Yugoslav Journal of Operations Research
13 (2003), Number 2, 217-227

CONVERGENCE OF ESTIMATED OPTIMAL INVENTORY LEVELS IN MODELS WITH PROBABILISTIC DEMANDS

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Abstract: The behavior of estimations of the optimal inventory level is analyzed. Two models are studied. The demands follow unknown probability distribution function. The included density functions are estimated and a plug-in rule is suggested for computing estimates of the optimal levels.
Two search algorithms are proposed and compared using Monte Carlo experiments.

Keywords: Backorders, simulated annealing, density function estimation.

1. INTRODUCTION

The inventory problems to be analyzed can be formulated as follows: "given set of demands of a certain number of periods, determine the parameters that implement a policy that ensures minimum costs at a long run".

We start with an inventory and at every time period t we examine the inventory position. The set-up cost $c(s)$ is associated to each placed order. A holding cost $c(h)$ is incurred per unit-time in the inventory stock. The backordered cost $c(b)$ is incurred per unit-time per backordered unit of demand. The cost of using a policy is a linear combination of set-up, holding and backordered costs.

We can consider this problem deterministic or we can assume that we face probabilistic demands. The later case is more realistic though a percent of the demands can be considered non-random. For example, the demands of a large buyer can be similar in any period. The level of the product in the firm should be set to reserve sufficient inventory in order to meet the deterministic demand. It is known before the next replenish. This problem is analyzed in Section 3 following the results of Haussmann-Thomas (1972). Section 4 is devoted to the analysis of an inventory model where all the demands during the stock out period are backordered or lost. Only a fraction α of the demands can be backordered. We follow the scheme used by Warrier-Shah (1997).

The classical approach is to observe asset of demands and to felicitate the needed probability distribution functions. Then an optimal inventory level r_0 is determined. It should minimize the expectation of the overall cost.

Iyer-Schrage (1992) analyzed the bad performance of this approach when the assumed distribution is not the real one. In this paper we propose to estimate the distribution function. The computed "optimal inventory level" is an estimate of r_0. We point out that its convergence depends on the convergence of the estimator of the density function. Non parametric density function estimation theory provides a frame for determining practical procedures for obtaining approximate solutions. A point wise converge is ensured. The examples worked out by Hausmann-Thomas (1972) and Warrier-Shah (1997) are reworked. Search algorithms are proposed. They allow determining integer solutions. The classical approach assumes the continuity of the random variables and a rounding-off permits to fix integer values of the "optimal inventor level". The behavior of two estimation procedures is analyzed in four examples.

2. SOME RESULTS ON DENSITY FUNCTION ESTIMATION

The estimation of a density function has its roots in the research of John Grant in 1661. The modern treatment of it is related with the use of nonparametric procedures; see Devroye-Gyorfy (1985) for a detailed discussion. A broad class of estimators of a density $g(x)$ is given by fixing a sequence of functions $\{\delta_h(x,u)\}$ so that $\delta : \Re \times \Re \to \Re$, which satisfies certain conditions. A class of delta sequence estimators is determined by

$$\hat{g}_h(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_h(x, X_i)$$

where X_i is the i-th observation of the random variable (rv) X. An adequate set C is defined by the properties:

(i) $\delta_h(x,u) \geq 0,\ \forall (x,u) \in \Re^2,\ h(n) = h \to \infty$

(ii) $\text{Sup}_u\, \delta_h(x,u) = O(h^{-1})$

(iii) $\forall n,\ \exists \xi_n \in \Re^+$ so that if $\| x-u \| \leq \xi_n \Rightarrow \delta_h(x,u) \geq O(h^{-1})$

(iv) $\text{Sup}_{\{\|x-u\| \geq c\xi_n\}} \delta_h(x,u) \leq \left(\dfrac{h^{-1}}{n}\right)$ with $c \geq 1$

(v) $n\xi_n \to \infty$ and $\xi_n \to 0$

(vi) $\int_{\Re} \delta_h(x,u)\,du = 1,\ \forall h$.

