and d. It is significant to note here that the deviation among the different approximate radio chromatic numbers obtained by Radio for graphs with the same characteristics was small.

For reason of comparison we show in Table 2 the result obtained by algorithm A , i.e., the mean approximate chromatic number of 10 running of algorithm A.

All invariants stated in the previous section belong to the class of *NP-complete* problems see [10] and the corresponding optimization cases fall in the *NP-hard* category, therefore only approximate values can be achieved, since the development of corresponding exact algorithms is unexpected. The values of the different invariants of the radio colorings of figures 2, 3 and 4 give a taste of the hard combinatorial character of the radio coloring concept.

Table 1: Approximate Radio chromatic Numbers - Algorithm Radio

N \ d	0.1	0.2	0.3	0.4	0.5
25	6	11	20	24	25
50	16	30	46	49	50
75	23	51	75	75	75
100	36	81	100	100	100

Table 2: Approximate chromatic Numbers - Algorithm A

N \ *d*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
25	3	4	6	6	6	8	9	11	15
50	4	6	7	9	10	12	15	18	23
75	5	8	9	12	14	16	21	24	31
100	7	9	11	14	16	21	25	31	39
125	7	10	12	16	21	24	29	35	45
150	7	11	15	18	23	26	34	42	55

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