The function h is the bandwidth and it weights the values of u in terms of its closeness to x. The appropriate selection of h plays an important role in achieving good estimators. Bootstrap can be used for choosing h, see Faraway-Jhun (1990). Assume that $\delta_h(x,u)$ satisfies the above given conditions and that

(a) $\dfrac{h^{-1}}{n} \to 0$

(b) $\{\Psi_h(x,u)\} = \{\mathrm{Sup}_{\{tt| \|x-t\| \le \|x-u\|\}} \delta_h(x,t)\} \in L(h^{-1}), \ \forall x$

(c) $\left\{ \int_{\Re} \Psi_h(x,u)du \right\} < \infty, \ \forall x$

(d) $\int_{\{\|x-u\| \ge c\xi_n\}} \delta_h(x,u)du \to 0, \ \forall x$

(e) $nh \to \infty$ and $\sum_{n=1}^{\infty} \exp\{-c(nh)\} < \infty, \ \forall c > 0$.

Then, with the point wise convergence of the corresponding estimator of $g(x)$ holds almost surely, see Lima (1998) for a detailed discussion. She worked out detailed proofs for some well known estimators.

3. DETERMINISTIC RANDOM DEMANDS

The problem can be characterized by a firm that has a production in order to satisfy two types of demands. The deterministic demand of some client is known before the next replenish. The demand of the other buyers is unpredictable. Then $c(s) > 0$, setup costs are positive. The demands are placed periodically. In practice the demand of some big customers and/or the internal demand of the firm can be considered as known. The notation to be used is:

$D =$ random demand per period with density function $f(d)$ so that its expectation is $E(d) = \delta$ and its variance is $\mathrm{var}(d) = \Delta^2$.

$P =$ known process demands (in periods).

$Q =$ number of units ordered each time in which an order is placed.

$R =$ inventory level at which an order is placed

$X =$ random demand over the lead period with a density function $g(x)$ so that its expected value is $E(X) = X_g$ and its variance is $\mathrm{var}(X) = V^2$.

$\lambda =$ mean total demand per period $= \delta + P/\rho$

$\Lambda =$ fixed lead time in periods.

$\mu = X_g + nP + \dfrac{P\rho}{2\lambda}\left[\left(\dfrac{\tau}{\rho} - n\right)P + \left(1 - \dfrac{\delta}{\lambda}\right)\right] =$ average total demand over the lead time

$\rho =$ time between process demands in periods

$c(i) =$ carrying cost rate per period (percentage)

$c(h,i) =$ inventory cost per unit backordered of the random demand

$c(b,b) =$ backordered cost per unit of the process demand
$c(b,d) =$ backordered cost per unit of the process demand
$n_0 =$ minimum number of process breakdowns during one leading time $= \lfloor \tau / \rho \rfloor$ where $\lfloor z \rfloor$ represents the greatest integer closer to z and $c(bd) >> c(bb)$.

The total cost per period, using an infinite planning horizon, is:

$$T_c = \frac{\lambda c(s)}{Q} + c(i)\left(r - \mu + \frac{Q}{2} \right) + c(bb)\frac{\lambda E(r)}{Q} \tag{3.1}$$

$E(r)$ is the expectation of r and represents the number of units of backorders in one ordering cycle. If the distribution of r is unknown, it should be estimated.

We seek for minimizing (3.1). If we differentiate it with respect to Q and we can have:

$$\frac{\partial T_c}{\partial r} = 0 \Rightarrow Q = \sqrt{\frac{2\lambda\left[c(s) + c(bb)E(r)\right]}{c(i)}} \tag{3.2}$$

and

$$\frac{\partial T_c}{\partial r} = D' + D'' + D''' = 0$$

where

$$D' = \frac{\delta(\Lambda\rho^{-1} - n)(G(r - nP) - 1)}{\lambda}$$

$$D'' = (1 - \Lambda\rho^{-1} + n)(G(r - nP) - 1)$$

$$D''' = \frac{P}{\rho\Lambda}\int_0^P G(r - \xi - nP)d\xi - P$$

and

$$G(t) = \int_{-\infty}^{t} g(x)dx$$

The solution of (3.1) is

$$Q_0 = \sqrt{\frac{2\lambda\left[c(s) + c(bb)E(r)\right]}{c(i)}} \tag{3.4}$$

Two situations may arise with respect to the inventory level. They are:

1. *r* is not tripped by a process drawdown, denoted $r \propto NT$.

Then, see Hausmann-Thomas (1972)

$$E(r \mid r \propto NT) = R' + R'' = M_1(r)$$

Defining

$$R' = \frac{\int_{r-(n+1)P}^{\infty} (x - r - (n+1)P) g(x) dx}{\Lambda \rho^{-1} + n}$$

and

$$R'' = \frac{\int_{r-(n+1)P}^{\infty} (x - r - nP) g(x) dx}{1 - \Lambda \rho^{-1} + n}$$

whenever $c(bd)$ is sufficiently large for requiring that $r_0 \geq (n-1)/P$

2. *r* is tripped by a process drawdown, denoted *r*∝*T*.

Hence, there are n additional process draw downs during Λ and there are $r < r^*$ units in the inventory and on order.

As a result each $r \in R^* = \{r^* - P + 1, ..., r^* - 1\}$ is equally probable. That is $1/P$ is the probability of observing a $r \in R^*$. Then

$$E(r | r \propto T) = \int_0^P \int_{r_0 - \xi - nP}^{\infty} \frac{x - r - \xi - nP}{P} g(x) dx d\xi = M_2(r) \tag{3.6}$$

We should consider the incidence of each situation.

Taking $\alpha \in [0,1]$ as the fraction of untripped inventory levels we have that the expectation of r is

$$E(r) = \alpha M_1(r) + (1 - \alpha) M_2(r) \tag{3.7}$$

If α is known (3.7) is fixed. When it is unknown, a solution is to accept that $-\pi = P / \rho$ and to represent

$$\alpha = \frac{\delta}{\lambda} = \frac{1}{1 + \frac{P}{\delta \rho}} = \frac{1}{1 - \frac{\pi}{\delta}} \tag{3.8}$$

δ is unknown and the estimator of α can be derived by plugging-in its estimator. Then we have

$$\hat{\alpha} = \frac{\hat{\delta}}{\lambda} = \frac{1}{1 - \frac{\pi}{\hat{\delta}}} \tag{3.9}$$

The Taylor's formula permits to expand this expression and

$$\hat{\alpha} \approx \frac{\pi}{\delta}\left[1-\left(\frac{\hat{\delta}-\delta}{\delta}\right)+\left(\frac{\hat{\delta}-\delta}{\delta}\right)^2+\cdots\right]$$

If the terms of order larger than 2 are considered as negligible

$$E(\hat{\alpha}) \approx \frac{\pi}{\delta}+\frac{\pi}{\delta}\left[\frac{Var(\hat{\delta})}{\delta^2}\right]=\alpha+\frac{\Delta^2}{m\alpha^2}$$

Therefore, a robust estimate of δ, see Jureckovà-Sen (1996), is a mean based estimator as

$$\hat{\delta}=\frac{\sum_{t=1}^{m} d_t}{m} \tag{3.10}$$

where d_t is the demand in the observed period t. As $m \to \infty$ is granted that (3.10) tends to δ and there exists a m_0 so that for any $\varepsilon > 0$

$$|\hat{\alpha}-\alpha| < \varepsilon$$

A solution to (3.3) is obtained by using numerical methods because closed forms of the solution are not available for most densities $g(x)$. If $G(t)$ is known, it is possible to use a treatment-error procedure or a heuristic method.

The use of an inadequate distribution to represent the data has a poor performance, see Iyer-Schrage (1992). If we estimate $G(t)$ and no serious distribution assumptions are made, the errors depend on the statistical procedure which can be characterized theoretically. Say, that the adequateness of the computed optimal inventory level is sustained by the properties of the density function estimator used. If the estimator satisfies (i)-(iv) and (a)-(e) we have that

$$E(\hat{r})=\int_0^{\infty} r g_m(r)dr=\frac{1}{m}\int_0^{\infty} r\sum_{i=1}^{m}\delta_h(r,r_i)dr \xrightarrow{Em\to\infty(r)} E(r)$$

Then we can plug-in this estimation in (3.4). Similarly

$$\hat{G}(z)=\int_{-\infty}^{z} g_m(t)dt \xrightarrow{Gm\to\infty(z)} G(z)$$

Then D', D'' and D''' can be estimated using consistent estimators obtained by plugging-in the estimation of the distribution function.

The use of the easy computing estimator as the Rectangular Kernel

$$\delta_h(r,r_i)=\begin{cases}\frac{1}{2} \text{ if } |r|<1\\ 0 \text{ otherwise}\end{cases}$$

or the Gaussian Kernel

$$\delta_h(r, r_i) = \frac{\exp\left(-\frac{r^2}{2}\right)}{\sqrt{2\pi}}, \qquad \forall r \in \Re$$

They satisfy the hypothesis that sustains the point wise convergence, see Lima (1998).

The solution of the optimization problem may be computed by using the following Simulated Annealing algorithm.

Algorithm 1. Computation of an optimal inventory level

Step 1 Evaluate $T_c(r_0^*)$

Step 2 Input $K = K_0, T, Temp, B$

Step 3 For $t = 1$ to T do

Step 4 $R_{01} = r_0^* - K_0,\ R_{02} = r_0^* + K_0$

Step 5 Evaluate $T_c^*(R_{0i}^*),\ i = 1,2$.

If $T_c^*(R_{0i}^*) - T_c(r_0^*) = H \le 0$ then $r_0^* = r_0$

Else if $\exp(-H/Temp) > \text{random}(0,1)$ then $r_0^* = r_0$

Step 6 While $K_0 > 0,\ K_0 = K_0 - 1$

$Temp = Temp/B$ go to Step 3

Step 7 $r_0^* = r_0$

END

At the second step of the algorithms the parameters needed for running the algorithm are given. The parameter B is larger than one; hence the temperature is decreased at the end of each transition. The convergence of this algorithm follows because its construction fits with the hypothesis that sustains the equivalence with a sequence of inhomogeneous Markov Chain, see Aarts-Korsts (1989). The starting point is the solution computed by plugging in the density function estimates in the estimate of r_0 .

A simulation experiment was conducted for evaluating g of the proposed approach. 100 samples of size m were generated. Three values of α were analyzed: 0.05, 0.5 and 0.95. We generated the random demands using the normal distributions $N' = N(9,100)$, $N'' = N(100,9)$ and two exponentials: $E' = \exp(9)$ and $E'' = \exp(100)$. The Gaussian Kernel and the Rectangular Kernel were utilized for estimating g. Sample sizes $m = 100,$ 1000 and 5000 were used. The optimal values of T_c .

Each sample was computed and compared with the estimations. The behavior of each distribution p was evaluated by performing 100 runs for each combination.

$$\Delta_p = \frac{\sum_{k=1}^{100m} \left| T_c(r_0) - \hat{T}_c(r_0) \right|_k}{m T_c(r_0)}$$

Table 3.1: Results for Δ when the Gaussian Kernel was used

	$\alpha=0.05$	$m=100$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=1000$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=5000$ $\alpha=0.5$	$\alpha=0.95$
N'	11,2	8.3	3.4	9.3	9.5	8.8	5.4	6.2	1.6
N''	25.4	16.9	7.1	7.6	7.4	7.3	5.3	4.6	2.5
E'	34.0	45.3	20.3	28.9	56.4	78.1	59.2	57.1	83.6
E''	89.3	71.2	89.6	23.5	95.2	67.2	85.3	73.9	53.0

The Gaussian kernel exhibits a good performance when the normal distributions generated the demands. The increase of the sample size ensures gains in accuracy. Similarly occurs with respect to α. Therefore it works better when the sample size and the fraction of untripped inventory level is large. When the distribution is exponential the results do no exhibit a regular pattern

Table 3.2: Results for Δ when the Rectangular Kernel was used

	$\alpha=0.05$	$m=100$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=1000$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=5000$ $\alpha=0.5$	$\alpha=0.95$
N'	33.4	28.7	12.7	29.6	29.5	28.5	25.9	23.9	21.8
N''	67.7	54.8	18.1	30.2	28.6	27.5	28.3	21.0	18.7
E'	55.8	59.3	59.3	67.4	70.1	73.9	77.8	78.5	78.1
E''	64.0	70.6	75.4	69.2	74.9	69.6	71.6	68.0	70.2

The results of Table 3.2 suggest that the Rectangular Kernel is better than the Gaussian for the exponentials. The accuracy is very similar for the different values ofα. and m. When the distribution is normal, the use of a larger sample size and a larger α is the best option.

Therefore we can argue that the Gaussian Kernel with large sample size is recommended when we expect that g to be normal and the rectangular exponential.

4. MIXTURE OF BACKORDERS AND LOST SALES

Another common problem is to consider the inventory model as a mixture of backorders and lost sales. All the demands are completely backordered during the stock out periods, or t is lost forever. Warrier-Shah (1997) studied probabilistic order level systems. The fraction of backordered demands $\alpha \in [0\,1]$ is known. They considered $\Lambda = 0$ and that the scheduling period $\Pi \in [0,1]$ is fixed. The existing inventory for $t \in \Pi$ is $Q(t\,|\,x)$ for the demand x during Π and $dQ(t\,|\,x)/dt = -x/T$ with $Q(0\,|\,x) = r$. The solution to the differential equation that describes the instantaneous state of the existing inventory is $Q(t\,|\,x) = r - xt/T,\ t \in \Pi$.

Two situations may arise with the demands:

1. Shortages do not occur during Π. **[$x \le r$].**

Then

$$I_1 = \frac{1}{T}\int_0^{t*} Q[t \mid x]dt = r - \frac{x}{2} \tag{4.1}$$

2. Shortages occur during Π. **[x>r].**

In this case shortages occur after a $t^* < T$ because r satisfies the demand in $[0, t^*]$. As $Q(t^* \mid x) = 0$ we have that $t^* = rT/x$. The counter part of (4.1) is

$$I_{12} = \frac{1}{T}\int_0^{t*} Q(t \mid x)dt = \frac{r^2}{2x}$$

as shortages are observed during $]t^*, T]$ we obtain $-Q(t \mid x)$.

A fraction αm of shortages is backlogged. Hence

$$I_{21} = \alpha\left(\frac{r^2}{2x} + \frac{x}{2} - r\right) = \frac{\alpha(r-x)^2}{2x}$$

and the lost sales per unit during Π is

$$I_{22} = \frac{(1-\alpha)x(T-t^*)^2}{T}$$

The expected total cost of the system depends on the random demands. Then the density function g should be known for calculating it. We have three cost sources:

1. Inventory cost during Π.

The total cost per unit time given the density function g is:

$$c(h,r \mid g) = c(h)\left[\int_0^r \left(r - \frac{x}{2}\right)g(x)dx + \int_r^\infty \frac{r^2}{2x}g(x)dx\right]$$

2. Backlogging cost during Π.

$$c\left(b,b \middle| g\right) = c(bb)\left[\alpha\int_0^\infty \frac{(r-x)^2}{2x}g(x)dx\right]$$

represents the total expected cost due to backorders

3. Lost sales during Π.

$$c(b,d,r \mid g) = c(bd)\left[\frac{(1-\alpha)}{T}\int_r^\infty (x-r)g(x)dx\right]$$

measures the expectation of the corresponding total cost per unit. Then the optimal inventory level r_0 is obtained by minimizing:

$$T_c(r \mid g) = c(h, r \mid g) + c(b, d \mid g) + c(b, d, r \mid g) \tag{4.2}$$

By differentiating it we have

$$\frac{\partial T_c(r \mid g)}{\partial r} = S' + S'' + S''' = T_1(r_0 \mid g) = T_1(r_1 \mid g) = 0$$

where

$$S' = c(h) \int_0^{r_0} g(x)\,dx$$

$$S'' = (c(h) + \alpha c(bb)) \int_{r_0}^{\infty} \frac{r_0}{x} g(x)\,dx$$

$$S''' = \left[\alpha c(bb) + \frac{(1-\alpha)c(bd)}{T} \right] \int_{r_0}^{\infty} g(x)\,dx$$

Then, $T_1(r \mid g)$ is an estimating equation.

We have that r_0 is a minimum therefore:

$$\frac{\partial^2 T_c(r \mid g)}{\partial r^2} = \left[c(h) + \alpha c(bb) \right] \int_{r_0}^{\infty} \left[\frac{g(x)}{x} dx \right] + \frac{(1-\alpha)c(bd)}{T} g(r_0) = T_2(r_0 \mid g) \geq 0 \ .$$

The use of an estimator of g permits to compute estimates using the functional obtained by plugging in the estimator g_h. Then consistent estimates of S', S'' and S''' may be computed.

Note that $T_1(\infty|g)>0$ and $T_2(0|g)<0$. taking $r_0(g_h) \cong r_0$ when a sufficiently large sample is observed. Then it is an adequate starting point in any search procedure. The following algorithm permits to obtain a good approximation to r in a small number of iterations.

Algorithm 4.1. Search of an optimal inventory level or a mixture of backorders and lost sales.

Step 1. Evaluate $T(r_0(g_h))$

$t = 1$.

If $T_1(r_0(g_h)) = 0$ then go to Step 4

Step 2. If $T_1(r_0(g_h)) < 0$ then $r_t = r_{t-1} + 1$ else $r_t = r_{t-1} - 1$

Step 3 Compute $T_1(r_t(g_h)) = 0$

If $T_1(t_{tr-1}(g_h)) < 0$ and $T_1(r_t(g_h)) \geq 0$ then $r_0(g_h) = r_t$

If $T_1(t_{tr-1}(g_h)) \geq 0$ and $T_1(r_t(g_h)) < 0$ then $r_0(g_h) = r_{t+1}$ else

$t = t + 1$

Go to step 2

Step 4. $r_0(g_h) = r_t$

END

A Monte Carlo experiment was performed. It followed the same lines of the experiment of Section 3. The results appear in Tables 4.1 and 4.2. The average of

$$\Delta^* = \frac{1}{m}\sum_{t=1}^{m}\frac{|T_c(r\mid g) - T_c(r\mid g_h)|_t}{T_c(r\mid g)}$$

Table 4.1: Results for Δ* when the Gaussian Kernel was used

	$\alpha=0.05$	$m=100$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=1000$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=5000$ $\alpha=0.5$	$\alpha=0.95$
N'	21.7	18.9	18.8	19.3	18.2	17.3	18.2	18.3	15.7
N''	29.5	27.4	16.4	18.5	17.4	17.0	16.3	16.2	15.9
E'	1.0	1.1	1.1	1.1	0.7	0.7	0.9	0.7	0.8
E''	0.8	0.7	0.7	0.8	0.8	0.8	0.7	0.6	0.6

Tables 4.1 and 4.2 establish that the exponentials are better approximated than the normal

Table 4.2: Results for Δ* when the Rectangular Kernel was used

	$\alpha=0.05$	$m=100$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=1000$ $\alpha=0.5$	$\alpha=0.95$	$\alpha=0.05$	$m=5000$ $\alpha=0.5$	$\alpha=0.95$
N'	42.3	42.5	40.5	43.2	42.1	39.6	38.2	37.7	33.8
N''	56.1	55.8	53.1	55.3	53.8	47.5	44.5	40.2	38.5
E'	2.3	2.1	2.2	3.4	3.2	3.5	3.1	2.9	2.7
E''	4.7	4.5	3.8	4.1	3.7	3.2	3.5	3.2	3.0

Acknowledgements: The author thanks for the support received from DAAD and TWAS for developing the theoretical research. The computation was developed during a visit to LINAE, Universidad Veracruzana, Mexico. This paper has been improved with the helpful comments of two anonymous referees.

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